Math 320-2: Midterm 1 Solutions
Northwestern University, Winter 2015

1. Give an example of each of the following. You do not have to justify your answer.
(a) A series $\sum a_n$ of numbers which converges such that $\sum a_n^2$ does not converge.
(b) A sequence of functions on $(0,1)$ which converges pointwise but not uniformly.
(c) A power series centered at 1 with radius of convergence 3.
(d) A function which is bounded and analytic on $\mathbb{R}$.

Solution. (a) For $a_n = \frac{(-1)}{\sqrt{n}}$, $\sum a_n$ converges but $\sum a_n^2 = \sum \frac{1}{n}$ does not.
(b) The sequence $f_n(x) = x^n$ converges pointwise but not uniformly on $(0,1)$ to the zero.
(c) The series $\sum \left(\frac{x-1}{3}\right)^n$ converges when $\left|\frac{x-1}{3}\right| < 1$, so when $|x-1| < 3$.
(d) Any constant function works, as do $\sin x$, $\cos x$, or $\frac{1}{1+x^2}$. ☐

2. Suppose that for each $n \in \mathbb{N}$, $f_n : [-2,2] \to \mathbb{R}$ is an increasing function and that the sequence (f_n) converges pointwise to the constant function $f(x) = 1$. Show that $f_n \to f$ uniformly on $[-2,2]$ as well. (Note: the assumption that each f_n is increasing is important.)

Proof. (We did a similar Warm-Up in class one day.) Let $\epsilon > 0$. Since $f_n \to 1$ pointwise, $f_n(2) \to 1$ and $f_n(-2) \to 1$. Thus there exist N_1 and N_2 such that

$$|f_n(-2) - 1| < \epsilon \text{ for } n \geq N_1 \text{ and } |f_n(2) - 1| < \epsilon \text{ for } n \geq N_2.$$

Since f_n is increasing for each n, we have

$$f_n(-2) \leq f_n(x) \leq f_n(2) \text{ for all } x \in [-2,2],$$

so

$$f_n(-2) - 1 \leq f_n(x) - 1 \leq f_n(2) - 1.$$

Thus for $n \geq \max\{N_1, N_2\}$, we get

$$-\epsilon < f_n(-2) - 1 \leq f_n(x) - 1 \leq f_n(2) - 1 < \epsilon,$$

which implies that $|f_n(x) - 1| < \epsilon$ for $n \geq \max\{N_1, N_2\}$ and all $x \in [-2,2]$. Thus $f_n \to 1$ uniformly as claimed. ☐

3. Determine, with justification, the value of the following limit.

$$\lim_{n\to\infty} \int_0^1 \left[1 + \sin\left(2\cos\frac{x}{n} - 2\right)\right] dx$$

You may use the fact that $|\sin y| \leq |y|$ for all $y \in \mathbb{R}$ without proof.

Proof. For a fixed $x \in [0,1]$,

$$f_n(x) := 1 + \sin\left(2\cos\frac{x}{n} - 2\right) \to 1 + \sin(2\cos 0 - 2) = 1,$$

so $f_n \to 1$ pointwise on $[0,1]$. We first show that this convergence is actually uniform. Note that for each n, $\cos\frac{x}{n}$ is decreasing on the interval $[0,1]$, so

$$\cos\frac{x}{n} \geq \cos\frac{1}{n} \text{ and hence } 0 \leq 1 - \cos\frac{x}{n} \leq 1 - \cos\frac{1}{n} \text{ for all } x \in [0,1].$$

Let $\epsilon > 0$. Since $\cos x$ is continuous, $\cos \frac{1}{n} \to \cos 0 = 1$ so there exists $N \in \mathbb{N}$ such that

$$\left|1 - \cos \frac{1}{n}\right| < \frac{\epsilon}{2} \text{ for } n \geq N.$$

Thus for $n \geq N$ we get

$$|f_n(x) - 1| = \left|\sin\left(2\cos\frac{x}{n} - 2\right)\right| \leq \left|2\cos\frac{x}{n} - 2\right| = 2\left|1 - \cos\frac{x}{n}\right| \leq 2\left|1 - \cos\frac{1}{n}\right| < 2\frac{\epsilon}{2} = \epsilon$$

for all $x \in [0,1]$, showing that $f_n \to 1$ uniformly on $[0,1]$.

Since the convergence is uniform, we can interchange limits and integration to get:

$$\lim_{n\to\infty} \int_0^1 \left[1 + \sin\left(2\cos\frac{x}{n} - 2\right)\right] dx = \int_0^1 \left(\lim_{n\to\infty} \left[1 + \sin\left(2\cos\frac{x}{n} - 2\right)\right]\right) dx = \int_0^1 1\, dx = 1,$$

which is the desired value. $\square$

4. Show that the series

$$\sum_{n=1}^{\infty} \frac{ne^{x/n} - 1}{n^3 + 1}$$

converges uniformly on $(2,4)$ to a differentiable function f such that $|f'(x)| \leq e^x$ for all $x \in (2,4)$. You may use the fact that $\sum_{n=1}^{\infty} \frac{1}{n^3+1} \leq 1$ without proof.

Proof. (This was indeed the toughest problem.) For $x \in (2,4)$, we have

$$\left|\frac{ne^{x/n} - 1}{n^3 + 1}\right| \leq \frac{ne^{x/n}}{n^3} = \frac{e^{x/n}}{n^2} \leq \frac{e^4}{n^2}.$$

Thus since $\sum \frac{e^4}{n^2}$ converges (as it is a constant multiple of the convergent series $\sum \frac{1}{n^2}$), our given series converges uniformly on $(2,4)$ by the Weierstrass M-test; denote the function defined by this series by f.

Now, the term-by-term derivative of our series is

$$\sum_{n=1}^{\infty} \frac{e^{x/n}}{n^3 + 1}.$$

To know that f is differentiable, we have to know that this term-by-term derivative series is also uniformly convergent. We have

$$\left|\frac{e^{x/n}}{n^3 + 1}\right| \leq \frac{e^{x/n}}{n^3} \leq \frac{e^4}{n^3} \text{ for } x \in (2,4),$$

so since $\sum \frac{e^4}{n^3}$ converges the Weierstrass M-test again implies that this term-by-term derivative series converges uniformly on $(2,4)$. Hence f is differentiable on $(2,4)$ and

$$f'(x) = \sum_{n=1}^{\infty} \frac{e^{x/n}}{n^3 + 1}.$$

Finally, we have

$$|f'(x)| = \left|\sum_{n=1}^{\infty} \frac{e^{x/n}}{n^3 + 1}\right| \leq \sum_{n=1}^{\infty} \left|\frac{e^{x/n}}{n^3 + 1}\right| \leq \sum_{n=1}^{\infty} \left|\frac{e^x}{n^3 + 1}\right| = e^x \sum_{n=1}^{\infty} \frac{1}{n^3 + 1} \leq e^x$$

for $x \in (2,4)$ as required, where we have used the fact that $\sum \frac{1}{n^3+1} \leq 1$. $\square$

5. Suppose that the series $\sum_{n=0}^{\infty} a_n(x-1)^n$ has radius of convergence $R > 0$. Find the radius of convergence of the series

$$\sum_{n=1}^{\infty} n a_n x^{3n}.$$

Proof 1. The series $\sum b_k x^k$ in question has

$$b_k = \begin{cases} n a_n & k = 3n \\ 0 & \text{else,} \end{cases}$$

so

$$|b_k|^{1/k} = \begin{cases} |n a_n|^{1/3n} & k = 3n \\ 0 & \text{else.} \end{cases}$$

Since the terms corresponding to $k = 3n$ are always nonnegative, the supremums defining $\limsup |b_k|^{1/k}$ can only come from the $k = 3n$ terms, so

$$\limsup |b_k|^{1/k} = \limsup |n a_n|^{1/3n} = \left(\lim n^{1/n}\right)^{1/3} \left(\limsup |a_n|^{1/n}\right)^{1/3} = \left(\limsup |a_n|^{1/n}\right)^{1/3}$$

since $n^{1/n} \to 1$. Since $\sum a_n(x-1)^n$ has radius convergence R, $\limsup |a_n|^{1/n} = \frac{1}{R}$, so $\limsup |b_k|^{1/k} = \frac{1}{\sqrt[3]{R}}$. Thus the series in question has radius of convergence $\sqrt[3]{R}$. □

Proof 2. Since $\sum a_n(x-1)^n$ has radius of convergence R, the derivative series $\sum n a_n (x-1)^{n-1}$ also has radius of convergence R and thus $\sum n a_n (x-1)^n$ does as well. Making first the substitution $y = x - 1$, we get that $\sum n a_n y^n$ converges for $|y| < R$, and then making the substitution $y = x^3$ we get that the series in question

$$\sum_{n=1}^{\infty} n a_n x^{3n}$$

converges when $|x^3| < R$, so when $|x| < \sqrt[3]{R}$. Hence the series in question has radius of convergence $\sqrt[3]{R}$, agreeing with the first approach. □