Notes on the Point-Set Topology of $\mathbb{R}$
Northwestern University, Fall 2014

These notes give an introduction to the notions of "open" and "closed" subsets of $\mathbb{R}$, which belong to the subject known as "point-set topology". These are concepts we'll come back to next quarter in a much more general setting, but I think they're worth introducing now in the simpler case of $\mathbb{R}$. The point is that many concepts we'll look at can be reformulated in terms of open and closed sets, and doing so illustrates important properties which might not have otherwise been so apparent. If nothing else, these concepts give further practice dealing with inequalities and convergent sequences.

Closed Sets

Definition of Closed. A subset A of $\mathbb{R}$ is said to be *closed* if whenever (x_n) is a convergent sequence of elements of A, the limit $x = \lim x_n$ of this sequence also belongs to A.

Intuitively, this definition says that we can never "jump outside of A" by taking limits of convergent sequences in A, so A is "closed" under the process of "taking limits". In other words, if $x \in \mathbb{R}$ has the property that we can get arbitrarily close to it using elements of A, then x must itself be in A.

Example 1. For $a < b$, the closed interval $[a, b]$ is closed. Indeed, if $(x_n) \to x$ and $x_n \in [a, b]$ for all n, the "comparison theorem" in the book implies that $x \in [a, b]$ as well:

$$a \leq x_n \leq b \text{ for all } n \implies a \leq x \leq b.$$

As a contrast, for $a < b$ the open interval (a, b) is not closed in the above sense. For instance, the sequence

$$x_n = a + \frac{b-a}{2n}$$

consists of terms which belong to (a, b), but the limit a of this sequence is no longer in (a, b). Similarly, we can find a sequence of terms in (a, b) which converges to b, which is not in (a, b).

Example 2. We claim that the set of natural numbers $\mathbb{N} \subseteq \mathbb{R}$ is closed. Indeed, let (x_n) be a convergent sequence of natural numbers. The key point is that then, since distinct natural numbers are always at a distance ≥ 1 apart, this sequence must be *eventually constant*, meaning that past some index all terms are the same.

To see this, suppose that $x_n \to x$. Then there exists $N \in \mathbb{N}$ such that

$$|x - x_n| \leq \frac{1}{4} \text{ for } n \geq N.$$

If $n, m \geq N$, then

$$|x_n - x_m| = |(x_n - x) + (x - x_m)| \leq |x_n - x| + |x - x_m| < \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

But x_n and x_m are both natural numbers, and thus must be the same since distinct natural numbers are never within $\frac{1}{2}$ apart from each other. This shows that all terms x_n beyond x_N are the same, so (x_n) is eventually constant.

Hence if $(x_n) \to x$ and x_n is eventually constant, x must equal that common term which the x_n's eventually equal:

$$x = x_N = x_{N+1} = x_{N+2} = \cdots ,$$

which was itself a natural number to being with. Thus the limit x of (x_n) is a natural number if all terms x_n are natural numbers, so the set of natural numbers is closed.

Example 3. The set $\mathbb{Q}$ of rationals is not closed. Indeed, we've seen that for any irrational $x \in \mathbb{R}$ there exists a sequence (r_n) of rationals which converges to x. But then, (r_n) is a sequence in $\mathbb{Q}$ which converges to something not in $\mathbb{Q}$, so $\mathbb{Q}$ is not closed.

We also have various ways of constructing new closed sets from old ones. In particular, we have:

Theorem. Suppose that $\{A_i\}_{i \in I}$ is a collection of closed subsets of $\mathbb{R}$, indexed by some indexing set I. Then the intersection $\bigcap_{i \in I} A_i$ of them is closed as well.

Proof. Suppose that (x_n) is a convergent sequence with limit $x \in \mathbb{R}$ such that $x_n \in \bigcap_{i \in I} A_i$ for all n. Then in particular, for any $i \in I$, $x_n \in A_i$. But since each A_i is closed, this implies that $x \in A_i$ for all $i \in I$, so $x \in \bigcap_{i \in I} A_i$. Hence $\bigcap_{i \in I} A_i$ is closed as claimed. $\square$

The analogous claim for unions requires some care:

Theorem. Suppose that $A_1, \ldots, A_m$ are finitely many closed subsets of $\mathbb{R}$. Then $A_1 \cup \cdots \cup A_n$ is closed as well.

Proof. Suppose that (x_n) is a convergent sequence with limit $x \in \mathbb{R}$ such that $x_n \in A_1 \cup \cdots \cup A_m$ for all n. Then at least one of the sets A_i must contain a subsequence of the given sequence (x_n). (Note: at the point that I wrote these notes up we hadn't spoken about subsequences in class yet, but that was done shortly after.) To see this, we pick one of the A_i's which x_1 belongs to, then pick one which x_2 belongs to, then x_3, and so on: since (x_n) consists of infinitely many terms, we will have picked at least one of the A_i's an infinite number of times during this process since there are only finitely many of them, and the terms in the sequence (x_n) which belong to this specific A_i then give the subsequence we want. Denote this specific A_i by A_k.

Since the original sequence (x_n) converges to x, this subsequence does as well, so this gives a convergent sequence in A_k; since A_k is closed, the limit x of this sequence is in A_k, and hence $x \in A_1 \cup \cdots \cup A_m$, as required in order to say that $A_1 \cup \cdots \cup A_m$ is closed. $\square$

Example 4. The union of infinitely many closed subsets of $\mathbb{R}$ is not necessarily closed. For example, the intervals

$$A_n = \left[\frac{1}{n}, 1 - \frac{1}{n}\right]$$

are all closed, but their union

$$\bigcup_{n \in \mathbb{N}} A_n = (0, 1)$$

is not. The above proof breaks down in this example in the claim that there is some A_n which contains an infinite number of terms from the given sequence: here, since we have infinitely many A_n, this is no longer guaranteed.

Open Sets

Definition of Open. A subset U of $\mathbb{R}$ is said to be *open* if for any $x \in U$ there exists $r > 0$ such that $(x - r, x + r) \subseteq U$. In other words, for any $x \in U$ there exists an interval around it which is fully contained in U.

Intuitively, if $x \in U$, then points which are "close enough" to x (as determined by the value of $r > 0$) will also belong to U. So, in a sense, an open set fully "surrounds" all of its points.

Example 5. For $a < b$, the open interval (a, b) is open. Indeed, for $x \in (a, b)$ the "radius"

$$r = \min\{x - a, b - x\}$$

satisfies the requirement in the definition of open. First, since $a < x < b$, $x - a$ and $b - x$ are both positive so $r > 0$ since it is the minimum of two positive numbers. If $y \in (x - r, x + r)$, then

$$a = x - (x - a) \leq x - r < y < x + r \leq x + (b - x) = b,$$

so $y \in (a, b)$ as well. Hence $(x - r, x + r) \subseteq (a, b)$ as required.

Visually, r is the smaller of the distances from x to either endpoint of (a, b), and it makes sense visually at least that an interval of this radius around x is fully contained within (a, b).

As a contrast, for $a < b$ the closed interval $[a, b]$ is not open since for $a \in [a, b]$ there is no interval around it which is fully contained in $[a, b]$, and similarly for $b \in [a, b]$.

Moral. Open intervals are open and closed intervals are closed, but open intervals are not closed and closed intervals are not open. This is good, since otherwise our use of the words "open" and "closed" in these settings would get very confusing.

For closed sets it was arbitrary intersections of closed sets which were always closed, but for open sets it is arbitrary unions:

Theorem. Suppose that $\{U_i\}_{i \in I}$ is a collection of open sets, indexed by a set I. Then the union $\bigcup_{i \in I} U_i$ is open as well.

Proof. Let $x \in \bigcup_{i \in I} U_i$. Then $x \in U_k$ for some $k \in I$. Since U_k is open, there exists $r > 0$ such that $(x - r, x + r) \subseteq U_k$, and since $U_k \subseteq \bigcup_{i \in I} U_i$ we have $(x - r, x + r) \subseteq \bigcup_{i \in I} U_i$ as well. Thus $\bigcup_{i \in I} U_i$ is open. $\square$

Example 6. Consider the complent $\mathbb{R} \backslash \mathbb{N}$ of $\mathbb{N}$ in $\mathbb{R}$. We claim that this is open. Indeed, this can be written as a union of infinitely many open intervals:

$$\mathbb{R} \backslash \mathbb{N} = (-\infty, 1) \cup (1, 2) \cup (2, 3) \cup (3, 4) \cup \cdots,$$

and so is open by the previous theorem.

As opposed to the case for closed sets, it is now in taking intersections of open sets where care is needed:

Theorem. Suppose that $U_1, \ldots, U_n$ are finitely many open sets. Then the intersection $U_1 \cap \cdots \cap U_n$ is open.

Proof. Let $x \in U_1 \cap \cdots \cap U_n$, so that $x \in U_k$ for all $k = 1, \ldots, n$. Since each U_k is open, for each $k = 1, \ldots, n$ there exists $r_k > 0$ such that $(x - r_k, x + r_k) \subseteq U_k$. Set

$$r = \min\{r_1, \ldots, r_n\}.$$

Since each r_k is positive their minimum r is positive as well, and since $r \leq r_k$ for each k we have

$$(x - r, x + r) \subseteq (x - r_k, x + r_k) \subseteq U_k \text{ for all } k.$$

Thus $(x - r, x + r) \subseteq U_1 \cap \cdots \cap U_n$ so $U_1 \cap \cdots \cap U_n$ is open. $\square$

Example 7. The intersection of an infinite number of open sets is not necessarily open. For instance, for each $n \in \mathbb{N}$ consider the intervals

$$U_n = \left(-\frac{1}{n}, \frac{1}{n}\right),$$

all of which are open. Their intersection is

$$\bigcap_{n \in \mathbb{N}} U_n = \{0\},$$

which is not open. The proof of the previous theorem breaks down in the choice of r: first, with an infinite number of r_i there may not be such a minimum, but this is easily fixed by considering their infimum instead, and now the problem is that even if each $r_i > 0$ their infimum might be zero, so $r =$ this infimum is not a valid "radius" to be used in the definition of open.

Up to this point "closed" and "open" seem like two separate concepts, and while the notion of a closed set is related to the sequences we've been looking at so far in class, the notion of an open set seems to be a different beast. The connection between the two is the following fact, which in a sense says that open and closed are "opposite" concepts:

Theorem. A subset U of $\mathbb{R}$ is open if and only if its complement $\mathbb{R} \backslash U$ is closed.

Proof. We justify the equivalent statement that U is not open if and only if $\mathbb{R} \backslash U$ is not closed, which is obtained by taking contrapositives of both directions. First suppose that U is not open. Then there exists $x \in U$ such that no interval around x is fully contained within U. In particular, for any $n \in N$, $(x - \frac{1}{n}, x + \frac{1}{n}) \nsubseteq U$ so there exists $y_n \in \mathbb{R} \backslash U$ such that $y_n \in (x - \frac{1}{n}, x + \frac{1}{n})$. This gives a sequence (y_n) of elements of $\mathbb{R} \backslash U$ such that

$$|y_n - x| < \frac{1}{n} \text{ for all } n,$$

which implies that $y_n \to x$. Thus $\mathbb{R} \backslash U$ is not closed since there is a sequence of elements from it, namely (y_n), which converges to something not in $\mathbb{R} \backslash U$, namely $x \in U$.

Conversely, suppose that $\mathbb{R} \backslash U$ is not closed. Then there exists a sequence (y_n) of elements of $\mathbb{R} \backslash U$ which converges to some x not in $\mathbb{R} \backslash U$, meaning that x is in U. But then for this $x \in U$ there is no interval around it fully contained in U. Indeed, for any $r > 0$ we can pick $N \in \mathbb{N}$ such that $\frac{1}{N} < r$, and then the term y_N from the sequence (y_n) satisfies

$$|x - y_N| < \frac{1}{N} < r,$$

so $y_N \in (x - r, x + r)$. But $y_N \in \mathbb{R} \backslash U$, so $(x - r, x + r) \nsubseteq U$ and hence U is not open. $\square$

Thus "open" and "closed" are complementary notions, which is why open sets show up naturally when considering closed sets and sequences. In particular, here is another way to see that the intersection of infinitely many closed sets is still closed assuming that the union of infinitely many open sets is open. If $\{A_i\}_{i\in I}$ is a collection of arbitrarily many closed sets, then $\{A_i^c\}_{i\in I}$ is a collection of arbitrarily many open sets, where A_i^c denotes $\mathbb{R}\backslash A_i$. Thus

$$\bigcup_{i\in I} A_i^c$$

is open. By one of DeMorgan's Laws (which you likely saw in Math 300), we have

$$\bigcup_{i\in I} A_i^c = \left(\bigcap_{i\in I} A_i\right)^c,$$

so $\left(\bigcap_{i\in I} A_i\right)^c$ is open, which implies that $\bigcap_{i\in I} A_i$ is closed as claimed.

Compact Sets

FINISH

More to come!

These notes will be updated with more tidbits as we go on, and in particular we'll outline ways of reformulating some of what we do in terms of open and closed sets. For now, you might wonder what the term "topology" in the title of these means. The way in which open sets behave under arbitrary unions and finite intersections leads to the definition of what's a called a *topology* on a set: a topology on X is a collection of subsets of X such that:

- for any family $\{U_i\}$ of sets in that collection, their union $\bigcup U_i$ is also in that collection,
- for any finite number of sets $U_1, \dots, U_n$ in that collection, their intersection $U_1 \cap \cdots \cap U_n$ is also in that collection, and
- both X and $\emptyset$ are in that collection.

"Point-set topology" is the basic study of such collections, which provide a framework for a very general notion of "continuity". We won't say anything about this in this class, but if you're interested in learning more you might consider taking Math 344 - Introduction to Topology at some point.