Math 320-3: Final Exam Solutions
Northwestern University, Spring 2015

1. Give an example of each of the following. No justification is required.

(a) A non-constant differentiable function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $D(f \circ f)(0,0)$ is not invertible.

(b) A region $D \subseteq \mathbb{R}^2$ such that $\iint_D x \, d(x,y) = \int_0^\pi \int_0^2 r^2 \cos\theta \, dr \, d\theta$.

(c) A C^1 surface in $\mathbb{R}^3$ which is not smooth at $(0,0,1)$.

(d) A non-conservative C^1 vector field $\mathbf{F} = (P,Q)$ on $\mathbb{R}^2 \setminus \{(0,0)\}$ such that $Q_x = P_y$.

(e) A C^1 vector field $\mathbf{F}$ on $\mathbb{R}^3$ such that $\iint_{\partial E} \mathbf{F} \cdot \mathbf{n} \, dS = \mathrm{Vol}(E)$, where E is the solid enclosed by the unit sphere centered at the origin and where ∂E has outward orientation.

Solution. (a) The function $f(x,y) = (x,0)$ works. This has Jacobian

$$Df(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

which is not invertible, so by the chain rule

$$D(f \circ f)(0,0) = Df(0,0)Df(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

is not invertible either.

(b) The top-half of the ball $B_2(0,0)$ works. In polar coordinates this is described by $0 \le r \le 2$ and $0 \le \theta \le \pi$, so the given integral equality follows by the change of variables formula.

(c) The surface with equation $z = 1 - \sqrt{x^2+y^2}$ works. This is an upside-down cone with its "point" at $(0,0,1)$.

(d) The field $\mathbf{F} = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$ works. You can check directly that $Q_x = P_y$, and this field is not conservative on $\mathbb{R}^2 \setminus \{(0,0)\}$ since its integral over the unit circle oriented counterclockwise is nonzero.

(e) The field $\mathbf{F} = (x,0,0)$ works, since $\iint_{\partial E} \mathbf{F} \cdot \mathbf{n} \, dS = \iint_E \operatorname{div} \mathbf{F} \, dV$ by the Divergence Theorem and $\operatorname{div} \mathbf{F} = 1$. □

2. Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function such that $f(2tx, 2ty) = t^2 f(x,y)$ for all $(x,y) \in \mathbb{R}^2$ and all $t \in \mathbb{R}$. Show that

$$2x \frac{\partial f}{\partial x}(2x, 2y) + 2y \frac{\partial f}{\partial y}(2x, 2y) = 2f(x,y)$$

for all $(x,y) \in \mathbb{R}^2$.

Proof. Differentiating both sides with respect to t using the chain rule gives:

$$\frac{\partial f}{\partial x}(2tx, 2ty) \frac{\partial (2tx)}{\partial t} + \frac{\partial f}{\partial y}(2tx, 2ty) \frac{\partial (2ty)}{\partial t} = 2tf(x,y),$$

or

$$2x \frac{\partial f}{\partial x}(2tx, 2ty) + 2y \frac{\partial f}{\partial t}(2tx, 2ty) = 2tf(x,y).$$

Setting $t = 1$ gives the desired equality. □

3. Let S_1 be the surface in $\mathbb{R}^3$ consisting of all points satisfying

$$xyz^2 = 0$$

and S_2 the surface consisting of all points satisfying

$$y - ze^{xy} = -1.$$

Show that the curve where S_1 and S_2 intersect is smooth at $(1, 0, 1)$. Hint: Start by showing that two of the variables (x, y, z) can be expressed as C^1 functions of the third.

Proof. Let $F : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by

$$F(xyz^2, y - ze^{xy} + 1)$$

and note that $F(1, 0, 1) = (0, 0)$. We have

$$DF_{(y,z)}(x, y, z) = \begin{pmatrix} xz^2 & 1 - xze^{xy} \\ 2xyz & -e^{xy} \end{pmatrix},$$

sp

$$DF_{(y,z)}(1, 0, 1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is invertible. By the Implicit Function Theorem, there exist C^1 functions $y(x), z(x)$ defined on some interval around $x = 1$ such that $y = y(x)$ and $z = z(x)$ satisfy $F(x, y(x), z(x)) = 0$. Thus

$$x = t,\ y = y(t),\ z = z(t)$$

give parametric equations for the curve in question. These equations in particular give $\mathbf{x}'(t)$ with first component equal to 1 at $t = 1$, so $\mathbf{x}'(1) \neq \mathbf{0}$ and hence the curve in question is smooth at $(1, 0, 1)$. $\square$

4. Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is a C^1 function such that $\|Df(x, y)\| \leq \|(x, y)\|$ for all (x, y). If $f(0, 0) = 0$, show that

$$\left| \iint_{B_2(0,0)} (x + y) f(x, y)\, d(x, y) \right| \leq 32\sqrt{2}\pi.$$

Hint: $|\cos\theta + \sin\theta| \leq \sqrt{2}$ for all θ, which you can use without justification.

Proof. The closed ball $\overline{B_2(0, 0)}$ is compact and convex, so the Mean Value Theorem implies that there exists $\mathbf{c} \in L(\mathbf{0}; (x, y))$ such that

$$f(x, y) - f(0, 0) = Df(\mathbf{c}) \begin{pmatrix} x \\ y \end{pmatrix}.$$

Taking norms and using the fact that $f(0, 0) = 0$ gives

$$\|f(x, y)\| \leq \left\| Df(\mathbf{c}) \begin{pmatrix} x \\ y \end{pmatrix} \right\| \leq \|Df(\mathbf{c})\| \, \|(x, y)\| \leq \|\mathbf{c}\| \, \|(x, y)\| \leq 2 \, \|(x, y)\|,$$

where in the last inequality we use the fact that $\|\mathbf{c}\| \leq 2$ since $\mathbf{c} \in B_2(0, 0)$. Thus:

$$\left| \iint_{B_2(0,0)} (x + y) f(x, y)\, d(x, y) \right| \leq \iint_{B_2(0,0)} |x + y| |f(x, y)| d(x, y)$$

$$\leq \iint_{B_2(0,0)} 2|x+y|\,\|(x,y)\|\,d(x,y).$$

Converting to polar coordinates gives

$$\iint_{B_2(0,0)} 2|x+y|\,\|(x,y)\|\,d(x,y) = \int_0^{2\pi}\int_0^2 2r^3|\cos\theta+\sin\theta|\,dr\,d\theta \leq \int_0^{2\pi}\int_0^2 4\sqrt{2}r^3\,dr\,d\theta = 32\sqrt{2}\pi$$

as required. (Note that there are better bounds you can get as opposed to $32\sqrt{2}\pi$.) $\square$

5. Suppose that $\phi : E \to \mathbb{R}^3$ and $\psi : D \to \mathbb{R}^3$ (where $E, D \subseteq \mathbb{R}^2$) are C^1 functions and that $\tau : D \to E$ is a one-to-one C^1 function whose image is all of E and such that $\psi = \phi \circ \tau$. If $\det D\tau(s,t) < 0$ at all points $(s,t) \in D$ except those where $s = 0$ or $t = 0$, show that

$$\iint_E \phi(u,v)\cdot(\phi_u(u,v)\times\phi_v(u,v))\,d(u,v) = -\iint_D \psi(s,t)\cdot(\psi_s(s,t)\times\psi_t(s,t))\,d(s,t).$$

Proof. By the change of variables formula, we have

$$\iint_D \phi(\tau(s,t))\cdot(\phi_u(\tau(s,t))\times\phi_v(\tau(u,v)))|\det D\tau(s,t)|\,d(s,t) = \iint_E \phi(u,v)\cdot(\phi_u(u,v)\times\phi_v(u,v))\,d(u,v),$$

where $(u,v) = \tau(s,t)$. Since $\det D\tau(s,t) < 0$ everywhere on D except for on a set of measure zero, we have

$$|\det D\tau(s,t)| = -(\det D\tau(s,t))$$

everywhere on D except for on a set of measure zero. Since values of integrals are unchanged when altering the integrand on a set of measure zero, we get that the integral on the left above is

$$-\iint_D \phi(\tau(s,t))\cdot(\phi_u(\tau(s,t))\times\phi_v(\tau(u,v)))(\det D\tau(s,t))\,d(s,t).$$

Since $\psi = \phi\circ\tau$, by the chain rule we have

$$\psi_s(s,t)\times\psi_t(s,t) = (\det D\tau(s,t))(\phi_u(\tau(s,t))\times\phi_v(\tau(u,v)))$$

as we saw in class. Thus the integral above becomes

$$-\iint_D \psi(s,t)\cdot(\psi_s(s,t)\times\psi_t(s,t))\,d(s,t)$$

as required. $\square$

6. Suppose that D is the unit disk $x^2+y^2 \leq 1$ in $\mathbb{R}^2$ and that $v : D \to \mathbb{R}$ is a C^2 function such that $v_{xx}+v_{yy} = 0$ on D. Show that if $u : D \to \mathbb{R}$ is any C^2 function, then

$$\int_{\partial D} u\nabla v\cdot(x,y)\,ds = \iint_D \nabla u\cdot\nabla v\,dA$$

where ∂D is oriented counterclockwise. Hint: At any point (x,y) on the unit circle ∂D, the vector (x,y) is normal to ∂D.

Proof. We have

$$u\nabla v = (uv_x, uv_y),$$

so

$$u\nabla v \cdot (x, y) = uv_x x + uv_y y = (-uv_y, uv_x) \cdot (-y, x).$$

Thus the integral on the left becomes

$$\int_{\partial D} u\nabla v \cdot (x, y)\, ds = \int_{\partial D} (-uv_y, uv_x) \cdot \mathbf{T}\, ds$$

where $\mathbf{T} = (-y, x)$ is the unit tangent vector along ∂D. By Green's Theorem,

$$\int_{\partial D} (-uv_y, uv_x) \cdot \mathbf{T}\, ds = \iint_D \left(\frac{\partial (uv_x)}{\partial x} - \frac{\partial (-uv_y)}{\partial y} \right) dA = \iint_D (u_x v_x + uv_{xx} + u_y v_y + uv_{yy})\, dA.$$

Since $v_{xx} + v_{yy} = 0$ on D, this final integral simplifies to

$$\iint_D (u_x v_x + u_y v_y)\, dA = \iint_D \nabla u \cdot \nabla v\, dA$$

as required. $\square$

7. Do **EITHER** (a) **OR** (b).

Extra Credit: (5 points) Do the other one, making clear which is the part you want to count for Problem 7 and which you want to count for extra credit.

(a) Define $f : [0, 1] \times [0, 1] \to \mathbb{R}$ by

$$f(x, y) = \begin{cases} 0 & \text{if } (x, y) \text{ is of the form } \left(\frac{p}{n}, \frac{q}{n}\right) \text{ for some } p, q, n \in \mathbb{N} \\ 2 & \text{otherwise.} \end{cases}$$

Show that the iterated integrals of f exist and are equal. Careful: Do not take it for granted that f is integrable on $[0, 1] \times [0, 1]$.

(b) Let $\mathbf{F}(x, y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$. Show that if C_1 and C_2 are two simple, closed smooth curves in $\mathbb{R}^2$ which do not pass through $(0, 0)$, do not intersect each other, and which are oriented clockwise, then $\int_{C_1} \mathbf{F} \cdot \mathbf{T}\, ds = \int_{C_2} \mathbf{F} \cdot \mathbf{T}\, ds$.

Proof. (a) Fix $x \in [0, 1]$. If x is not of the form $\frac{p}{n}$, then $f(x, y) = 2$ for all $y \in [0, 1]$, so

$$\int_0^1 f(x, y)\, dy = \int_0^1 2\, dy = 2$$

for such x. Suppose now that $x = \frac{p}{n}$ for some $p, n \in \mathbb{N}$. Then the only values of y which are of the form $\frac{q}{n}$ are

$$\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, \frac{n}{n}.$$

Thus $f(x, y) = 0$ only finitely often and $f(x, y) = 2$ infinitely often for such x as y-varies, so the single variable function $y \mapsto f(x, y)$ is integrable over $[0, 1]$ for such x (it has finitely many discontinuities), and

$$\int_0^1 f(x, y)\, dy = \int_0^1 2\, dy = 2.$$

Thus

$$\int_0^1 \int_0^1 f(x,y)\,dy\,dx = \int_0^1 2\,dx = 2.$$

Switching the roles of x and y in the argument above shows that $\int_0^1 \int_0^1 f(x,y)\,dx\,dy$ also exists and equals 2 as well.

(b) First we prove this using Green's Theorem. Let D be the region between C_1 and C_2, so that ∂D consists of both C_1 and C_2. Suppose that C_2 is within C_1, and switch the orientation of C_1. Then Green's Theorem applies to give

$$\int_{-C_1+C_2} \mathbf{F}\cdot\mathbf{T}\,ds = \iint_D (Q_x - P_y)\,dA$$

where

$$P = \frac{x}{x^2+y^2} \quad \text{and} \quad Q = \frac{y}{x^2+y^2}.$$

A direct computation shows that $Q_x = P_y$, so the integral on the right above is zero. Hence

$$\int_{-C_1} \mathbf{F}\cdot\mathbf{T}\,ds + \int_{C_2} \mathbf{F}\cdot\mathbf{T}\,ds = 0,$$

so

$$\int_{C_2} \mathbf{F}\cdot\mathbf{T}\,ds = -\int_{-C_1} \mathbf{F}\cdot\mathbf{T}\,ds = \int_{C_1} \mathbf{F}\cdot\mathbf{T}\,ds$$

as required.

Alternatively, we can recognize that $\mathbf{F}$ is the gradient of

$$f(x,y) = \frac{1}{2}\ln(x^2+y^2).$$

Then $\int_{C_1} \mathbf{F}\cdot\mathbf{T}\,ds$ and $\int_{C_2} \mathbf{F}\cdot\mathbf{T}\,ds$ are both zero since the integral of any conservative field over any closed curve is zero. $\square$