Math 320-3: Midterm 2 Solutions
Northwestern University, Spring 2015

1. Give an example of each of the following. No justification is required.
(a) A bounded subset of $\mathbb{R}^3$ which is not a Jordan region.
(b) A bounded function on $[0,1]\times[0,1]$ which is not integrable.
(c) A non-constant function on $[0,1]\times[0,1]$ whose iterated integrals exist and are equal.
(d) A function $\phi:\mathbb{R}^2\to\mathbb{R}^2$ which sends the rectangle $[0,1]\times[0,2\pi]$ to the ellipse $x^2+2y^2\leq 1$.

Solution. (a) The subset of points in $[0,1]\times[0,1]\times[0,1]$ where all coordinates are rational works. The boundary of this is all of $[0,1]\times[0,1]\times[0,1]$, which does not have volume zero.

(b) The function which is 1 at each point with rational coordinates and 0 elsewhere works, since for any grid the upper sum is always 1 and the lower sum is always 0 by the denseness of $\mathbb{Q}$ and of $\mathbb{R}\setminus\mathbb{Q}$ in $\mathbb{R}$.

(c) Any non-constant function to which Fubini's Theorem applies will work, such as $f(x,y)=xy$. Fubini's Theorem applies because this is continuous.

(d) The function $\phi(r,\theta)=(r\cos\theta,\frac{1}{\sqrt{2}}r\sin\theta)$ works. Indeed, with $x=r\cos\theta$ and $y=\frac{1}{\sqrt{2}}r\sin\theta$, we have

$$x^2+2y^2=r^2\cos^2\theta+r^2\sin^2\theta=r^2,$$

so as r ranges from 0 to 1 and θ from 0 to 2π these (x,y) values fill out the ellipse $x^2+2y^2\leq 1$. $\square$

2. For a Jordan region E of $\mathbb{R}^2$, let $(1,1)+E$ denote the set obtained by adding $(1,1)$ to each point of E:

$$(1,1)+E:=\{(1+x,1+y)\mid(x,y)\in E\}.$$

Show that $(1,1)+E$ is also a Jordan region.

Proof. First we claim that the boundary of $(1,1)+E$ is $(1,1)+\partial E$. Indeed, let $p\in\partial E$ and let U be an open ball around $(1,1)+p$. Then $(-1,-1)+U$ is an open ball around p, so since p is a boundary point of E, this ball contains an element $q\in E$ and an element $z\notin E$. Then $(1,1)+q\in U$ and $(1,1)+z\in U$, so U contains a point $(1,1)+q$ of $(1,1)+E$ and a point $(1,1)+z$ not of $(1,1)+E$. Hence $(1,1)+p$ is in the boundary of $(1,1)+E$. By the same reasoning, if a is in the boundary of $(1,1)+E$, then $(-1,-1)+a$ is in the boundary of E, so any point in the boundary of $(1,1)+E$ is of the form "$(1,1)$ plus a point in the boundary of E", so

$$\partial((1,1)+E)=(1,1)+\partial E$$

as claimed.

Let $\epsilon>0$ and let R be a rectangle containing E. Since E is a Jordan region there exists a grid G on R such that $V(\partial E,G)<\epsilon$. The translation $(1,1)+R$ is then a rectangle containing $(1,1)+E$, and translating the grid G gives a grid $(1,1)+G$ on $(1,1)+R$. By what we showed above, a subrectangle of $(1,1)+G$ which intersects the boundary of $(1,1)+E$ is of the form $(1,1)+R_i$ where R_i is a subrectangle of G; if $R_i=[a,b]\times[c,d]$, then $(1,1)+R_i=[a+1,b+1]\times[c+1,d+1]$ so

$$|(1,1)+R_i|=((b+1)-(a+1))((d+1)-(c+1))=(b-a)(d-c)=|R_i|.$$

Hence

$$V(\partial((1,1)+E));(1,1)+G)=\sum_{((1,1)+R_i)\cap\partial((1,1)+E))}|(1,1)+R_i|$$

$$
\begin{aligned}
&= \sum_{R_i \cap \partial E} |(1,1) + R_i| \\
&= \sum_{R_i \cap \partial E} |R_i| \\
&= V(\partial E; G) \\
&< \epsilon,
\end{aligned}
$$

so the boundary of $(1,1)+E$ has volume zero and thus $(1,1)+E$ is a Jordan region as claimed. $\square$

3. Define $f : [0,1] \times [0,1] \to \mathbb{R}$ by

$$
f(x,y) = \begin{cases} x & \text{if } y = \frac{1}{n} \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}
$$

Show that f is integrable over $[0,1] \times [0,1]$ and determine the value of its integral.

Proof. Given any grid G on $[0,1] \times [0,1]$, on any subrectangle there is a point not of the form $(x, \frac{1}{n})$, and so a point where $f = 0$. Thus the infimum of f over any subrectangle is 0 and hence

$$
L(f,G) = 0
$$

for any grid G.

Let $\epsilon > 0$ and pick $N \in \mathbb{N}$ such that $\frac{1}{N+1} \leq \frac{\epsilon}{2}$. For each $n = 1, \ldots, N$, pick a rectangle R_n of width 1 and height smaller than $\frac{\epsilon}{2N}$ which contains the horizontal line $y = \frac{1}{n}$; if need be make these rectangles smaller to ensure that they do not intersect each other. Let G be the grid on $[0,1] \times [0,1]$ determined by these rectangles and the rectangle $[0,1] \times [0, \frac{\epsilon}{2}]$ along the bottom. Then only nonzero contributions to $U(f,G)$ come from the rectangle at the bottom of height $\frac{\epsilon}{2}$ and the R_n defined above. On any of these, $\sup f \leq 1$, so:

$$
\begin{aligned}
U(f,G) &= (\sup f \text{ on bottom rectangle})\frac{\epsilon}{2} + \sum_{R_n} (\sup f)|R_n| \\
&< \frac{\epsilon}{2} + \sum_{n=1}^{N} \frac{\epsilon}{2N} \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}
$$

Thus $U(f,G) - L(f,G) < \epsilon$, so f is integrable as claimed. Since the supremum of all lower sums is zero, the integral of f over $[0,1] \times [0,1]$ is zero as well. $\square$

4. Suppose that A and B are two Jordan regions in $\mathbb{R}^2$ such that for any vertical line L, the intersection of A with L has the same length as the intersection of B with L. Show that A and B have the same area.

Proof. Since A and B are Jordan regions, they are bounded so we can find some a and b such that both A and B lie fully within the vertical lines $x = a$ and $x = b$. Since the constant function 1 is continuous, it is integrable over both A and B as are the single variable constant functions obtained by holding x or y fixed. Thus Fubini's Theorem gives:

$$\text{area}(A) = \iint_A 1\, dA = \int_a^b \int_{A \cap L_x} 1\, dy\, dx$$

and

$$\text{area}(B) = \iint_A 1\, dA = \int_a^b \int_{B \cap L_x} 1\, dy\, dx$$

where L_x is the vertical line at a fixed value of x. The inner single-variable integrals give the lengths (measured vertically) of $A \cap L_x$ and $B \cap L_x$, so

$$\text{area}(A) = \int_a^b \text{length}(A \cap L_x)\, dx \quad \text{and} \quad \text{area}(B) = \int_a^b \text{length}(B \cap L_x)\, dx.$$

Note that for certain x, the intersections $A \cap L_x$ or $B \cap L_x$ could have length zero, say if the intersection was empty or just a finite number of points. By our given assumption,

$$\text{length}(A \cap L_x) = \text{length}(B \cap L_x) \text{ for any } x \in [a, b],$$

so the above integrals are equal and thus $\text{area}(A) = \text{area}(B)$ as claimed. $\square$

5. Show that for any strictly positive continuous function $f : \mathbb{R}^2 \to \mathbb{R}$, we have

$$\int_{B_2(1,1)} 2f(x, y)\, d(x, y) > \int_{B_1(0,0)} 6f(1 + 2u, 1 + 2v)\, d(u, v).$$

To be clear, $B_2(1, 1)$ denotes the disk of radius 2 centered at $(1, 1)$ and $B_1(0, 0)$ the disk of radius 1 centered at $(0, 0)$.

Proof. Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be the change of variables function

$$\phi(u, v) = (1 + 2u + 1 + 2v),$$

which is C^1, one-to-one, and has Jacobian equal to twice the identity matrix everywhere, which is invertible. Since $\phi(B_1(0, 0)) = B_2(1, 1)$, the change of variables formula gives

$$\int_{B_2(1,1)=\phi(B_1(0,0))} 2f(x, y)\, d(x, y) = \int_{B_1(0,0)} 2f(1 + 2u, 1 + 2v)4\, d(u, v),$$

since $\det D\phi = 4$ everywhere. Since f is positive, $8f > 6f$ and thus since f is continuous this final integral

$$\int_{B_1(0,0)} 8f(1 + 2u, 1 + 2v)\, d(u, v)$$

is strictly larger than

$$\int_{B_1(0,0)} 6f(1 + 2u, 1 + 2v)6\, d(u, v).$$

Thus

$$\int_{B_2(1,1)} 2f(x, y)\, d(x, y) > \int_{B_1(0,0)} 6f(1 + 2u, 1 + 2v)\, d(u, v)$$

as claimed. $\square$