Math 320-1: Midterm 1 Solutions
Northwestern University, Fall 2014

1. Give an example of each of the following. You do not have to justify your answer.

(a) A subset of $\mathbb{Q}$ with an irrational infimum and no supremum.

(b) A strictly decreasing sequence which converges to π.

(c) A Cauchy sequence (x_n) of nonzero numbers such that the sequence $\left(\frac{1}{x_n}\right)$ is not Cauchy.

(d) A nonconstant convergent sequence (x_n) and bounded sequence (y_n) such that $(x_n y_n)$ does not converge.

Solution. (a) $\mathbb{Q} \cap (-\sqrt{2}, \infty)$ works since it has infimum $-\sqrt{2}$ and is unbounded above.

(b) $x_n = \pi + \frac{1}{n}$ works.

(c) $x_n = \frac{1}{n}$ works since, being convergent, it is Cauchy but $\frac{1}{x_n} = n$ is unbounded and hence not Cauchy.

(d) Taking $x_n = 1 + \frac{1}{n}$ and $y_n = (-1)^n$ works since $x_n y_n = (-1)^n(1 + \frac{1}{n})$ does not converge because the even-indexed terms converge to 1 while the odd-indexed terms converge to -1. $\square$

2. Determine the supremum of

$$A = \left\{ \frac{n-1}{2n-1} \;\middle|\; n \in \mathbb{N} \right\}$$

and prove that your answer is correct.

Proof. We claim that $\sup A = \frac{1}{2}$. Indeed, since $2n - 2 < 2n - 1$ for all $n \in \mathbb{N}$, we have

$$\frac{2(n-1)}{2n-1} < 1, \text{ so } \frac{n-1}{2n-1} < \frac{1}{2} \text{ for all } n \in \mathbb{N},$$

so $\frac{1}{2}$ is an upper bound of A. For $\epsilon > 0$, picking $N \in \mathbb{N}$ such that

$$\frac{1}{4N-2} < \epsilon$$

gives

$$\frac{1}{2} - \frac{N-1}{2N-1} = \frac{1}{4N-2} < \epsilon,$$

so

$$\frac{1}{2} - \epsilon < \frac{N-1}{2N-1}.$$

This shows that nothing smaller than $\frac{1}{2}$ is an upper bound of A, so $\frac{1}{2}$ is the least upper bound as claimed. $\square$

3. Suppose that (x_n) is a sequence which converges to -1. Show that the sequence $(\sqrt[3]{x_n})$ of cube roots also converges to -1. Hint: For any $a, b \in \mathbb{R}$, $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$.

Proof. Let $\epsilon > 0$. Since $x_n \to -1$, there exists $N_1 \in \mathbb{N}$ such that

$$|x_n + 1| < \epsilon \text{ if } n \geq N_1.$$

Also, there exists $N_2 \in \mathbb{N}$ such that

$$|x_n + 1| < 1 \text{ for } n \geq N_2,$$

which implies that $x_n < 0$ (since $x_n + 1 < 1$) for $n \geq N_2$. Thus for $n \geq \max\{N_1, N_2\}$ we have

$$|x_n + 1| < \epsilon \text{ and } |\sqrt[3]{x_n^2} - \sqrt[3]{x_n} + 1| > 1,$$

where is the second inequality is true since $\sqrt[3]{x_n^2}$ and $-\sqrt[3]{x_n}$ are both positive for $n \geq \max\{N_1, N_2\}$. Hence if $n \geq \max\{N_1, N_2\}$, we have

$$|\sqrt[3]{x_n} + 1| = \frac{|x_n + 1|}{|\sqrt[3]{x_n^2} - \sqrt[3]{x_n} + 1|} < |x_n + 1| < \epsilon,$$

so $\sqrt[3]{x_n} \to -1$ as claimed. $\square$

4. Show that the sequence (x_n) defined by

$$x_n = (-1)^n \left(\cos(\sin n) - \frac{2\cos n}{n} + \frac{\sin n}{n^2} \right)$$

has a convergent subsequence.

Proof. For any $n \in \mathbb{N}$ we have

$$\begin{aligned} |x_n| &= \left| \cos(\sin n) - \frac{2\cos n}{n} + \frac{\sin n}{n^2} \right| \\ &\leq |\cos(\sin n)| + \left| \frac{2\cos n}{n} \right| + \left| \frac{\sin n}{n^2} \right| \\ &\leq 1 + 2|\cos n| + |\sin n| \\ &\leq 1 + 2 + 1 \\ &= 4 \end{aligned}$$

since the absolute values of both cos and sin are bounded by 1. Hence the sequence (x_n) is bounded, so it has a convergent subsequence by the Bolzano-Weierstrass Theorem. $\square$

5. Suppose that (a_n) is a sequence of positive numbers such that the sequence (x_n) defined by

$$x_n = a_1 + a_2 + \cdots + a_n$$

converges. If (b_n) is a bounded sequence, show that the sequence (y_n) defined by

$$y_n = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

also converges.

Proof. Let $\epsilon > 0$ and let $M > 0$ be a positive bound for $|b_n|$, which exists since (b_n) is bounded. Since (x_n) converges, it is Cauchy so there exists $N \in \mathbb{N}$ such that for $k \geq 1$ we have

$$|x_{n+k} - x_n| = |a_{n+k} + \cdots + a_{n+1}| < \frac{\epsilon}{M}.$$

Thus for $n \in \mathbb{N}$ and $k \geq 1$ we get:

$$\begin{aligned} |y_{n+k} - y_n| &= |a_{n+k} b_{n+k} + \cdots + a_{n+1} b_{n+1}| \\ &\leq |a_{n+k}||b_{n+k}| + \cdots |a_{n+1}||b_{n+1}| \end{aligned}$$

$$
\begin{aligned}
&\leq a_{n+k}M + \cdots + a_{n+1}M \\
&= (a_{n+k} + \cdots + a_{n+1})M \\
&= |a_{n+k} + \cdots + a_{n+1})M \\
&< \frac{\epsilon}{M}M \\
&= \epsilon
\end{aligned}
$$

where we use the fact that the a_i are positive to say that $|a_i| = a_i$ and then $a_{n+k} + \cdots + a_{n+1} = |a_{n+k} + \cdots + a_{n+1}|$. Thus (y_n) is Cauchy as well, so it converges. $\square$