Math 320-2: Midterm 2
Northwestern University, Winter 2015

Name: ____

1. (10 points) Give an example of each of the following. You do not have to justify your answer.
(a) A nonempty subset of $\mathbb{R}\backslash\mathbb{Q}$ which is both closed and open in $\mathbb{R}\backslash\mathbb{Q}$.
(b) A Cauchy sequence in $C[0,1]$ relative to the sup metric.
(c) A metric on $\mathbb{Q}$ relative to which $\mathbb{Q}$ is complete.
(d) A dense subset of $\mathbb{R}$ whose boundary consists of a single point.
Note: in parts (a) and (d) we are considering the standard absolute value metric.

2. (10 points) Suppose that X is a metric space and $S = \{p_1, \ldots, p_n\}$ is a finite subset of X. Show, using only the definition of open, that the complement of S in X is open. (In other words, you cannot use the fact that S is closed in X and the complement of a closed set is open.)

3. (10 points) Consider $\mathbb{R}^2$ and let $D = \{(x, x^2) \mid x \in \mathbb{R}\}$ be the subset consisting of all points satisfying $y = x^2$. Show that D is complete with respect to whichever of the Euclidean, taxicab, or box metrics you prefer.

4. (10 points) Let $C_b[0,\pi]$ denote the space of bounded real-valued functions on $[0,\pi]$ equipped with the sup metric. For each of the following functions, determine with justification whether or not it belongs to the *open* ball of radius π^2 centered at the function $f \in C_b[0,\pi]$ defined by $f(x) = x$.

$$g(x) = x\sin(2x) \qquad \text{and} \qquad h(x) = \begin{cases} x - x^2 & \text{if } x \in \mathbb{Q} \\ x & \text{if } x \notin \mathbb{Q} \end{cases}$$

5. (10 points) Suppose that A is a dense subset of a metric space X and let $p \in A^c$ be an element of its complement in X. Show that any open ball around p contains *infinitely* many points of A. (Careful: a sequence converging to p does not necessarily consist of infinitely many *distinct* points.)