Notes on Compactness

These are some notes which supplement the material on compactness in the book. The book only gives proofs of the main properties of compact spaces using the sequence characterization of compactness, so here I reprove these main properties using the open cover point of view. I feel that this point of view is actually more natural and more intuitively captures the idea of what "compact" is supposed to mean, which is somehow that a compact space is not "too large". On exams and in the homework, you can use whichever point of view you find simpler to work with.

First we recall the open cover characterization of compactness:

Definition. Let M be a metric space and $S \subset M$ a subspace. An *open cover* of S is a collection $\{U_\alpha\}$ of open subsets of M such that S is contained in their union—meaning that any element of S is in at least one of the U_α. A *subcover* of an open cover $\{U_\alpha\}$ is an open cover $\{V_\beta\}$ of S so that each V_β occurs in the collection $\{U_\alpha\}$. An open cover is *finite* if it contains finitely many sets.

Definition. A subspace K of a metric space M is said to be *compact* if any open cover of K has a finite subcover—i.e. given any collection of open subsets $\{U_\alpha\}$ of M whose union contains K, there exist finitely many sets $U_1, \ldots, U_n$ in the collection whose union also contains K.

Note that since we can consider a metric space to be a subspace of itself, it also makes sense to say that a metric space M is itself compact. For each result below, try drawing a picture of what the conclusion is saying, and a picture illustrating how the proof works.

Proposition. *A compact subspace of a metric space is closed and bounded.*

Proof. Let K be a compact subspace of a metric space M. The "open cover" proof that K is closed is left for the fourth homework. Here we show that K is bounded.

Pick any $p \in K$ and consider the collection $\{M_r(p) \mid r > 0\}$ of all balls centered at p of any positive radius. This is an open cover of K since the element $q \in K$ is contained in the ball of radius $d(q, p) + 1$ around p.

Since K is compact, this has a finite subcover—let $r_1, \ldots, r_n$ be the radii of the balls in this finite subcover, and set $r = \max\{r_1, \ldots, r_n\}$. Then each of the balls $M_{r_i}(p)$ is contained in the ball $M_r(p)$, and since these balls cover K, it follows that K itself is contained in $M_r(p)$. Hence K is bounded. □

Proposition. *A closed subspace of a compact metric space is compact. (This is problem 2.47 in the book)*

Proof. Suppose that M is a compact metric space and that $S \subset M$ is a closed subspace. Note that then $M \backslash S$ is open in M. Let $\{U_\alpha\}$ be an open cover of S—to be clear, each U_α is an open subset of M and their union contains S. Then

$$\{U_\alpha\} \cup (M \backslash S)$$

is an open cover of M. Since M is compact, this has a finite subcover—let $U_1, \ldots, U_n$ be the elements of this finite subcover which come from the collection $\{U_\alpha\}$. Then $\{U_1, \ldots, U_n\}$ is a finite subcover of the open cover $\{U_\alpha\}$ of S. Thus any open cover of S has a finite subcover, so S is compact. □

The point above is that using the fact that M is compact gives a finite subcover, and then if we just throw away the open set $M \backslash S$ if it happens to be in in there, we are left with a finite cover of S which is a subcover of the open cover of S we started with.

Proposition. *The image of a compact subspace under a continuous map is compact. (This is problem 2.48 in the book)*

Proof. Suppose that $f : M \to N$ is a continuous function between metric spaces M and N and let $K \subset M$ be a compact subspace. We must show that $f(K) \subset N$ is compact. Let $\{U_\alpha\}$ be an open cover of $f(K)$. Since f is continuous, each preimage $f^{-1}(U_\alpha)$ is open in M. Since the U_α cover $f(K)$, it follows that the $f^{-1}(U_\alpha)$ cover K. Hence $\{f^{-1}(U_\alpha)\}$ is an open cover of K, so since K is compact, this has a finite subcover—say

$$\{f^{-1}(U_1), \ldots, f^{-1}(U_n)\}.$$

It follows that $\{U_1, \ldots, U_n\}$ is an open cover of $f(K)$, and this is then a finite subcover of the open cover $\{U_\alpha\}$ of $f(K)$. We conclude that $f(K)$ is compact. $\square$

Notice the power of the open cover definition of compactness used in the final result. At each point of our metric space M, we have a certain ball with some property; compactness of M then allows us to reduce this possibly infinite number of balls to a finite collection, and we can do things like take the minimum of their radii. It is a good idea to see exactly what goes wrong in the proof below if M is not compact.

Proposition. *A continuous function on a compact metric space is uniformly continuous.*

Proof. Suppose that $f : M \to N$ is a continuous function between metric spaces M and N and that M is compact. Let $\epsilon > 0$. Since f is continuous, for each $p \in M$ there exists $\delta_p > 0$ (which may depend on p) such that

$$d_M(q, p) < \delta_p \text{ implies } d_N(f(q), f(p)) < \frac{\epsilon}{2}.$$

We want to find a $\delta > 0$ satisfying this condition for any p, so a δ independent of p.

The collection $\{M_{\delta_p/2}(p)\}$, as p ranges over all points of M, is then an open cover of M. Since M is compact, this has a finite subcover—say

$$\{M_{\delta_{p_1}/2}(p_1), \ldots, M_{\delta_{p_n}/2}(p_n)\}.$$

Set $\delta = \min\{\delta_{p_1}/2, \ldots, \delta_{p_n}/2\}$. Note that since each $\delta_{p_i} > 0$ and there are only finitely many, the minimum of this set exists and is positive.

Suppose that q and q' are any two points of M such that

$$d_M(q, q') < \delta.$$

Since the radii $\delta_{p_1}/2, \ldots, \delta_{p_n}/2$ give balls which cover M, q' is in one of these balls—without loss of generality say that $q' \in M_{\delta_{p_1}/2}(p_1)$. Note that then

$$d_M(q, p_1) \le d_M(q, q') + d_M(q', p_1) < \delta + \delta_{p_1}/2 \le \delta_{p_1}/2 + \delta_{p_1}/2 \le \delta_{p_1}.$$

Thus $d_M(q, p_1) < \delta_{p_1}$ and $d_M(q', p_1) < \delta_{p_1}/2 < \delta_{p_1}$, so by the choice of δ_{p_1} we have

$$d_N(f(q), f(q')) \le d_N(f(q), f(p)) + d_N(f(q'), f(p)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence we have that

$$d_M(q, q') < \delta \text{ implies } d_N(f(q), f(q')) < \epsilon,$$

so we conclude that f is uniformly continuous. $\square$

Again, the point above is that the δ that was constructed does not depend on which point we are checking continuity at—the same δ works for any $p \in M$, which is what uniform continuity requires.

Let me also point out another "good" property of compact spaces. Given a function $f : \mathbb{R} \to \mathbb{R}$, you have probably seen before that integrals of the form

$$\int_{-\infty}^{\infty} f(x)\, dx$$

are not always defined, even if f is continuous. So, you cannot always integrate functions over all of $\mathbb{R}$. However, you will (or may) see in later courses that integrals over compact spaces of continuous functions are *always* defined; for example, integrals of continuous functions $f : [a, b] \to \mathbb{R}$ are always defined since $[a, b]$ is compact. This turns out to perhaps be the most down-to-earth useful property of compact spaces, once you figure out what it actually means to define integration over more general types of spaces.

Finally, we give first a proof that nonempty, closed intervals are compact using sequences but different than the one given in the book, and then a proof of the same fact using open covers.

Lemma. *Any sequence (x_n) in $\mathbb{R}$ has a monotone subsequence.*

Proof. Consider those indices n with the property that x_n is larger than every term in the sequence coming after it. There are two possibilies:

First, suppose there were infinitely many such indices, and list them in increasing order:

$$n_1 < n_2 < n_3 < \cdots .$$

Then by the property above which these indices satisfy, it follows that the subsequence (x_{n_k}) of (x_n) is decreasing:

$$x_{n_1} > x_{n_2} > x_{n_3} > \cdots ,$$

and hence monotone.

Second, suppose there were finitely many such indices (this includes the possibility that there are no such indices), and let m_1 be an index larger than all of them. Since m_1 then does not satisfy the given property, there is an index $m_2 > m_1$ such that $x_{m_1} \leq x_{m_2}$. Similarly, m_2 does not satisfy the above mentioned property, so there is an index $m_3 > m_2$ such that $x_{m_2} \leq x_{m_3}$. Continuing in this manner produces an increasing (and hence monotone) subsequence (x_{m_k}) of (x_n). ☐

Theorem (Bolzano-Weierstrass)**.** *Any bounded sequence (x_n) of real numbers has a convergent subsequence.*

Proof. By the lemma, the sequence (x_n) has a monotone subsequence (x_{n_k}). Since the original sequence is bounded, so is this subsequence. Thus this subsequence converges since monotone and bounded sequences always converge. ☐

Corollary. *For any $a < b$, the interval $[a, b]$ is compact.*

Proof. Let (x_n) be any sequence in $[a, b]$. Since $[a, b]$ is bounded, this sequence is bounded as well. Thus by the Bolzano-Weierstrass Theorem, it has a convergent subsequence, showing that $[a, b]$ is compact. ☐

The open cover proof of this is trickier, but very nice; this is problem 2.46 in the book. This is not something you will be expected to know how to do on the midterm, but is a good exercise nonetheless since it really requires you to know what "least upper bound" and "open cover" *really* mean. I encourage you to go through it step-by-step and convince yourself that it works.

Theorem. *For any $a < b$, the interval [a,b] is compact.*

Proof. Let $\{U_\alpha\}$ be an open cover of $[a, b] \subset \mathbb{R}$. Define the set C by

$$C = \{x \in [a, b] \mid \text{finitely many of the } U_\alpha \text{ cover } [a, x]\}.$$

This set is nonempty (since $a \in C$ because $[a, a] = \{a\}$ is covered by a single set in the given collection) and bounded above by b. Thus it has a supremum—call it $u \leq b$. We claim that $u \in C$ and $u = b$; if so, then this shows that $b \in C$, so by the definition of C it follows that the original open cover indeed has a finite subcover, and we will be done.

Since $\{U_\alpha\}$ forms an open cover of $[a, b]$, at least one set U' in this collection contains the element $u \in [a, b]$. Since U' is open, there is some $\epsilon > 0$ so that $(u - \epsilon, u + \epsilon) \subset U$. Now, $u - \epsilon$ is not an upper bound of C, so there is some $x \in C$ such that $u - \epsilon < x$. If $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$ are finitely many sets in our collection which cover $[a, x]$ (which exist since $x \in C$), then

$$\{U_{\alpha_1}, \ldots, U_{\alpha_n}, U'\}$$

are finitely many sets in our collection which cover $[a, u]$. Thus $u \in C$.

If $u < b$, then $(u - \epsilon, u + \epsilon)$ contains an element z larger than u and smaller than b. But then the collection

$$\{U_{\alpha_1}, \ldots, U_{\alpha_n}, U'\}$$

from above will also cover $[a, z]$, showing that $z \in C$ and contradicting the fact that u is an upper bound of C. Thus we must have $u = b$ as claimed. We conclude, as mentioned before, that $[a, b]$ is compact. □

Note in the proof above exactly where we used that u was an upper bound of C and where we used that it was the *least* upper bound—both facts were essential.