Math 104 - Worked Examples
Lecture 2, Summer 2010

This is a collection of various examples worked out in full detail. It is not easy to be able to come up with such proofs on your own at first, but the idea is that after having seen these techniques being applied, you should be able to recall and use them later on. What you may want to do is read through the proofs presented below, and then try to recreate them on your own. Feel free to ask about any questions you may have, and please let me know if you find any mistakes!

Claim. *Let $C([a,b])$ denote the set of continuous, real-valued functions on $[a,b]$. The function $d : C([a,b]) \times C([a,b]) \to \mathbb{R}$ given by*

$$d(f,g) := \sup_{x\in[a,b]} |f(x) - g(x)|$$

defines a metric on $C([a,b])$.

Proof. First, since the absolute value of a real number is always nonnegative, the supremum of such numbers is also nonnegative so for any $f,g \in C([a,b])$ we clearly have $d(f,g) \geq 0$. Now,

$$d(f,g) = \sup_{x\in[a,b]} |f(x) - g(x)| = 0$$

if and only if each quantity $|f(x) - g(x)|$ is zero. This happens if and only if $f(x) - g(x) = 0$ for all $x \in [a,b]$, so if and only if $f(x) = g(x)$ for all $x \in [a,b]$. Hence $d(f,g) = 0$ if and only if $f = g$.

Second, since $|f(x) - g(x)| = |g(x) - f(x)|$ for any $f,g \in C([a,b])$ and any $x \in [a,b]$, we have that $d(f,g) = d(g,f)$ for any $f,g \in C([a,b])$.

Finally, let $f,g,h \in C([a,b])$. We must show that

$$d(f,g) \leq d(f,h) + d(h,g).$$

To do this, it suffices to show that

$$|f(x) - g(x)| \leq d(f,h) + d(h,g)$$

for any $x \in [a,b]$. This would show that the right-hand side was an upper bound for the set of numbers $\{|f(x) - g(x)| \ : \ x \in [a,b]\}$, and so the right-hand side would be greater than or equal to their supremum, which is $d(f,g)$. For any $x \in [a,b]$, we have

$$|f(x) - g(x)| \leq |f(x) - h(x)| + |h(x) - g(x)|$$

by the triangle inequality for the usual absolute value. Since

$$|f(x) - h(x)| \leq d(f,h) \text{ and } |h(x) - g(x)| \leq d(h,g)$$

for any $x \in [a,b]$, we get

$$|f(x) - g(x)| \leq |f(x) - h(x)| + |h(x) - g(x)| \leq d(f,h) + d(g,h)$$

for any $x \in [a,b]$, which is what we wanted to show. We conclude that d is a metric on $C([a,b])$ as claimed. □

Claim. *Suppose that the sequences (x_n) and (y_n) of real numbers converge to x and y respectively. Then the sequence $(x_n y_n)$ converges to xy.*

Thoughts. Given $\epsilon > 0$, we want to find an index N large enough so that for $n \geq N$, we have

$$|x_n y_n - xy| < \epsilon.$$

The idea is the same one that often pops up with these "ϵ"-proofs: find a way to bound $|x_n y_n - xy|$ by something which you can force to be smaller than ϵ.

Since the only thing we know at the start is how to come up with bounds on $|x_n - x|$ and $|y_n - y|$ (using the assumption that the given sequences converge), we should be looking for a way to bound $|x_n y_n - xy|$ using these expressions somehow. Note that the triangle inequality implies

$$|x_n y_n - xy| \leq |x_n y_n - x_n y| + |x_n y - xy|.$$

(We can also see this by adding and subtracting $x_n y$ inside $|x_n y_n - xy|$ and then using the usual triangle inequality for the absolute value.) Thus we have

$$|x_n y_n - xy| \leq |x_n||y_n - y| + |x_n - x||y|.$$

Now we are in business, and since we have two terms to work with we try and "$\epsilon/2$-trick".

The second term is easy to bound: since $(x_n) \to x$, we know there exists $N_1 \in \mathbb{N}$ such that for $n \geq N_1$,

$$|x_n - x| < \frac{\epsilon}{2|y|}.$$

This will give us $|x_n - x||y| < \epsilon/2$. However, note that this only works if $y \neq 0$ since otherwise we can't divide by $|y|$. So, we will have to consider the $y = 0$ case separately. Let's skip this for now.

Now we have that for $n \geq N_1$,

$$|x_n y_n - xy| \leq |x_n||y_n - y| + |x_n - x||y| < |x_n||y_n - y| + \frac{\epsilon}{2}.$$

The first term looks almost as easy to bound, and a first guess may be to use the fact that $(y_n) \to y$ to pick $N_2 \in \mathbb{N}$ so that for $n \geq N_2$,

$$|y_n - y| < \frac{\epsilon}{2|x_n|}.$$

However, this is bad since the right hand side is changing as n does because of the x_n term. We need to find a way to bound $|x_n||y_n - y|$ by something which does not depend on n. To do this, note that first we can bound $|x_n|$ as follows. Since for $|x_n - x| < \epsilon/2|y|$ for $n \geq N_1$, we also have that (you should convince yourselves that this is true)

$$|x_n| < |x| + \frac{\epsilon}{2|y|}$$

for $n \geq N_1$. This gives us

$$|x_n||y_n - y| < \left(|x| + \frac{\epsilon}{2|y|}\right)|y_n - y|$$

for $n \geq N_1$, and now we can apply our $\epsilon/2$-trick as we did before since the only thing depending on n now is $|y_n - y|$. This will give us a natural number N_2, and to make sure that all our bounds hold we need to guarantee that the n's we consider are larger than both N_1 and N_2. Let's proceed to our final proof.

Proof of Claim. Let $\epsilon > 0$. First we consider the case $y = 0$. Since (x_n) converges to x, there exists $N_1 \in \mathbb{N}$ such that for $n \geq N_1$,

$$|x_n - x| < 1, \text{ so } |x_n| < |x| + 1.$$

Since (y_n) converges to $y = 0$, there exists $N_2 \in \mathbb{N}$ such that

$$|y_n| < \frac{\epsilon}{|x| + 1} \text{ for } n \geq N_2.$$

Thus for $n \geq \max\{N_1, N_2\}$ we have

$$|x_n y_n - 0| = |x_n||y_n| < (|x| + 1)|y_n| < (|x| + 1)\frac{\epsilon}{|x| + 1} = \epsilon.$$

We conclude that if $y = 0$, then $(x_n y_n)$ converges to $xy = 0$.

Now, suppose that $y \neq 0$. Choose $N_1 \in \mathbb{N}$ such that

$$|x_n - x| < \frac{\epsilon}{2|y|} \text{ if } n \geq N_1.$$

Note that then also

$$|x_n| < |x| + \frac{\epsilon}{2|y|} \text{ if } n \geq N_1.$$

Next, choose $N_2 \in \mathbb{N}$ such that

$$|y_n - y| < \frac{\epsilon}{2\left(|x| + \frac{\epsilon}{2|y|}\right)} \text{ if } n \geq N_2.$$

If $n \geq \max\{N_1, N_2\}$, we then have:

$$\begin{aligned} |x_n y_n - xy| &\leq |x_n||y_n - y| + |x_n - x||y| \\ &< \left(|x| + \frac{\epsilon}{2|y|}\right) \frac{\epsilon}{2\left(|x| + \frac{\epsilon}{2|y|}\right)} + \frac{\epsilon}{2|y|}|y| \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

We conclude that $(x_n y_n)$ converges to xy as claimed. □

Claim. *For any natural number $n \geq 2$, the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^n$ is continuous. (This is a rephrasing of problem 1.14 in Pugh.)*

Thoughts. We will use the ϵ-δ definition of continuity to show that f is continuous at any $y \in \mathbb{R}$. Let $\epsilon > 0$. The goal is to find $\delta > 0$ so that

$$|u - y| < \delta \text{ implies } |f(u) - f(y)| < \epsilon.$$

The δ we want should only depend on the known quantities in the problem, which are ϵ and y. In particular, δ should not depend on u. If you have trouble following what comes below, try working out the $n = 2$ case first.

We want to find a way to bound $|u^n - y^n|$ by something depending on δ, which we can then set to be less than or equal to ϵ. This is what will help us actually find the δ we want. Also, note that the assumption $|u - y| < \delta$ gives an additional bound we can try to use.

Now, we have the identity

$$f(u) - f(y) = u^n - y^n = (u - y)(u^{n-1} + u^{n-2}y + \cdots + y^{n-1}).$$

Notice that after taking absolute values, we can bound the first term on the right side by δ. So we have the bound

$$|u^n - y^n| < \delta|u^{n-1} + u^{n-2}y + \cdots + y^{n-1}|.$$

The point is that we've gotten rid of at least one u on the right side. The goal is now to find a way to bound the second term on the right in a way which does not involve u. Using the triangle inequality a few times, we can further bound the right by

$$\delta|u^{n-1} + u^{n-2}y + \cdots + y^{n-1}| < \delta(|u|^{n-1} + |u|^{n-2}|y| + \cdots + |y|^{n-1})$$

by "bringing all the absolute values inside".

If $|u - y| < \delta$, then $|u| < |y| + \delta$. (You should convince yourselves that this is true. This is a good type of bound to keep in mind for the rest of the course.) Thus we also have

$$|u|^k < (|y| + \delta)^k \text{ for any } k \in \mathbb{N}.$$

Hence we can bound the right side of the inequality we had above by

$$\delta(|u|^{n-1}+|u|^{n-2}|y|+\cdots+|y|^{n-1}) < \delta[(|y|+\delta)^{n-1}+ \text{ other stuff you get by replacing } |u| \text{ with } |y| + \delta].$$

After doing so, we have a bound on $|u^n - y^n|$ which does not depend on $|u|$. We want to find a δ to make the resulting expresssion smaller than or equal to ϵ. But since we end up so many δ's on the right side, it is not at all clear that such a δ can be found.

Here is how we get around this, which is a common technique used in these kinds of arguments. Suppose it happened to be the case that $|u - y| < 1$. Again this gives $|u| < |y| + 1$. Using the same inequalities above, we get the bound

$$|u^n - y^n| < \delta[(|y| + 1)^{n-1} + (|y| + 1)^{n-2}|y| + \cdots + |y|^{n-1}].$$

Now we are good: since the right side now only has a single δ in it, to make this smaller than or equal to ϵ we only need to pick $\delta > 0$ so that it satisfies

$$\delta \leq \frac{\epsilon}{(|y| + 1)^{n-1} + (|y| + 1)^{n-2}|y| + \cdots + |y|^{n-1}}.$$

Note that the denominator is positive, so the right side is indeed a positive number. However, this only works if $|u - y| < 1$, so what happens otherwise?

Here is the key point: to be able to use the bounds on $|u^n - y^n|$ we used above, we need to have both $|u - y| < 1$ and $|u - y| <$ the right side of the inequality above with ϵ in the numerator and that whole mess involving $|y|$ in the denominator. To get this to work, we will choose our actual (and final) value of δ to be

$$\delta = \min\left\{1, \frac{\epsilon}{(|y| + 1)^{n-1} + (|y| + 1)^{n-2}|y| + \cdots + |y|^{n-1}}\right\}.$$

This will give us exactly the bounds we want. Follow the proof given below to see that it does indeed work.

Proof of Claim. Let $y \in \mathbb{R}$ and let $\epsilon > 0$. Set

$$\delta = \min\left\{1, \frac{\epsilon}{(|y|+1)^{n-1} + (|y|+1)^{n-2}|y| + \cdots + |y|^{n-1}}\right\}.$$

Note that both terms above are positive, so their minimum δ is also positive. Suppose that $u \in \mathbb{R}$ satisfies

$$|u - y| < \delta.$$

Then we have $|u| < |y| + \delta$, so

$$\begin{aligned}
|u^n - y^n| &= |u-y||u^{n-1} + u^{n-2}y + \cdots + y^{n-1}| \\
&< \delta|u^{n-1} + u^{n-2}y + \cdots + y^{n-1}| \\
&\leq \delta(|u|^{n-1} + |u|^{n-2}|y| + \cdots + |y|^{n-1}) \\
&< \delta[(|y|+\delta)^{n-1} + (|y|+\delta)^{n-2}|y| + \cdots + |y|^{n-1}] \\
&\leq \delta[(|y|+1)^{n-1} + (|y|+1)^{n-2}|y| + \cdots + |y|^{n-1}] \\
&\leq \frac{\epsilon}{(|y|+1)^{n-1} + (|y|+1)^{n-2}|y| + \cdots + |y|^{n-1}}[(|y|+1)^{n-1} + (|y|+1)^{n-2}|y| + \cdots + |y|^{n-1}] \\
&= \epsilon.
\end{aligned}$$

In the fifth line we used the fact that $\delta \leq 1$ while in the sixth we used that

$$\delta \leq \frac{\epsilon}{(|y|+1)^{n-1} + (|y|+1)^{n-2}|y| + \cdots + |y|^{n-1}},$$

which are both true by the choice of δ as the minimum of these two numbers. Hence for this choice of $\delta > 0$, we have that

$$|u - y| < \delta \text{ implies } |u^n - y^n| < \epsilon,$$

so we conclude that f is continuous at y. Since $y \in \mathbb{R}$ was arbitrary, f is continuous. □

Claim. *Suppose that every monotone and bounded sequence in $\mathbb{R}$ converges. Then any nonempty and bounded above subset S of $\mathbb{R}$ has a supremum. (This is the hard part of problem 30b in Pugh.)*

Thoughts. Here is a failed attempt at a proof. This is the one I originally had in mind when I first assigned the problem, only to realize later that it doesn't quite work.

By the mentioned analog for supremums of problem 1, it is enough to construct a sequence of elements of S which converge to an upper bound of S. This upper bound will then be the supremum. Pick any $x_1 \in S$. If x_1 is the supremum, we are done; otherwise, there exists $x_2 \in S$ such that $x_1 < x_2$. If x_2 is the supremum of of S, we are done; otherwise pick $x_3 \in S$ such that $x_2 < x_3$. Continuing in this manner either gives the supremum at some step or produces a strictly increasing sequence (x_n) of elements of S. Since S is bounded above, this sequence is bounded to by our assumption it converges to some $x \in S$.

Here is the problem: this x may in fact not be an upper bound of S! If it is, then we are done, but there is no way to guarantee that it will be. This is what I missed when I first assigned the problem. To get around this, you have to be more careful about how you construct the x_n—you want to construct them in a way that ensures that their limit will be an upper bound of S. If you have trouble following the proof below, draw a picture of what is going on. We start with an interval $[x_1, y_1]$, and then construct an interval $[x_2, y_2]$—what does it look like? What can you say about its length? What about the interval $[x_3, y_3]$ constructed next? And so on.

Proof of Claim. If S itself had a largest element, i.e. a maximum, then that maximum would be the supremum and there is nothing to show. So, suppose that S does not have a maximum.

Pick any element $x_1 \in S$ and any upper bound y_1 of S. Let c_1 be the midpoint of the interval $[x_1, y_1]$. If c_1 is an upper bound of S, let $x_2 = x_1$ and $y_2 = c_1$; if c_1 is not an upper bound of S, let x_2 be an element of S larger than c_1 and let $y_2 = y_1$. Now let c_2 be the midpoint of the interval $[x_2, y_2]$. As before, if c_2 is an upper bound of S, let $x_3 = x_2$ and $y_3 = c_2$; if c_2 is not an upper bound of S, let x_3 be an element of S larger than c_2 and let $y_3 = y_2$. Continuing in this manner produces a collection of intervals $[x_n, y_n]$, the left endpoints of which give an increasing sequence (x_n) of elements in S. Note that each of the right endpoints, y_n, by construction is an upper bound of S and none of them can be in S since we are assuming that S does not have a maximum. Also note that if $\epsilon = y_1 - x_1 > 0$ is the length of the first interval, the length of the interval $[x_n, y_n]$ is less than $\epsilon/2^{n-1}$ since we constructed each of these intervals by using midpoints of the previous interval.

Now, since S is bounded, the sequence (x_n) is bounded, so by our assumption it converges to some number x. Note that since the sequence (x_n) is increasing, it follows that $x_n \leq x$ for all n. We claim that $x = \sup S$. Using the mentioned analog for supremums of problem 1, since we have a sequence of elements in S converging to x, it suffices to show that x is an upper bound of S. Note that since each y_n is an upper bound of S, we must have $x \leq y_n$ for all n since if $y_M < x$ for some M, then the terms in the sequence (x_n) would be bounded away from x by a distance of at least $x - y_M > 0$ and so could not converge to x, which they do.

Finally, to show that x is an upper bound of S, let $s \in S$. We must show that $s \leq x$. If $s = x$, there is nothing to show, so assume that $s \neq x$. Since the lengths $\epsilon/2^{n-1}$ of the intervals $[x_n, y_n]$ converge to 0, there exists $N \in \mathbb{N}$ such that the length of $[x_N, y_N]$ is smaller than $|x - s|$. Since x itself is in this interval, it follows that $s \notin [x_N, y_N]$. But y_N is an upper bound of S, so we cannot have $y_N < s$ and thus we must have

$$s < x_N \leq x.$$

We conclude that x is an upper bound of S and hence that x is the supremum of S. □

Claim. *Let $f : M \to N$ be a continuous function between two metric spaces M and N. Suppose that f has the following property: a sequence (p_n) converges in M if and only if $(f(p_n))$ converges in N. Prove that the image of a closed set in M is closed in N.*

Thoughts. This is actually quite easy to prove, but I just wanted to point out a subtle point. A map with the property that the image of closed set is closed is itself said to be *closed*. The point here is that continuous functions are not necessarily closed. For example, let $f : (0, 2) \to \mathbb{R}$ be the function given by $f(x) = x$. This is continuous, but the image of the interval $(0, 1]$ (which is closed as a subset of the domain $(0, 2)$) is not closed in $\mathbb{R}$, so f is not a closed map. It is a good idea to look at the quick proof below, and see what goes wrong for this function, and in general to figure out why we need to make the additional assumptions we make in the claim.

Similarly, a map $g : M \to N$ is said to be *open* if the image of any open set is open. Again, continuous functions are not necessarily open—you should try to come up with a counterexample and think of additional assumptions we can make on a continuous function that would guarantee it was open.

Proof of Claim. Let V be a closed subset of M and let (q_n) be a sequence of points in $f(V)$ which converges to some $q \in N$. We want to show that $q \in f(V)$. Now, since each $q_n \in f(V)$, we can find points $p_n \in V$ so that $f(p_n) = q_n$. By our assumption, since $(f(p_n))$ converges in N, (p_n) converges to some $p \in M$.

Now, since V is closed and each $p_n \in V$, it follows that $p \in V$. Since f is continuous, we know that $(f(p_n))$ then converges to $f(p)$. Since $(f(p_n)) = (q_n)$ also converges to q, it must be that $q = f(p)$. Thus $q \in f(V)$, so we conclude that $f(V)$ is closed in N. □

Claim. *The set $\mathbb{R}$ of real numbers, with respect to the usual Euclidean distance, is complete. (This proof is the one we gave in class, which is different than the one given in the book.)*

Proof. Let (x_n) be a Cauchy sequence in $\mathbb{R}$. For each n, define b_n to be

$$b_n = \inf\{x_k \mid k \geq n\}.$$

Note that for each n, this set is nonempty and bounded since (x_n) itself, being Cauchy, is bounded. Hence these infimums all exist. Since for each n, the set for which b_n is the infimum contains the one for which b_{n+1} is the infimum, it follows that $b_n \leq b_{n+1}$ for each n. Hence (b_n) is an increasing and bounded sequence, so it converges to some $b \in \mathbb{R}$. We claim that (x_n) also converges to b.

To see this, let $\epsilon > 0$. Since (x_n) is Cauchy, there exists $N \in \mathbb{N}$ such that $|x_n - x_m| < \epsilon/2$ for $n, m \geq N$. In particular, for $n \geq N$ we have

$$|x_n - x_N| < \frac{\epsilon}{2}.$$

This implies that

$$x_n \in [x_N - \epsilon/2, x_N + \epsilon/2] \text{ for } n \geq N.$$

Thus the infimum of any set contained in this interval is also in this interval (here we use the fact that the interval is closed), so

$$b_n \in [x_N - \epsilon/2, x_N + \epsilon/2] \text{ for } n \geq N.$$

Again since this interval is closed, it follows that the limit b of (b_n) is also in this interval, so

$$|x_N - b| \leq \frac{\epsilon}{2}.$$

Thus, for $n \geq N$, we have

$$|x_n - b| \leq |x_n - x_N| + |x_N - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

We conclude that (x_n) converges to b, and hence that $\mathbb{R}$ is complete. □

Claim. *Suppose that M is a compact metric space and (p_n) is a sequence in M with the property that every convergent subsequence of it converges to the same $p \in M$. Then (p_n) itself converges to p.*

Proof. By way of contradiction, suppose that (p_n) did not converge to p. Then there exists $\epsilon > 0$ such that for every $N \in \mathbb{N}$, there exists $n > N$ such that

$$d(p_n, p) \geq \epsilon.$$

First pick such an $n_1 > 1$, so that $d(p_{n_1}, p) \geq \epsilon$. Then there exists $n_2 > n_1$ such that $d(p_{n_2}, p) \geq \epsilon$, and then $n_3 > n_2$ so that $d(p_{n_3}, p) \geq \epsilon$. Continuing in this manner produces a subsequence (p_{n_k}) of (p_n) so that

$$d(p_{n_k}, p) \geq \epsilon \text{ for all } k.$$

Since M is compact, this has a convergent subsequence $(p_{n_{k_\ell}})$, which converges to p since this is also a convergent subsequence of the original sequence (p_n). This is a contradiction since each term in this subsequence is at a distance at least ϵ away from p, and so cannot converge to p. We conclude that (p_n) converges to p. □

Claim. *A subset S of a metric space M is dense in M if and only if every open ball in M contains an element of S.*

Proof. Suppose that S is dense in M, Let $p \in M$ and $\epsilon > 0$, and consider the ball $M_\epsilon(p)$. Since S is dense in M, there is a sequence (p_n) in S converging to p. Then, for large enough n, $d(p_n, p) < \epsilon$. Hence in particular, $M_\epsilon(p)$ contains an element of this sequence, which is an element of S.

Conversely, suppose that every open ball in M contains an element of S. Let $p \in M$. To show that S is dense in M, we must show that there is a sequence of elements of S converging to p. For each $n \in \mathbb{N}$, the ball $M_{1/n}(p)$ contains an element of S—call it p_n. These points then satisfy

$$d(p_n, p) < \frac{1}{n},$$

implying that (p_n) converges to p. Since $p \in M$ was arbitrary, we conclude that $\overline{S} = M$, so S is dense in M. □

Claim. *A function $f : M \to N$ is uniformly continuous if and only if $d_N(f(p_n), f(q_n)) \to 0$ for any sequences (p_n) and (q_n) in M such that $d_M(p_n, q_n) \to 0$.*

Proof. Suppose that f is uniformly continuous and let (p_n) and (q_n) be two sequences in M such that $(d_M(p_n, q_n))$ converges to 0. Let $\epsilon > 0$. Then there exists $\delta > 0$ such that

$$d_N(f(x), f(y)) < \epsilon \text{ whenever } d_M(x, y) < \delta.$$

Since $d_M(p_n, q_n) \to 0$, there exists $N \in \mathbb{N}$ so that $d_M(p_n, q_n) < \delta$ if $n \geq N$. Thus if $n \geq N$, we also have

$$d_N(f(p_n), f(q_n)) < \epsilon,$$

showing that $(d_N(f(p_n), f(q_n)))$ converges to 0 as claimed.

To prove the converse, we instead prove its contrapositive: if f is not uniformly continuous, then there exist sequences (p_n) and (q_n) in M such that $d_M(p_n, q_n) \to 0$ but $d_N(f(p_n), f(q_n))$ does not converge to 0. So, suppose f is not uniformly continuous. Then there exists $\epsilon > 0$ so that for any $\delta > 0$ there exist points $p, q \in M$ such that

$$d_M(p, q) < \delta \text{ but } d_N(f(p), f(q)) \geq \epsilon.$$

In particular, for each $n \in N$, there exist points p_n and q_n such that

$$d_M(p_n, q_n) < \frac{1}{n} \text{ but } d_N(f(p_n), f(q_n)) \geq \epsilon.$$

It follows that for these two sequences, $(d_M(p_n, q_n))$ does converge to 0 but $(d_N(f(p_n), f(q_n)))$ does not since each term in the latter sequence is bounded away from 0 by at least $\epsilon > 0$. □