Math 320-2: Final Exam Solutions
Northwestern University, Winter 2015

1. Give an example of each of the following. You do not have to justify your answer.

(a) A pointwise convergent sequence of functions on $[1,2]$ which is not uniformly convergent.
(b) A subset of $\mathbb{R}$ with empty interior and closure equal to $[0,1]$ under the Euclidean metric.
(c) A disconnected subset of $\mathbb{R}^2$ which is compact with respect to the taxicab metric.
(d) A bounded continuous function $f : \mathbb{R}^2 \to \mathbb{R}^2$ with respect to the box metric.
(e) A metric on $\mathbb{R}$ with respect to which $(1,2)$ is open but not connected.

Solution. (a) The sequence $f_n(x) = (x-1)^n$ works, analogously to how $g_n(x) = x^n$ is such an example on $[0,1]$.

(b) The set $[0,1] \cap \mathbb{Q}$ works: this is dense in $[0,1]$ so its closure is $[0,1]$ and has empty interior since it contains no irrationals.

(c) The subset consisting of the closed ball of radius 1 around $(0,0)$ and the closed ball of radius 1 around $(5,5)$ works.

(d) Any constant function works.

(e) The discrete metric is an example: since every subset if open with respect to a discrete metric, $(1,2)$ is open but not connected since $(1,2) = (1,\frac{3}{2}] \cup (\frac{3}{2},2)$ is a union of disjoint, open, nonempty sets. $\square$

2. For each $n \in \mathbb{N}$, define the function $f_n : [0,2] \to \mathbb{R}$ by

$$f_n(x) = x \sin\left(\frac{x}{n}\right) + \sqrt{x^2 + \frac{1}{n}}.$$

Show that (f_n) converges uniformly on $[0,2]$.

Proof. First, for a fixed $x \in [0,2]$, we have

$$f_n(x) \to x\sin(0) + \sqrt{x^2} = \sqrt{x^2}$$

so (f_n) converges pointwise to $f(x) = \sqrt{x^2}$. To show that this convergence is uniform, let $\epsilon > 0$ and pick $N \in \mathbb{N}$ such that $\frac{25}{n} < \epsilon^2$. Then if $n \geq N$, we have for $x \in [0,2]$:

$$\begin{aligned}
|f_n(x) - f(x)| &= \left| x\sin\frac{x}{n} + \sqrt{x^2 + \frac{1}{n}} - \sqrt{x^2} \right| \\
&\leq \left| x\sin\frac{x}{n} \right| + \left| \sqrt{x^2 + \frac{1}{n}} - \sqrt{x^2} \right| \\
&= |x| \left| \sin\frac{x}{n} \right| + \left| \sqrt{\frac{1}{n}} \right| \\
&\leq \frac{|x|^2}{n} + \frac{1}{\sqrt{n}} \\
&\leq \frac{4}{\sqrt{n}} + \frac{1}{\sqrt{n}} \\
&= \frac{5}{\sqrt{n}} \\
&< \epsilon.
\end{aligned}$$

Thus $f_n \to f$ uniformly as claimed. $\square$

3. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is analytic on all of $\mathbb{R}$ and that its 3rd derivative satisfies

$$f^{'''}\left(\frac{1}{n}\right) = 0 \text{ for all } n \in \mathbb{N}.$$

Show that f is a polynomial of degree at most 2. Hint: Argue that $f^{'''}(0) = 0$ and use this to show that $f^{'''}$ must be zero everywhere. Also recall that derivatives of analytic functions are analytic.

Proof. Since f is analytic, so is f', and hence f'', and hence f''' as well. Analytic functions are continuous, so f''' is continuous and thus since $\frac{1}{n} \to 0$, we have

$$0 = f'''\left(\frac{1}{n}\right) \to f'''(0),$$

so $f'''(0) = 0$. If f''' was not the zero function, there would thus exist an interval around 0 on which f was zero only at $x = 0$, which is not possible since any such interval will contain a number of the form $x = \frac{1}{n}$ at which f is also zero. Thus f''' must be the zero function. (Said more succinctly using fancier language, a nonzero analytic function has isolated zeroes, and $x = 0$ in this case is not an isolated zero of the analytic function f''', so f''' must be the zero function.)

If f''' is the zero function, all higher order derivatives are also identically zero. Thus the Taylor series of f around 0 looks like

$$f(x) = a_0 + a_1 x + a_2 x^2 + \text{ a bunch of terms which are all zero,}$$

so f is a polynomial of degree at most 2 as claimed. $\square$

4. Equip $\mathbb{R}^2$ with the Euclidean metric and let $\mathbb{Q}^2$ denote the set of points in $\mathbb{R}^2$ whose coordinates are both rational. Determine, with justification, the boundary of $\mathbb{Q}^2$ in $\mathbb{R}^2$.

Solution. We claim that the boundary of $\mathbb{Q}^2$ in $\mathbb{R}^2$ is all of $\mathbb{R}^2$. Note that it suffices to show this using the box metric on $\mathbb{R}^2$ since the boundary with respect to the Euclidean metric is the same as the boundary with respect to the box metric. Take any $(a, b) \in \mathbb{R}^2$ and any ball $B_r(a, b)$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exist $m, n \in \mathbb{Q}$ such that $m \in (a - r, a + r)$ and $n \in (b - r, b + r)$, and thus $(m, n) \in B_r(a, b) = (a - r, a + r) \times (b - r, b + r)$. (Note that if we had stuck with a Euclidean ball, the resulting (m, n) might not be in $B_r(a, b)$ since in this case this ball is smaller than the rectangle $(a - r, a + r) \times (b - r, b + r)$. You would have to be a bit more careful about how you pick m and n.)

Similarly, since the irrationals are dense in $\mathbb{R}$, there exist $x, y \in \mathbb{R}\backslash\mathbb{Q}$ such that $x \in (a - r, a + r)$ and $y \in (b - r, b + r)$, so that $(x, y) \in B_r(a, b)$. Thus any ball around (a, b) contains something in $\mathbb{Q}^2$ and something in $\mathbb{R}^2\backslash\mathbb{Q}^2$, so (a, b) is a boundary point of $\mathbb{Q}^2$ as claimed. $\square$

5. Let Y be a discrete metric space and suppose that $\mathbb{R}^3$ is equipped with the Euclidean metric. Show that any continuous function $f : \mathbb{R}^3 \to Y$ is constant.

Proof. Since $\mathbb{R}^3$ is connected and f is continuous, the image $f(\mathbb{R}^3)$ of f is connected as well. But the only nonempty connected subsets of a discrete space are those consisting of single points, so $f(\mathbb{R}^3)$ consists of a single point, meaning that f is constant.

To justify that the only nonempty connected subspaces of a discrete space are singletons, suppose that $S \subseteq Y$ is connected and nonempty. Take $p \in S$. Then

$$S = \{p\} \cup (S\backslash\{p\})$$

is a union of two disjoint open subsets (since any subset of a discrete space is open), so one set on the right must be empty. Thus $S\backslash\{p\}$ is empty, meaning that $S = \{p\}$ as required. $\square$

6. Suppose that K is a compact metric space and that $f : K \to \mathbb{R}$ is a function which is *locally bounded*, meaning that for every $p \in K$ there exists an open ball $B_r(p)$ around p on which f is bounded. Show that f is bounded on all of K.

(Careful: we are not assuming that f is continuous. Also, we do not know beforehand that the bound on f over one open ball must be the same as the bound it has over a different open ball.)

Proof. Since f is locally bounded, for any $p \in K$ there exists $B_{r(p)}(p)$ on which f is bounded, say by $M_p \in \mathbb{R}$. (The subscript is used to emphasize the dependence on p.) The open balls $B_{r(p)}(p)$ together form an open cover of K, so since K is compact this has a finite subcover, say

$$K \subseteq B_{r(p_1)}(p_1) \cup \cdots \cup B_{r(p_n)}(p_n).$$

Let M be the maximum of the corresponding local bounds $M_{p_1}, \ldots, M_{p_n}$. For any $q \in K$, q is in some open ball $B_{r(p_i)}(p_i)$, so $|f(q)| \leq M_{p_i} \leq M$. Thus $|f(q)| \leq M$ for all $q \in K$, so f is bounded on all of K. $\square$

7. Let D be the subset of $\mathbb{R}^2$ given by the inequality $(x-2)^2 + (y-3)^2 \leq 1$, so D consists of the circle $(x-2)^2 + (y-3)^2 = 1$ and the region it encloses, and let $f : D \to \mathbb{R}$ be the function $f(x, y) = xy - x^2 + ye^{xy}$. Show that there exists $(a, b) \in D$ such that

$$f(x, y) \leq f(a, b) \text{ for all } (x, y) \in D.$$

Proof. First, D is bounded since it is contained in the open ball $B_2(2, 3)$ taken with respect to the Euclidean metric, and it is closed since it contains its boundary circle. (Or, we can also say that it is closed since it equals the *closed* ball of radius 1 around $(2, 3)$, and closed balls in any metric space are always closed.) Thus D is compact by the Heine-Borel Theorem.

For $(x_n, y_n) \to (x, y)$ in D, we have $x_n y_n \to xy$, $x_n^2 \to x^2$, and $e^{x_n y_n} \to e^{xy}$ since the single-variable exponential function is continuous. Thus $f(x_n, y_n) \to f(x, y)$, so f is continuous and thus the Extreme Value Theorem implies that f has a maximum, which is what is asked for. $\square$