# Math 320-2: Final Exam
## Northwestern University, Winter 2015

**Name**: ____________________

**1.** (10 points) Give an example of each of the following. You do not have to justify your answer.

(a) A pointwise convergent sequence of functions on $[1, 2]$ which is not uniformly convergent.

(b) A subset of $\mathbb{R}$ with empty interior and closure equal to $[0, 1]$ under the Euclidean metric.

(c) A disconnected subset of $\mathbb{R}^2$ which is compact with respect to the taxicab metric.

(d) A bounded continuous function $f : \mathbb{R}^2 \to \mathbb{R}^2$ with respect to the box metric.

(e) A metric on $\mathbb{R}$ with respect to which $(1, 2)$ is open but not connected.

**2.** (15 points) For each $n \in \mathbb{N}$, define the function $f_n : [0, 2] \to \mathbb{R}$ by

$$f_n(x) = x \sin\left(\frac{x}{n}\right) + \sqrt{x^2 + \frac{1}{n}}.$$

Show that $(f_n)$ converges uniformly on $[0, 2]$.

**3.** (15 points) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is analytic on all of $\mathbb{R}$ and that its 3rd derivative satisfies

$$f'''\left(\frac{1}{n}\right) = 0 \text{ for all } n \in \mathbb{N}.$$

Show that $f$ is a polynomial of degree at most 2. Hint: Argue that $f'''(0) = 0$ and use this to show that $f'''$ must be zero everywhere. Also recall that derivatives of analytic functions are analytic.

**4.** (15 points) Equip $\mathbb{R}^2$ with the Euclidean metric and let $\mathbb{Q}^2$ denote the set of points in $\mathbb{R}^2$ whose coordinates are both rational. Determine, with justification, the boundary of $\mathbb{Q}^2$ in $\mathbb{R}^2$.

**5.** (15 points) Let $Y$ be a discrete metric space and suppose that $\mathbb{R}^3$ is equipped with the Euclidean metric. Show that any continuous function $f : \mathbb{R}^3 \to Y$ is constant.

**6.** (15 points) Suppose that $K$ is a compact metric space and that $f : K \to \mathbb{R}$ is a function which is *locally bounded*, meaning that for every $p \in K$ there exists an open ball $B_r(p)$ around $p$ on which $f$ is bounded. Show that $f$ is bounded on all of $K$.

(Careful: we are not assuming that $f$ is continuous. Also, we do not know beforehand that the bound on $f$ over one open ball must be the same as the bound it has over a different open ball.)

**7.** (15 points) Let $D$ be the subset of $\mathbb{R}^2$ given by the inequality $(x-2)^2 + (y-3)^2 \leq 1$, so $D$ consists of the circle $(x-2)^2 + (y-3)^2 = 1$ and the region it encloses, and let $f : D \to \mathbb{R}$ be the function $f(x, y) = xy - x^2 + ye^{xy}$. Show that there exists $(a, b) \in D$ such that

$$f(x, y) \leq f(a, b) \text{ for all } (x, y) \in D.$$