Notes on Supremums and Infimums

The purpose of these notes is to elaborate on the notions of supremums and infimums discussed in the book. The book gives some very basic definitions, but these topics deserve much more attention paid to them. Here we give some further characterizations and properties of these two ideas.

Supremums

Definition. The *supremum* (or *least upper bound*) of a set $S \subseteq \mathbb{R}$ which is bounded above is an upper bound $b \in \mathbb{R}$ of S such that $b \leq u$ for any upper bound u of S. We use the notation $b = \sup S$ for supremums.

Note that there are two definining properties of $\sup S$: (i) it is an upper bound of S, and (ii) it is smaller than or equal to any other upper bound of S. Both of these are crucial.

The following justifies us talking about *the* supremum of a set as opposed to *a* supremum:

Proposition. *The supremum of a set, if it exists, is unique.*

Proof. Suppose that $S \subseteq \mathbb{R}$ is bounded above and that $a, b \in \mathbb{R}$ are supremums of S. Note that in particular both a and b are then upper bounds of S.

Since a is a least upper bound of S and b is an upper bound of S, $a \leq b$. Similarly, since b is a least upper bound and a an upper bound of S, $b \leq a$. Thus $a = b$, showing that the supremum of a set is unique. □

Intuitively, another way of stating the definition of supremum is that no number smaller than the supremum can be an upper bound of the given set. The following makes this precise:

Proposition. *An upper bound b of a set $S \subseteq \mathbb{R}$ is the supremum of S if and only if for any $\epsilon > 0$ there exists $s \in S$ such that $b - \epsilon < s$.*

For practice, try to give a precise proof of this, but the intuition is the following. The statement "there exists $s \in S$ such that $b - \epsilon < s$" means exactly that $b - \epsilon$ is not an upper bound of S; in other words, this is the *negation* of what it means to say that $u \in \mathbb{R}$ is an upper bound of S: for any $s \in S$, $s \leq u$. As ϵ varies over all positive real numbers, $b - \epsilon$ varies over all real numbers smaller than b, so the condition given in the proposition precisely says that for an upper bound b of S, $b = \sup S$ if and only if no number smaller than b is an upper bound of S.

This next proposition requires material on convergent sequences, which we will discuss in Chapter 2:

Proposition. *Suppose that $S \subseteq \mathbb{R}$ is bounded above and that $b \in \mathbb{R}$ is an upper bound of S. Then $b = \sup S$ if and only if there exists a sequence (x_n) of elements in S converging to b.*

Proof. Suppose that $b = \sup S$. For any $n \in \mathbb{N}$, the previous proposition tells us that there exists $x_n \in S$ such that

$$b - \frac{1}{n} < x_n.$$

Since also $x_n \leq b$ because b is an upper bound of S, this implies that $|x_n - b| < \frac{1}{n}$, from which it follows that the sequence (x_n) thus obtained converges to b as required.

Conversely, suppose that there is a sequence (x_n) of elements of S converging to b. For any $\epsilon > 0$, there exists $N \in \mathbb{N}$ so that

$$|x_N - b| < \epsilon.$$

Unwinding this inequality gives

$$-\epsilon < x_N - b < \epsilon,$$

and so in particular $b - \epsilon < x_N$. Hence b satisfies the condition in the previous proposition which is equivalent to b being the supremum of S, so $b = \sup S$ as claimed. □

Infimums

All of the above statements have analogs for infimums:

Definition. The *infimum* (or *greatest lower bound*) of a set $S \subseteq \mathbb{R}$ which is bounded below is a lower bound $a \in \mathbb{R}$ of S such that $\ell \leq a$ for any lower bound ℓ of S. We use the notation $a = \inf S$ for infimums.

Proposition. *The infimum of a set, if it exists, is unique.*

The following says that no number smaller than an infimum can be a lower bound of the given set:

Proposition. *A lower bound a of a set $S \subseteq \mathbb{R}$ is the infimum of S if and only if for any $\epsilon > 0$ there exists $s \in S$ such that $s < a + \epsilon$.*

Proposition. *Suppose that $S \subseteq \mathbb{R}$ is bounded below and that $a \in \mathbb{R}$ is a lower bound of S. Then $a = \inf S$ if and only if there exists a sequence (x_n) of elements in S converging to a.*

You should try to prove that above facts for practice. They are similar to the proofs for the corresponding facts about supremums with slight modifications.

Here is useful relationship between the above notions:

Proposition. *Suppose that $S \subseteq \mathbb{R}$ is nonempty and bounded above and let $-S := \{-x \mid x \in S\}$. Then $-S$ is bounded below and $\inf(-S) = -\sup S$.*

Proof. First we show that $-S$ is bounded below. Let u be an upper bound of S, so that

$$s \leq u \text{ for all } s \in S.$$

Then $-s \geq -u$ for all $s \in S$, so $-u$ is less than or equal to anything in $-S$. Hence $-u$ is a lower bound of $-S$, so $-S$ is bounded below.

Now, to show that $\inf(-S) = -\sup S$, we show that $-\sup S$ satisfies the defining properties of $\inf(-S)$. First, since $\sup S$ is an upper bound of S, what we just showed above tells us that $-\sup S$ is indeed a lower bound of $-S$. Let $\ell \in R$ be a lower bound of $-S$; then

$$\ell \leq -s \text{ for all } s \in S.$$

Multiplying through by -1 gives

$$s \leq -\ell \text{ for all } s \in S,$$

so $-\ell$ is an upper bound of S. Hence $\sup S \leq -\ell$ by definition of supremum, so $\ell \leq -\sup S$. Thus $-\sup S$ is greater than or equal to any lower bound of $-S$, so we conclude that $-\sup S = \inf(-S)$ as claimed. □

The moral of the above result is that changing signs exchanges supremums and infimums.

Examples

Claim. $\inf(0,\infty) = 0$

Proof. Since $(0,\infty)$ consists of all real numbers greater than 0, 0 is a lower bound of $(0,\infty)$. Let $\epsilon > 0$. Then $\frac{\epsilon}{2} \in (0,\infty)$ and

$$\frac{\epsilon}{2} < 0 + \epsilon.$$

Hence 0 satisifes the alternate characterization of infimums given in one of the propositions, so $0 = \inf(0,\infty)$ as claimed. □

Claim. $\sup\left\{1 - \frac{1}{n} \mid n \in \mathbb{N}\right\} = 1$

Thoughts. Let's call this set S. It should be clear that 1 is an upper bound of S, since 1 minus something positive is always smaller than 1. To show that 1 is the supremum of S, we will use the characterization of supremums given in one of the propositions.

So, given any $\epsilon > 0$, we want to find an element $s \in S$ such that $1 - \epsilon < s$. Again, this will say that for any $\epsilon > 0$, $1 - \epsilon$ is not an upper bound S, so nothing smaller than 1 is an upper bound of S and thus 1 must be the least upper bound.

Now, the s we want to find will be of the form

$$s = 1 - \frac{1}{N} \text{ for some } N \in \mathbb{N}$$

since these is precisely what elements of S looks like. So we want to find something of the form $1 - \frac{1}{N}$ so that

$$1 - \epsilon < 1 - \frac{1}{N}.$$

But this inequality is the same as $\epsilon > \frac{1}{N}$, and this finally tells us how to choose N. All of this is scratch work telling us how to find the element s we need, and now we can give the final proof.

Proof of Claim. First, since $\frac{1}{n} > 0$ for all $n \in \mathbb{N}$, $1 - \frac{1}{n} < 1$ for all $n \in \mathbb{N}$ so 1 is an upper bound of the given set. Now, let $\epsilon > 0$. Pick $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$; such a natural number exists by the Archimedean Property of $\mathbb{R}$. Then $-\frac{1}{N} > -\epsilon$ so

$$1 - \epsilon < 1 - \frac{1}{N}.$$

Since $1 - \frac{1}{N}$ is an element of the given set, this shows that no number smaller than 1 can be an upper bound of the given set—i.e. 1 satisfies the condition given in the alternate characterization of supremums in one of the propositions. Thus $\sup\left\{1 - \frac{1}{n} \mid n \in \mathbb{N}\right\} = 1$ as claimed. □

Claim. *Suppose that A and B are subsets of $\mathbb{R}$ which are nonempty and bounded below. Then* $\inf(A \cup B) = \min\{\inf A, \inf B\}$.

Proof. Since $\inf A$ is a lower bound of A and $\inf B$ is a lower bound of B, the smaller of these two is a lower bound of $A \cup B$. If $t \in \mathbb{R}$ is any lower bound of $A \cup B$, it is in particular a lower bound of A, so $t \leq \inf A$, and it is a lower bound of B, so $t \leq \inf B$. Hence $t \leq \min\{\inf A, \inf B\}$, so we conclude that $\inf(A \cup B) = \min\{\inf A, \inf B\}$ since the latter is a lower bound of $A \cup B$ which is greater than or equal to any other lower bound. □

To leave you with something to think about: if in the above situation $A \cap B \neq \emptyset$, so that $A \cap B$ has an infimum, what can we say about $\inf(A \cap B)$ in relation to $\inf a$ and $\inf B$, if anything?