Math 320-3: Final Exam
Northwestern University, Spring 2015

Name: ______________________________

1. (10 points) Give an example of each of the following. No justification is required.

(a) A non-constant differentiable function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $D(f \circ f)(0, 0)$ is not invertible.

(b) A region $D \subseteq \mathbb{R}^2$ such that $\iint_D x \, d(x, y) = \int_0^\pi \int_0^2 r^2 \cos\theta \, dr \, d\theta$.

(c) A C^1 surface in $\mathbb{R}^3$ which is not smooth at $(0, 0, 1)$.

(d) A non-conservative C^1 vector field $\mathbf{F} = (P, Q)$ on $\mathbb{R}^2 \setminus \{(0, 0)\}$ such that $Q_x = P_y$.

(e) A C^1 vector field $\mathbf{F}$ on $\mathbb{R}^3$ such that $\iint_{\partial E} \mathbf{F} \cdot \mathbf{n} \, dS = \mathrm{Vol}(E)$, where E is the solid enclosed by the unit sphere centered at the origin and where ∂E has outward orientation.

Problem	Score
1	
2	
3	
4	
5	
6	
7	
Total	

2. (15 points) Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function such that $f(2tx, 2ty) = t^2 f(x, y)$ for all $(x, y) \in \mathbb{R}^2$ and all $t \in \mathbb{R}$. Show that

$$2x\frac{\partial f}{\partial x}(2x, 2y) + 2y\frac{\partial f}{\partial y}(2x, 2y) = 2f(x, y)$$

for all $(x, y) \in \mathbb{R}^2$.

3. (15 points) Let S_1 be the surface in $\mathbb{R}^3$ consisting of all points satisfying

$$xyz^2 = 0$$

and S_2 the surface consisting of all points satisfying

$$y - ze^{xy} = -1.$$

Show that the curve where S_1 and S_2 intersect is smooth at $(1, 0, 1)$. Hint: Start by showing that two of the variables (x, y, z) can be expressed as C^1 functions of the third.

4. (15 points) Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is a C^1 function such that $\|Df(x,y)\| \leq \|(x,y)\|$ for all (x,y). If $f(0,0) = 0$, show that

$$\left| \iint_{B_2(0,0)} (x+y) f(x,y) \, d(x,y) \right| \leq 32\sqrt{2}\pi.$$

Hint: $|\cos\theta + \sin\theta| \leq \sqrt{2}$ for all θ, which you can use without justification.

5. (15 points) Suppose that $\phi : E \to \mathbb{R}^3$ and $\psi : D \to \mathbb{R}^3$ (where $E, D \subseteq \mathbb{R}^2$) are C^1 functions and that $\tau : D \to E$ is a one-to-one C^1 function whose image is all of E and such that $\psi = \phi \circ \tau$. If $\det D\tau(s,t) < 0$ at all points $(s,t) \in D$ except those where $s = 0$ or $t = 0$, show that

$$\iint_E \phi(u,v) \cdot (\phi_u(u,v) \times \phi_v(u,v))\, d(u,v) = -\iint_D \psi(s,t) \cdot (\psi_s(s,t) \times \psi_t(s,t))\, d(s,t).$$

6. (15 points) Suppose that D is the unit disk $x^2 + y^2 \leq 1$ in $\mathbb{R}^2$ and that $v : D \to \mathbb{R}$ is a C^2 function such that $v_{xx} + v_{yy} = 0$ on D. Show that if $u : D \to \mathbb{R}$ is any C^2 function, then

$$\int_{\partial D} u \nabla v \cdot (x, y)\, ds = \iint_D \nabla u \cdot \nabla v \, dA$$

where ∂D is oriented counterclockwise. Hint: At any point (x, y) on the unit circle ∂D, the vector (x, y) is normal to ∂D.

7. (15 points) Do **EITHER** (a) **OR** (b).

Extra Credit: (5 points) Do the other one, making clear which is the part you want to count for Problem 7 and which you want to count for extra credit.

(a) Define $f : [0,1] \times [0,1] \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} 0 & \text{if } (x,y) \text{ is of the form } \left(\frac{p}{n}, \frac{q}{n}\right) \text{ for some } p,q,n \in \mathbb{N} \\ 2 & \text{otherwise.} \end{cases}$$

Show that the iterated integrals of f exist and are equal. Careful: Do not take it for granted that f is integrable on $[0,1] \times [0,1]$.

(b) Let $\mathbf{F}(x,y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$. Show that if C_1 and C_2 are two simple, closed smooth curves in $\mathbb{R}^2$ which do not pass through $(0,0)$, do not intersect each other, and which are oriented clockwise, then $\int_{C_1} \mathbf{F} \cdot \mathbf{T}\, ds = \int_{C_2} \mathbf{F} \cdot \mathbf{T}\, ds$.