Math 320-3: Midterm 1 Solutions
Northwestern University, Spring 2015

1. Give an example of each of the following. No justification is required.
(a) A function $f : \mathbb{R}^2 \to \mathbb{R}$ which is not continuous at $\mathbf{0}$ but such that $f_x(\mathbf{0})$ and $f_y(\mathbf{0})$ exist.
(b) A function $f : \mathbb{R}^2 \to \mathbb{R}^2$ which is differentiable at $(0,0)$ but not at $(1,1)$.
(c) A differentiable function $g : \mathbb{R}^2 \to \mathbb{R}$ such that for $f(x,y) = (y,x)$, $D(g \circ f)(\mathbf{0}) = \begin{pmatrix} 1 & 0 \end{pmatrix}$
(d) A single-variable function $f(x)$ such that $g(x,y) = (f(x)e^y, ye^{xy})$ is invertible near $\mathbf{0}$.

Solution. (a) Define $f(x,y)$ to be 0 along the x- and y-axes and 1 otherwise. Then $f(x,0) = 0$ for all x and $f(0,y) = 0$ for all y so $f_x(\mathbf{0}) = 0$ and $f_y(\mathbf{0})$ but $\lim_{x \to 0} f(x,x) = 1 \neq f(0,0)$ so f is not continuous at $\mathbf{0}$.

(b) Define $f(x,y)$ to be $(1,1)$ at $(1,1)$ and $(0,0)$ elsewhere. Then near $(0,0)$ the function f is constant so it is differentiable at $(0,0)$, but f is not continuous at $(1,1)$ and so cannot be differentiable there.

(c) Since

$$D(g \circ f)(\mathbf{0}) = Dg(f(\mathbf{0}))Df(\mathbf{0}) = \begin{pmatrix} g_x(\mathbf{0}) & g_y(\mathbf{0}) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} g_y(\mathbf{0}) & g_x(\mathbf{0}) \end{pmatrix},$$

we need a function with $g_y(\mathbf{0}) = 1$ and $g_x(\mathbf{0}) = 0$. The function $g(x,y) = y$ is one such example.

(d) We have

$$Dg(x,y) = \begin{pmatrix} f'(x)e^y & f(x)e^y \\ y^2e^{xy} & e^{xy} + xye^{xy} \end{pmatrix}, \text{ so } Dg(\mathbf{0}) = \begin{pmatrix} f'(0) & f(0) \\ 0 & 1 \end{pmatrix}.$$

The Inverse Function Theorem implies that g is invertible near $\mathbf{0}$ as long as this Jacobian matrix is invertible, so we only need a function with $f'(0) \neq 0$. The function $f(x) = x$ is one example. $\square$

2. Determine whether or not the following function $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $\mathbf{0}$.

$$f(x,y) = \begin{cases} x + 2y + \frac{x^2y^3 - xy^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Solution. We have $f(x,0) = x$ for all x and $f(0,y) = 2y$ for all y, so $f_x(\mathbf{0}) = 1$ and $f_y(\mathbf{0}) = 2$. Thus $Df(\mathbf{0})$ must be $\begin{pmatrix} 1 & 2 \end{pmatrix}$. We have with $\mathbf{h} = (h,k)$:

$$\frac{f(\mathbf{0}+\mathbf{h}) - f(\mathbf{0}) - Df(\mathbf{0})\mathbf{h}}{\|\mathbf{h}\|} = \frac{h + 2k + \frac{h^2k^3 - hk^2}{h^2+k^2} - 0 - h - 2k}{\sqrt{h^2+k^2}} = \frac{h^2k^3 - hk^2}{(h^2+k^2)^{3/2}}.$$

We can convert to polar coordinates to see that the limit of this expression does not exist, or argue as follows. Along $h = k$ this becomes

$$\frac{h^5 - h^3}{(2h^2)^{3/2}} = \frac{1}{\sqrt{8}} \left(\frac{h^5 - h^3}{|h|^3} \right)$$

As $h \to 0^+$ this gives a limit of $-\frac{1}{\sqrt{8}}$ while if $h \to 0^-$ the limit is $\frac{1}{\sqrt{8}}$, so the limit does not exist and hence f is not differentiable at $\mathbf{0}$. $\square$

3. Suppose that $f : \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable and that $g : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies

$$\|g(\mathbf{x})\| \le \sqrt{\|f(\mathbf{x})\| \, \|\mathbf{x}\|} \text{ for all } \mathbf{x} \in \mathbb{R}^2.$$

Assuming that $f(\mathbf{0}) = \mathbf{0}$ and $Df(\mathbf{0}) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, show that g is differentiable at $\mathbf{0}$.

Proof. First, we have

$$\|g(\mathbf{0})\| \le \sqrt{\|f(\mathbf{0})\| \, \|\mathbf{0}\|} = 0,$$

so $g(\mathbf{0}) = \mathbf{0}$. Since f is differentiable, $f(\mathbf{0}) = \mathbf{0}$, and $Df(\mathbf{0})$ is equal to the zero matrix, we have

$$\lim_{\mathbf{h} \to \mathbf{0}} \frac{f(\mathbf{0} + \mathbf{h}) - f(\mathbf{0}) - Df(\mathbf{0})\mathbf{h}}{\|\mathbf{h}\|} = \lim_{\mathbf{h} \to \mathbf{0}} \frac{f(\mathbf{h})}{\|\mathbf{h}\|} = \mathbf{0}.$$

Since

$$\frac{\|g(\mathbf{h})\|}{\|\mathbf{h}\|} \le \frac{\sqrt{\|f(\mathbf{h})\| \, \|\mathbf{h}\|}}{\|\mathbf{h}\|} = \sqrt{\frac{\|f(\mathbf{h})\|}{\|\mathbf{h}\|}}$$

and the limit of the final expression is 0 by the continuity of the square root function, we have

$$\lim_{\mathbf{h} \to \mathbf{0}} \frac{g(\mathbf{h}) - g(\mathbf{0}) - \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) \mathbf{h}}{\|\mathbf{h}\|} = \mathbf{0}$$

by the squeeze theorem, showing that g is differentiable at $\mathbf{0}$ with $Dg(\mathbf{0}) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. $\square$

4. Suppose that $f : \mathbb{R}^2 \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}^2$ are C^1 and that $\mathbf{x}, \mathbf{a} \in \mathbb{R}^2$ satisfy $g(\mathbf{x}) = \mathbf{x}$ and $g(\mathbf{a}) = \mathbf{a}$. If $\|Dg(\mathbf{y})\| \le \frac{1}{2}$ for all $\mathbf{y} \in \mathbb{R}^2$ and $\|Df(\mathbf{z})\| \le 4$ for all $\mathbf{z} \in \mathbb{R}^2$, show that

$$\|f(\mathbf{x}) - f(\mathbf{a})\| \le 2 \|\mathbf{x} - \mathbf{a}\|.$$

Proof. By the Mean Value Theorem we have

$$\|f(g(\mathbf{x})) - f(g(\mathbf{a}))\| \le M \|\mathbf{x} - \mathbf{a}\|$$

where M is the supremum of the norms $\|D(f \circ g)(\mathbf{y})\|$. The chain rule gives $D(f \circ g)(\mathbf{y}) = Df(g(\mathbf{y}))Dg(\mathbf{y})$, so

$$\|D(f \circ g)(\mathbf{y})\| = \|Df(g(\mathbf{y}))Dg(\mathbf{y})\| \le \|Df(g(\mathbf{y}))\| \, \|Dg(\mathbf{y})\| \le 4\left(\frac{1}{2}\right) = 2$$

for any $\mathbf{y}$, which implies that $M \le 2$. Thus

$$\|f(g(\mathbf{x})) - f(g(\mathbf{a}))\| \le M \|\mathbf{x} - \mathbf{a}\| \le 2 \|\mathbf{x} - \mathbf{a}\|,$$

and since $g(\mathbf{x}) = \mathbf{x}$ and $g(\mathbf{a}) = \mathbf{a}$, the left side is $\|f(\mathbf{x}) - f(\mathbf{a})\|$, which gives the claim. $\square$

5. Consider the curve in $\mathbb{R}^3$ consisting of the points which satisfy the equations

$$x^2 - x + y^2 + z^2 = 1 \quad \text{and} \quad xy + z = 1.$$

Show that near the point $(1, 0, 1)$ the curve can be described by parametric equations of the form

$$x = x(t), \ y = t, \ z = z(t)$$

where $x, z : (a, b) \to \mathbb{R}$ are differentiable functions on some open interval $(a, b) \subseteq \mathbb{R}$, and then compute the derivatives $x'(0)$ and $z'(0)$. Hint for the last part: consider the function $g : (a, b) \to \mathbb{R}^2$ defined by $g(t) = (x(t), z(t))$.

Solution. Let $F : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by

$$F(x, y, z) = (x^2 - x + y^2 + z^2 - 1, xy + z - 1),$$

so that the curve in question consists of all points satisfying $F(x, y, z) = (0, 0)$. Also,

$$DF_{(x,z)}(x, y, z) = \begin{pmatrix} 2x - 1 & 2z \\ y & 1 \end{pmatrix}.$$

Since F is C^1 and

$$DF_{(x,z)}(1, 0, 1) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

is invertible, the Implicit Function Theorem implies that near $(1, 0, 1)$ there exists a C^1 function $g : (a, b) \to \mathbb{R}^2$ on some interval $(a, b) \subseteq \mathbb{R}$ such that

$$F(x(y), y, z(y)) = (0, 0)$$

where $g(y) = (x(y), z(y))$ are the components of g. Setting $y = t$ then gives the parametric equations

$$x = x(t),\ y = t,\ z = z(t)$$

which are asked for.

Now, we have

$$Dg(t) = -[DF_{(x,z)}(x(t), t, z(t))]^{-1} DF_y(x(t), t, z(t)$$

for all t, also by the Implicit Function Theorem. Since

$$DF_y = \begin{pmatrix} 2y \\ x \end{pmatrix},$$

we get

$$Dg(0) = -[DF_{(x,z)}(1, 0, 1)]^{-1} DF_y(1, 0, 1) = -\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

On the other hand, $Dg(0) = \left(\begin{smallmatrix} x'(0) \\ z'(0) \end{smallmatrix}\right)$, so $x'(0) = 2$ and $z'(0) = -1$. $\square$