Worked Examples, Fall 2011

This is a collection of various examples worked out in full detail, along the lines of the "Worked Examples" handout from Summer 2010. It is not easy to be able to come up with such proofs on your own at first, but the idea is that after having seen these techniques being applied, you should be able to recall and use them later on. What you may want to do is read through the proofs presented below, and then try to recreate them on your own. Feel free to ask about any questions you may have, and please let me know if you find any mistakes!

Claim. *The sequence of real numbers defined recursively by*

$$x_1 = \sqrt{2},\ x_{n+1} = \sqrt{2 + x_n} \text{ for } n > 1$$

converges and its limit is 2.

Thoughts. If we want to show this sequence converges using the definition of convergence itself, we would need to have a candidate as to what the sequence should converge to. Finding this is not actually difficult given the recursive definition: we essentially take the limit of both sides of

$$x_{n+1} = \sqrt{2 + x_n}$$

and use the fact that (x_{n+1}) converges to the same thing as (x_n) to get an equation which the limit must satisfy, and then solve that equation. See the proof below for precise reasoning as to why this technique is valid.

However, even once we have a guess for the value of the limit, it will be difficult to show that this is correct using the definition of convergence. Indeed, it is the recursive definition of (x_n) which makes this tough. The point is that in this case we can actually show the limit exists via a different method, without actually knowing what the limit is going to be beforehand. The key is that the given sequence is actually *monotone* and *bounded*, and we have seen that such sequences always converge. Once we know this sequence converges, we can use the technique mentioned above to actually find the value of the limit.

Proof. First we show that this sequence is bounded and nondecreasing. To show that (x_n) is nondecreasing, we proceed by induction. Certainly

$$x_1 = \sqrt{2} < x_2 = \sqrt{2 + \sqrt{2}}.$$

If $x_{n-1} < x_n$, then

$$x_n = \sqrt{2 + x_{n-1}} < \sqrt{2 + x_n} = x_{n+1}.$$

By induction we conclude that $x_n < x_{n+1}$ for all n. To see that (x_n) is bounded, we may also proceed via induction. We first have

$$0 < x_1 = \sqrt{2} < 2.$$

If $0 < x_n < 2$, then

$$0 < x_{n+1} = \sqrt{2 + x_n} < \sqrt{2 + 2} = 2.$$

Thus $0 < x_n < 2$ for all n, so (x_n) is bounded. Since (x_n) is a monotone and bounded sequence of real numbers, it converges to some $x \in \mathbb{R}$.

Now that we know this sequence converges, we can compute the value of its limit x as follows. Since (x_n) converges to x, the subsequence (x_{n+1}) formed by starting at the second term also converges to x. On the other hand, the sequence $(2+x_n)$ converges to $2+x$ and hence the sequence $(\sqrt{2+x_n})$ converges to $\sqrt{2+x}$. But this final sequence is the same as (x_{n+1}), and since limits of a sequence are unique we must have

$$x = \sqrt{2+x}.$$

Thus $x^2 = 2+x$, so the limit x of (x_n) satisfies the quadratic equation $x^2 - x - 2 = 0$. This implies that

$$x = \frac{1 \pm 3}{2} = -1 \text{ or } 2.$$

But the limit cannot be -1 since $x_n > 0$ for all n, so we conclude that $x = 2$ and hence (x_n) converges to 2. $\square$

Claim. *The function $f : [0,1] \to \mathbb{R}$ defined by $f(x) = x$ for $x \in [0,1]$ is integrable and its integral over $[0,1]$ is $\frac{1}{2}$.*

Of course, this is a simple consequence of the fact that continuous functions over closed intervals are always integrable and that one can use the Fundamental Theorem of Calculus to calculate the value of the integral. The point here, however, is to do this using only the definition of integrability itself.

Proof. For each $n \in \mathbb{N}$, let P_n be the partition of $[0,1]$ obtained by breaking $[0,1]$ up into subintervals of length $\frac{1}{n}$; to be concrete, P_n is given by

$$P_n := \left\{0 < \frac{1}{n} < \frac{2}{n} < \cdots < \frac{n-1}{n} < 1\right\}.$$

Since f is increasing, on each subinterval $[\frac{k}{n}, \frac{k+1}{n}]$, the infimum of f occurs at the left endpoint and the supremum at the right endpoint. Thus we have

$$\begin{aligned}
U(f, P_n) &= \sum_{\text{subintervals}} \left(\sup_{x \in \text{subinterval}} f(x) \right) (\text{length of subinterval}) \\
&= \sum_{k=0}^{n-1} \left(\sup_{x \in [\frac{k}{n}, \frac{k+1}{n}]} x \right) \left(\frac{k+1}{n} - \frac{k}{n} \right) \\
&= \sum_{k=0}^{n-1} \left(\frac{k+1}{n} \right) \left(\frac{1}{n} \right) \\
&= \frac{1}{n^2} \sum_{k=0}^{n-1} (k+1) = \frac{1}{n^2}(1 + 2 + \cdots + n) = \frac{n(n+1)}{2n^2},
\end{aligned}$$

and

$$\begin{aligned}
L(f, P_n) &= \sum_{\text{subintervals}} \left(\inf_{x \in \text{subinterval}} f(x) \right) (\text{length of subinterval}) \\
&= \sum_{k=0}^{n-1} \left(\inf_{x \in [\frac{k}{n}, \frac{k+1}{n}]} x \right) \left(\frac{k+1}{n} - \frac{k}{n} \right) \\
&= \sum_{k=0}^{n-1} \left(\frac{k}{n} \right) \left(\frac{1}{n} \right) \\
&= \frac{1}{n^2} \sum_{k=0}^{n-1} k = \frac{1}{n^2}(0 + 1 + 2 + \cdots + (n-1)) = \frac{(n-1)n}{2n^2}.
\end{aligned}$$

Finally, note that

$$U(f,P_n) - L(f,P_n) = \frac{n(n+1)}{2n^2} - \frac{(n-1)n}{2n^2} = \frac{2n}{2n^2} = \frac{1}{n}.$$

Now, let $\epsilon > 0$ and pick N such that $\frac{1}{N} < \epsilon$. Then for the partition P_N, we have

$$U(f,P_N) - L(f,P_N) = \frac{1}{N} < \epsilon,$$

so we conclude that f is integrable over $[0,1]$. Alternatively, since $U(f,P_n) \to \frac{1}{2}$ and $L(f,P_n) \to \frac{1}{2}$, we must have

$$U(f) = \inf\{U(f,P) \mid P \text{ is a partition of } [0,1]\} \leq \frac{1}{2},$$

and

$$L(f) = \sup\{L(f,P) \mid P \text{ is a partition of } [0,1]\} \geq \frac{1}{2},$$

so $\frac{1}{2} \leq L(f) \leq U(f) \leq \frac{1}{2}$. Thus $\frac{1}{2} = L(f) = U(f)$, showing that f is integrable over $[0,1]$ and that the value of its integral is $\frac{1}{2}$. $\square$