Math 320-3: Midterm 1
Northwestern University, Spring 2015

Name: ____________________

1. (10 points) Give an example of each of the following. No justification is required.

(a) A function $f : \mathbb{R}^2 \to \mathbb{R}$ which is not continuous at $\mathbf{0}$ but such that $f_x(\mathbf{0})$ and $f_y(\mathbf{0})$ exist.

(b) A function $f : \mathbb{R}^2 \to \mathbb{R}^2$ which is differentiable at $(0, 0)$ but not at $(1, 1)$.

(c) A differentiable function $g : \mathbb{R}^2 \to \mathbb{R}$ such that for $f(x, y) = (y, x)$, $D(g \circ f)(\mathbf{0}) = \begin{pmatrix} 1 & 0 \end{pmatrix}$

(d) A single-variable function $f(x)$ such that $g(x, y) = (f(x)e^y, ye^{xy})$ is invertible near $\mathbf{0}$.

Problem	Score
1	
2	
3	
4	
5	
Total	

2. (10 points) Determine whether or not the following function $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $\mathbf{0}$.

$$f(x,y) = \begin{cases} x + 2y + \frac{x^2y^3 - xy^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

3. (10 points) Suppose that $f : \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable and that $g : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies

$$\|g(\mathbf{x})\| \leq \sqrt{\|f(\mathbf{x})\|\,\|\mathbf{x}\|} \text{ for all } \mathbf{x} \in \mathbb{R}^2.$$

Assuming that $f(\mathbf{0}) = \mathbf{0}$ and $Df(\mathbf{0}) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, show that g is differentiable at $\mathbf{0}$.

4. (10 points) Suppose that $f : \mathbb{R}^2 \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}^2$ are C^1 and that $\mathbf{x}, \mathbf{a} \in \mathbb{R}^2$ satisfy $g(\mathbf{x}) = \mathbf{x}$ and $g(\mathbf{a}) = \mathbf{a}$. If $\|Dg(\mathbf{y})\| \leq \frac{1}{2}$ for all $\mathbf{y} \in \mathbb{R}^2$ and $\|Df(\mathbf{z})\| \leq 4$ for all $\mathbf{z} \in \mathbb{R}^2$, show that

$$\|f(\mathbf{x}) - f(\mathbf{a})\| \leq 2\,\|\mathbf{x} - \mathbf{a}\|\,.$$

5. (10 points) Consider the curve in $\mathbb{R}^3$ consisting of the points which satisfy the equations

$$x^2 - x + y^2 + z^2 = 1 \quad \text{and} \quad xy + z = 1.$$

Show that near the point $(1, 0, 1)$ the curve can be described by parametric equations of the form

$$x = x(t), \ y = t, \ z = z(t)$$

where $x, z : (a, b) \to \mathbb{R}$ are differentiable functions on some open interval $(a, b) \subseteq \mathbb{R}$, and then compute the derivatives $x'(0)$ and $z'(0)$. Hint for the last part: consider the function $g : (a, b) \to \mathbb{R}^2$ defined by $g(t) = (x(t), z(t))$.