The Riemann-Lebesgue Theorem
(or, a brief introduction to Measure Theory)

Our study of integration naturally leads us to ask: which functions are integrable? This is silly, since the answer is "those satisfying the definition of integrability". A better question to ask is: is there a (quick) way to tell, just by looking at a function and without too much work, whether or not it is integrable? As long as we can determine where the function is continuous, which is often simpler to do than trying to establish integrability from scratch, the Riemann-Lebesgue Theorem will give us a way to do this. The key notion is that of a *zero set*, which fits into the broader framework of *measure theory*. The final section on Lebesgue integration is not material which may appear on the final.

1 Zero Sets

Intuitively, zero sets are the subsets of $\mathbb{R}$ which have zero "length". To make this precise, we have to define what we mean by the "length" of an arbitrary subset of $\mathbb{R}$. This is easy to do for intervals or unions of intervals, but trickier to do for general subsets; for example, what is the "length" of the set of rational numbers, or of the set of irrational numbers?

For now, let us define what mean by "zero length"; we will come back to arbitrary "length" a bit later when we talk about measure theory.

Definition 1. A subset $Z \subseteq \mathbb{R}$ is a *zero set* if for any $\epsilon > 0$ there exists a countable collection $\{(a_i, b_i)\}$ of intervals which cover Z such that

$$\sum_{i=1}^{\infty}(b_i - a_i) \leq \epsilon.$$

This sum is called the *total length* of the collection $\{(a_i, b_i)\}$. We will also say that a zero set has *measure* zero.

Let us wrap our heads around this definition. Given a countable collection of open intervals, its total length is exactly what is sounds like: we are just adding up the lengths of all intervals in the collection. (Of course we should only consider collections where this sum actually exists, i.e. such that $\sum(b_i - a_i)$ converges.) If a set Z is covered by such a collection, clearly the "length" of Z should be smaller than or equal to the total length of the collection. The above definition says that a zero set is a set whose "length" is smaller than or equal to any $\epsilon > 0$, so that the "length" of a zero set should actually be zero.

To summarize: a zero set is one which can be covered by collections of open intervals of arbitrarily small total length.

Example 1. Any finite subset of $\mathbb{R}$ is a zero set. Indeed, suppose that $Z = \{x_1, \dots, x_n\}$ is a finite subset of $\mathbb{R}$ and fix $\epsilon > 0$. For each i, let $I_i = (x_i - \frac{\epsilon}{2n}, x_i + \frac{\epsilon}{2n})$ be the interval of radius $\frac{\epsilon}{2n}$ around x_i. Then the the collection $\{I_i\}$ covers Z and its total length is

$$\sum \text{length}(I_i) = \sum_{i=1}^{n} \frac{\epsilon}{n} = \epsilon.$$

Hence Z is a zero set. Note that this makes sense intuitively: the "length" of a single point is zero, and the "length" of Z is obtained by adding together the lengths of $\{x_i\}$ for each i.

Example 2. More interestingly, any countable set is a zero set. We have already shown this for countable sets which are finite, so suppose that Z is countably infinite. Since Z is countable, we can list its elements as

$$x_1, x_2, x_3, \ldots.$$

Let $\epsilon > 0$ and for each i let I_i be an interval of length $\frac{\epsilon}{2^i}$ around x_i; so, I_1 is an interval of length $\frac{\epsilon}{2}$ around x_1, I_2 is an interval of length $\frac{\epsilon}{4}$ around x_2, I_3 has length $\frac{\epsilon}{8}$ around x_3, and so on. Then the collection $\{I_i\}$ covers Z and its total length is

$$\sum_{n=1}^{\infty} \text{length}(I_i) = \sum_{n=1}^{\infty} \frac{\epsilon}{2^i} = \epsilon \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^i = \epsilon,$$

where we use properties of geometric series to compute the final sum. Thus Z is a zero set.

Thus, for instance, the set of rational numbers is a zero set. The set of irrational numbers, however, is not a zero set, since if it were its union with $\mathbb{Q}$ would be a zero set as a consequence of the following proposition; this union is all of $\mathbb{R}$, and $\mathbb{R}$ is not a zero set since it has "infinite length".

Proposition 1. *The countable union of zero sets is a zero set, as is any subset of a zero set.*

Proof. Suppose first that Z_1 and Z_2 are both zero sets, and let $\epsilon > 0$. Then we can countable covers $\{I_i\}$ ad $\{J_i\}$ of Z_1 and Z_2 respectively each of total length less than or equal to $\frac{\epsilon}{2}$. The collection we get by taking the union of $\{I_i\}$ and $\{J_i\}$ then covers $Z_1 \cup Z_2$ and has total length less than or equal to ϵ, so $Z_1 \cup Z_2$ is a zero set. The same idea works for a finite union of zero sets, and for the union of countably many zero sets we use the same "$\frac{\epsilon}{2^i}$"-trick we used to show that a countably infinite set was a zero set.

If $A \subset Z$ and Z is a zero set, then for any $\epsilon > 0$ we can find a collection $\{I_i\}$ of open intervals covering Z with total length $\leq \epsilon$. This same collection also covers A, so it follows that A is also a zero set. □

Note that the above facts are perfectly intuitive: the finite (or countable) union of sets of "length" zero should still have length zero, as should any subset of a set of "length" zero. It may seem surprising that there are uncountable zero sets, but there are—see the optional problem on the final homework assignment for an example.

2 Riemann-Lebesgue Theorem

Now we can give a complete characterization of integrable functions. For a function $f : [a, b] \to \mathbb{R}$, let $D(f)$ denote its *discontinuity set*:

$$D(f) = \{x \in [a, b] \mid f \text{ is not continuous at } x\}.$$

So, for example, a continuous function has an empty discontinuity set.

Theorem 1. *A function $f : [a, b] \to \mathbb{R}$ is (Riemann) integrable if and only if it is bounded and its set of discontinuity points $D(f)$ is a zero set.*

So, whether or not a function is integrable is completely determined by whether or not it is discontinuous at "too many" points, or by whether or not the set of points where it is discontinuous has "length" zero. In particular, if a function is integrable on $[a, b]$, then it is in fact continuous at an uncountable number of points in $[a,b]$; more precisely, it is continuous "outside of a set of measure zero", so we say that it is continuous *almost everywhere.*

The proof of this theorem is not easy but is still pretty manageable, it just requires a bit of setup. To save time, we will just assume this to be true and leave proofs to Wikipedia or other texts. As the following examples now show, this theorem in general gives us a quicker way of determining integrability.

Example 3. Since the discontinuity set of a continuous function is empty and the empty set has measure zero, the Riemann-Lebesgue theorem immediately implies that continuous functions on closed intervals are always integrable.

Example 4. A piecewise continuous function has a finite set of discontinuity points. Since finite sets are always zero sets, Riemann-Lebesgue again implies that a piecewise continuous function on $[a,b]$ is integrable.

Example 5. Consider the function $f:[0,1]\to\mathbb{R}$ defined by

$$f(x)=\begin{cases}1 & \text{if } x\in\mathbb{Q}\\ 0 & \text{if } x\in\mathbb{R}\backslash\mathbb{Q}.\end{cases}$$

Then f is discontinuous everywhere, so $D(f)=[0,1]$ is not a zero set. Thus f is not integrable.

Example 6. My favorite function on $[0,1]$ is discontinuous at each rational. Thus its set of discontinuity points is contained in a zero set (the set of all rationals) and so is a zero set itself. Hence my favorite function on $[0,1]$ is integrable by the Riemann-Lebesgue Theorem.

The moral is that an integrable function is one whose discontinuity set is not "too large" in the sense that it has length zero.

3 Lebesgue Integration

Here is another way to think about the Riemann-Lebesgue Theorem. Suppose that $f:[a,b]\to\mathbb{R}$ is bounded. If f were integrable, we could "split" its integral up into one over the subset of points of $[a,b]$ where f is continuous and the subset $D(f)$ where it is not:

$$\int_a^b f \text{ ``}=\text{''} \int_{[a,b]\backslash D(f)} f+\int_{D(f)} f.$$

Of course, so far we have only defined what it means for a function to be integrable on an interval, so integrating over arbitrary subsets of $\mathbb{R}$ (as in the two expressions on the right) does not (yet) make sense. However, let's ignore this for now.

Since f is continuous on $[a,b]\backslash D(f)$, the first integral above exists. Thus, whether or not f is integrable over $[a,b]$ is completely determined by whether or not it is "integrable" over $D(f)$. If $D(f)$ is a zero set, *any* integral over it is zero since an integral is supposed to give the area under the graph of the function, and such an area is always zero if the "base" of the region has "length" zero. In this case then, $\int_{D(f)} f=0$, so the points in $D(f)$ contribute nothing to the integral of f. Thus, the Riemann-Lebesgue theorem says that an integrable function is one for which the points where it is not continuous contribute nothing to the value of integral.

To make this precise would require us to develop a theory of integration over more general subsets of $\mathbb{R}$. In fact, Example 5 also shows that we need such a theory: the function f given there is not integrable and yet I claim there is a well-defined area under its graph.

Example 7. Consider the function f of Example 5. The region under its graph consists of vertical line segments lying over each rational in $[0,1]$. Intuitively, I claim that the "area" of this region is zero. Indeed, as we know, the set of rationals in $[0,1]$ is a zero set, so for any $\epsilon > 0$ we can find a countable collection of intervals $\{I_i\}$ covering $[0,1] \cap \mathbb{Q}$ such that

$$\sum \text{length}(I_i) \leq \epsilon.$$

Consider the collection $\{R_i\}$ of rectangles of height one where R_i has I_i as its base. Then the area of R_i is height $\cdot$ length $= 1 \cdot \text{length}(I_i) = \text{length}(I_i)$. Thus the total area enclosed by these rectangles is

$$\sum \text{area}(R_i) = \sum \text{length}(I_i) \leq \epsilon.$$

But the region A under the graph of f is contained in the region enclosed by all the rectangles R_i, so the area of A is $\leq \epsilon$. Since this is true for all $\epsilon > 0$, we must have that the area of A is zero as claimed.

The point again is that the region under the graph of this function has a well-defined area, but the theory of Riemann integration is not strong enough to detect this. We need a theory of integration which can "integrate" the same functions which Riemann integration does, but can also handle other functions which "should" be integrable. This consideration leads to what is called the *Lebesgue integral* and gives a glimpse into what is more generally known as *measure theory.* This is essentially the most general theory of integration available, and allows one to define integration over a vast variety of different types of spaces all at once.

We will outline how this works in the case of $\mathbb{R}$ via the Lebesgue integral. The starting point is defining a general notion of "length". We mimick the definition we gave for zero sets, only modified to allow for positive measure:

Definition 2. Given $A \subseteq \mathbb{R}$, consider all possible collections of intervals covering A. The *Lebesgue (outer) measure* of A is the infimum $\mu(A)$ of the total lengths of all such collections:

$$\mu(A) := \inf \left\{ \sum (b_i - a_i) \;\middle|\; \text{the collection } \{(a_i, b_i)\} \text{ covers } A \right\}.$$

If this infimum does not exist—i.e. if all possible total lengths $\sum(b_i - a_i)$ are infinite—then we say that A has *infinite* measure.

The idea is similar to the one given after the definition of a zero set: the measure (or "length") of A should be $\leq$ the total length of any countable covering of A by open intervals, and the actual measure of A is the smallest possible such total length. A zero set is then one which has measure zero.

We use "outer" to describe this measure since we are measuring the length of A from the "outside" by looking at collections of intervals which contain A in their union; there is a similar notion of *inner measure* where we measure the length of A from the "inside" by looking at collections of intervals whose union is contained in A. This distinction between outer and inner measure is important in the full theory of Lebesgue integration and is related to the notion of what it means for a set to be *measurable*, but we will skip this distinction here and focus on outer measure.

Example 8. An interval $[a,b]$, (a,b), $(a,b]$, or $[a,b)$ has measure $b-a$, as one would expect of a notion of "length". $\mathbb{R}$ has infinite measure, and since $\mathbb{Q}$ has measure zero, $\mathbb{R}\backslash\mathbb{Q}$ also has infinite measure. However, since $[0,1]$ has measure 1, the set of irrationals in $[0,1]$ also has measure 1.

Now that we have a general notion of "length", we can outline how this leads to a more general theory of integration. To start, let $A \subseteq \mathbb{R}$ and consider the function χ_A defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

We call this the *indicator* function of A since it "picks out" which numbers belong to A. The region under the graph of χ_A consists of vertical line segments lying above each element of A, and as a result the "area" of this region should intuitively be the height 1 times the length of the "base" A, so we define

$$\int_{\mathbb{R}} \chi_A \, d\mu := \mu(A)$$

and call this the *Lebesgue integral* of the function χ_A. (The $d\mu$ is just notation referring to the fact that we are integrating with respect to Lebesgue measure.)

Example 9. The function of Example 5 is precisely the indicator function of $\mathbb{Q} \cap [0,1]$, and hence its Lebesgue integral is $\mu(\mathbb{Q} \cap [0,1]) = 0$. So, although this function is not Riemann integrable, it *is* Lebesgue integrable (a notion which we admittedly have not and will not define) and its Lebesgue integral is precisely what the area of the region under its graph should be, namely 0.

Example 10. As a consequence of the fact that $\mu(\mathbb{Q} \cap [0,1]) = 0$ and $\mu([0,1]) = 1$, we see that $\mu((\mathbb{R} \backslash \mathbb{Q}) \cap [0,1]) = 1$. Thus the function g on $[0,1]$ which is zero at all rationals and 1 at all irrationals has Lebesgue integral 1.

To define the integrals of more general functions, we proceed as follows. First, for any linear combination $n\chi_A + m\chi_B$ of indicator functions (called a *simple* function), we define its Lebesgue integral as

$$\int (n\chi_A + m\chi_B) \, d\mu := n \int \chi_A \, d\mu + m \int \chi_B d\mu = n\mu(A) + m\mu(B).$$

Of course, this "linearity" property is one we would expect an integral to have, so we are defining the Lebesgue integral precisely so that this property is forced to hold.

Note that now we already know how to integrate step functions (since we can express such a function as a linear combination of indicator functions), and the result is going to be equal to the total area enclosed by rectangles whose bases are the intervals where the "steps" occur. Finally, given a nonnegative function f, we define its integral to be the supremum of the integrals of all simple functions $\leq f$:

$$\int f \, d\mu := \sup \left\{ \int s \, d\mu \;\middle|\; s \text{ is a simple function such that } s \leq f \right\}.$$

In other words, to define the Lebesgue integral of a nonnegative function, we approximate it from below by simple functions whose integrals we know, and take the supremum of the values of these integrals. One can then go on to define integrals for functions which take on both positive and negative values. The key result is the following:

Theorem 2. *A function which is Riemann integrable is Lebesgue integrable and its Lebesgue integral agrees with its Riemann integral.*

So, the upshot is that if a function is Riemann integrable to begin with, it remains integrable in the Lebesgue sense, but now Lebesgue integration allows one to integrate functions which are not Riemann integrable.

This is just the tip of the iceberg, and measure theory in general has vast applications. In particular, if you ever take a more advanced probability or statistics course, expect to see some of these ideas again.