Math 320-2: Midterm 2 Solutions
Northwestern University, Winter 2015

1. Give an example of each of the following. You do not have to justify your answer.
(a) A nonempty subset of $\mathbb{R}\backslash\mathbb{Q}$ which is both closed and open in $\mathbb{R}\backslash\mathbb{Q}$.
(b) A Cauchy sequence in $C[0,1]$ relative to the sup metric.
(c) A metric on $\mathbb{Q}$ relative to which $\mathbb{Q}$ is complete.
(d) A dense subset of $\mathbb{R}$ whose boundary consists of a single point.
Note: in parts (a) and (d) we are considering the standard absolute value metric.

Solution. (a) The set $(1,2)\cap(\mathbb{R}\backslash\mathbb{Q})$ of irrational between 1 and 2 works, as does any subset of the form $(a,b)\cap(\mathbb{R}\backslash\mathbb{Q})$ where $a<b$ are both rational.

(b) Any constant sequence works, say $f_n(x)=1$ for all n and all x, or more generally any uniformly convergent sequence.

(c) The discrete metric works, since the only Cauchy sequences with respect to a discrete metric are ones which are eventually constant.

(d) The set $\mathbb{R}\backslash\{0\}$ of nonzero real numbers is dense in $\mathbb{R}$ (since any nonempty open interval contains a nonzero number) and has boundary $\{0\}$. □

2. Suppose that X is a metric space and $S=\{p_1,\ldots,p_n\}$ is a finite subset of X. Show, using only the definition of open, that the complement of S in X is open. (In other words, you cannot use the fact that S is closed in X and the complement of a closed set is open.)

Proof. Let $q\in S^c$. Then q is different from each p_i, so each distance $d(q,p_i)$ is positive. Set $r=\min\{d(q,p_1),\ldots,d(q,p_n)\}$, which is thus positive. We claim that the open ball $B_r(q)$ is contained in S^c. Indeed, if $x\in B_r(q)$, we have $d(x,q)<r$ so for each $i=1,\ldots,n$:

$$d(p_i,x)\geq d(p_i,q)-d(p,x)>d(p_i,q)-r\geq 0$$

since $r\geq d(p_i,q)$. Thus $d(p_i,x)>0$ so $x\neq p_i$ for each i, and hence $x\in S^c$. Since then $B_r(q)\subseteq S^c$, S^c is open in X as claimed. □

3. Consider $\mathbb{R}^2$ and let $D=\{(x,x^2)\mid x\in\mathbb{R}\}$ be the subset consisting of all points satisfying $y=x^2$. Show that D is complete with respect to whichever of the Euclidean, taxicab, or box metrics you prefer.

Proof. The set D is a closed subset of $\mathbb{R}^2$ since, for instance, it is the graph of the continuous function $f(x)=x^2$, and any closed subset of a complete metric space is itself complete. Thus since $\mathbb{R}^2$ is complete, D is as well. □

4. Let $C_b[0,\pi]$ denote the space of bounded real-valued functions on $[0,\pi]$ equipped with the sup metric. For each of the following functions, determine with justification whether or not it belongs to the *open* ball of radius π^2 centered at the function $f\in C_b[0,\pi]$ defined by $f(x)=x$.

$$g(x)=x\sin(2x)\qquad\text{and}\qquad h(x)=\begin{cases}x-x^2 & \text{if } x\in\mathbb{Q}\\ x & \text{if } x\notin\mathbb{Q}\end{cases}$$

Solution. We claim that $g \in B_{\pi^2}(f)$ but $h \notin B_{\pi^2}(f)$. First, we have:

$$|g(x) - f(x)| = |x||\sin 2x - 1| \le \pi(|\sin 2x| + 1) \le 2\pi$$

for all $x \in [0, \pi]$. Thus the supremum of all expressions $|g(x) - f(x)|$ is smaller than or equal to 2π, which is smaller than π^2, so $d(g, f) < \pi^2$ where d is the sup metric. Hence g is in the open ball $B_{\pi^2}(f)$ as claimed.

Now, we have:

$$|h(x) - f(x)| = \begin{cases} x^2 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

for each $x \in [0, \pi]$. The distance $d(h, f)$ is the supremum of such expressions, and we claim that this supremum is equal to π^2. Indeed, for any $x \in [0, \pi] \cap \mathbb{Q}$ we have $x^2 < \pi^2$, so π^2 is an upper bound for the set expressions which defines $d(h, f)$. Now, for $0 < \epsilon < \pi^2$, we have $\sqrt{\pi^2 - \epsilon} < \sqrt{\pi^2}$. Thus since $\mathbb{Q}$ is dense in $\mathbb{R}$ there exists $r \in \mathbb{Q}$ such that

$$\sqrt{\pi^2 - \epsilon} < r < \pi, \text{ for } \pi^2 - \epsilon < r^2 < \pi^2.$$

Since $|h(r) - f(r)| = r^2$, this shows that nothing smaller than π^2 can be an upper bound for the set of expressions $|h(x) - f(x)|$ for $x \in [0, \pi]$, so π^2 is indeed the least upper bound. Thus $d(h, f) = \pi^2$ so h is not in $B_{\pi^2}(f)$. (It would be in the corresponding closed ball, however.) $\square$

5. Suppose that A is a dense subset of a metric space X and let $p \in A^c$ be an element of its complement in X. Show that any open ball around p contains *infinitely* many points of A. (Careful: a sequence converging to p does not necessarily consist of infinitely many *distinct* points.)

Proof 1. Take any open ball $B_r(p)$ around p. Since A is dense in X, this ball contains some $a_1 \in A$. Now, take the smaller open ball $B_{d(p,a_1)}(p)$ of radius $d(p, a_1) > 0$. Again since A is dense in X, this ball contains some $a_2 \in A$, and we must have $a_1 \neq a_2$ since $d(p, a_2) < d(p, a_1)$. Take the even smaller open ball $B_{d(p,a_2)}(p)$, which again contains some $a_3 \in A$ such that $a_3 \neq a_2$ and $a_3 \neq a_1$ since

$$d(p, a_3) < d(p, a_2) < d(p, a_1).$$

Continuing in this manner, taking balls at each step of small enough radii to exclude the previously chosen point of A, we get an entire sequence of distinct elements of A

$$a_1, a_2, a_3, \ldots,$$

all of which are in the original ball $B_r(p)$ we started with. Thus $B_r(p)$ contains infinitely many points of A as claimed. $\square$

Proof 2. For a contradiction, suppose that there exists an open ball $B_r(p)$ around p which contains only finitely many points of A. Call these finitely many points $a_1, \ldots, a_n$. Then the ball of radius

$$\min\{d(p, a_1), \ldots, d(p, a_n)\} > 0$$

around p contains none of the a_i, so it contains no element of A. This contradicts A being dense in X, so we conclude that no such ball $B_r(p)$ can exist. $\square$