MATH 110 - SOLUTIONS TO FINAL
LECTURE 1, SUMMER 2009

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1. (10 points) Give two equivalent definitions (or characterizations) of each of the following.
(a) A normal operator on an inner-product space V.
(b) A generalized eigenvector of an operator T.
(c) A positive operator on an inner-product space V.
(d) An isometry on an inner-product space V.

Solution. Here are some possible answers.
(a) An operator T such that $TT^* = T^*T$; or, an operator T such that $\|Tv\| = \|T^*v\|$ for all $v \in V$.
(b) A vector v such that for some eigenvalue λ of T there exists $k \geq 1$ such that $(T-\lambda I)^k v = 0$; or, a vector v such that for some eigenvalue λ of T, $(T - \lambda I)^{\dim V} v = 0$.
(c) A self-adjoint operator T such that $\langle Tv, v\rangle \geq 0$ for all $v \in V$; or, an operator T so that there exists $S \in \mathcal{L}(V)$ such that $T = S^*S$.
(d) An operator T such that $\|Tv\| = \|v\|$ for all $v \in V$; or, an operator T such that $T^*T = I$. □

2. (15 points) Give examples, with brief justification, of each of the following.
(a) An operator on $\mathbb{R}^2$ which is not self-adjoint with respect to the standard inner product.
(b) An isometry on $\mathbb{R}^4$ with no (real) eigenvalues.
(c) An operator on $\mathbb{C}^4$ whose characteristic polynomial equals the square of its minimal polynomial.

Solution. Here are some possible answers.
(a) With respect to the standard inner product, the adjoint of a matrix is just its tranpose, so any non-symmetric matrix would work — say

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

(b) First note that the rotation by 90° operator on $\mathbb{R}^2$:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

is an isometry and has no eigenvalues. So, the following analog of this on $\mathbb{R}^4$:

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

is an isometry on $\mathbb{R}^4$ with no eigenvalues.

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(c) The following matrix in Jordan form:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

works. The characteristic polynomial is $(z-1)^4$, and the minimal polynomial is $(z-1)^2$ since the largest Jordan block is of size 2. $\square$

3. (20 points) Suppose that P is an operator on a finite-dimensional inner-product space V such that $P^2 = P$. Prove that P is an orthogonal projection if and only if it is self-adjoint.

Proof. This was on the sixth homework. For completeness sake, here is a proof.

Suppose that P is an orthogonal projection, so that $V = \operatorname{range} P \oplus \operatorname{null} P$ and these two subspaces are the orthogonal complements of each other. We must show that

$$\langle Pv, u\rangle = \langle v, Pu\rangle \text{ for any } v, u \in V.$$

Let $v, u \in V$ and write them as

$$v = v_1 + w_1,\ u = v_2 + w_2 \text{ with } v_1, v_2 \in \operatorname{range} P \text{ and } w_1, w_2 \in \operatorname{null} P.$$

Then using the fact that $\operatorname{null} P$ and $\operatorname{range} P$ are orthogonal to each other, and the fact that $P|_{\operatorname{range} P} = I$ (since $P^2 = P$), we have

$$\langle Pv, u\rangle = \langle P(v_1 + w_1), v_2 + w_2\rangle = \langle v_1, v_2 + w_2\rangle = \langle v_1, v_2\rangle$$

and

$$\langle v, Pu\rangle = \langle v_1 + w_1, P(v_2 + w_2)\rangle = \langle v_1 + w_1, v_2\rangle = \langle v_1, v_2\rangle.$$

Thus $\langle Pv, u\rangle = \langle v, Pu\rangle$ for any $v, u \in V$, so $P = P^*$.

Conversely, suppose that P is self-adjoint. To show that P is an orthogonal projection, it is enough to show by problem 6.17 of an earlier homework that anything in $\operatorname{null} P$ is orthogonal to anything in $\operatorname{range} P$. To this end, let $u \in \operatorname{null} P$ and $w = Pw' \in \operatorname{range} P$. Then, since P is self-adjoint, we have

$$\langle u, w\rangle = \langle u, Pw'\rangle = \langle Pu, w'\rangle = \langle 0, w'\rangle = 0.$$

We conclude that P is an orthogonal projection.

Here is another proof of the backwards direction. Since $P = P^*$, we have

$$\operatorname{null} P = (\operatorname{range} P^*)^\perp = (\operatorname{range} P)^\perp,$$

so $V = \operatorname{range} P \oplus \operatorname{null} P$ is exactly a decomposition of the form subspace direct sum its orthogonal complement, so P is an orthogonal projection. $\square$

4. (20 points) Suppose that T is a self-adjoint operator on a inner-product space V such that there exists $v \in V$ with $\|v\| = 1$ such that $\langle Tv, v\rangle > 1$. Prove that there exists an eigenvalue of T which is larger than 1.

Proof. By the Spectral Theorem, there exists an orthonormal basis $(e_1, \ldots, e_n)$ of V consisting of eigenvectors of T — let $\lambda_1, \ldots, \lambda_n$ be the corresponding eigenvalues. Note that since T is self-adjoint, all of these are real. We then have

$$v = \langle v, e_1\rangle e_1 + \cdots + \langle v, e_n\rangle e_n$$

and

$$Tv = \lambda_1\langle v, e_1\rangle e_1 + \cdots + \lambda_n\langle v, e_n\rangle e_n.$$

Thus

$$\langle Tv, v\rangle = \lambda_1|\langle v, e_1\rangle|^2 + \cdots + \lambda_n|\langle v, e_n\rangle|^2 > 1.$$

Now, since $\|v\| = 1$,

$$\|v\|^2 = |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_n\rangle|^2 = 1.$$

Let μ denote the largest eigenvalue of T. Then

$$\begin{aligned}\mu &= \mu(|\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_n\rangle|^2)\\ &\geq \lambda_1|\langle v, e_1\rangle|^2 + \cdots + \lambda_n|\langle v, e_n\rangle|^2\\ &> 1,\end{aligned}$$

so $\mu > 1$ as required.

Here's another way of doing this last step. If all eigenvalues satisfied $\lambda_i \leq 1$, then

$$\begin{aligned}\langle Tv, v\rangle &= \lambda_1|\langle v, e_1\rangle|^2 + \cdots + \lambda_n|\langle v, e_n\rangle|^2\\ &\leq |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_n\rangle|^2 = 1,\end{aligned}$$

so $\langle Tv, v\rangle \leq 1$, a contradiction. Thus at least one eigenvalue of T must be larger than 1. □

5. (20 points) Let V be a finite-dimensional complex vector space. If you get stuck on part (a) below, assume it is true and use it in part (b).

(a) Prove that if N is a nilpotent operator on V, then $N + I$ has a square root.

(b) Prove that any invertible operator T on V has a square root.

Proof. (a) This is in the book. For completeness sake, here is a proof.

We guess that there is a square root of $N + I$ of the form

$$I + a_1N + a_2N^2 + \cdots + a_nN^n.$$

Note that such a guess comes from thinking about a series expansion of $\sqrt{1+x}$ and from the fact that N is nilpotent, so that high enough powers of it are zero. We claim that there do exist scalars $a_1, \ldots, a_n$ such that

$$N + I = (I + a_1N + a_2N^2 + \cdots + a_nN^n)^2.$$

Expanding the right hand side gives

$$I + 2a_1N + (a_1^2 + 2a_2)N^2 + \cdots + (\text{something})N^n$$

since higher powers of N are zero.

Now, we first see that a choice of $a_1 = 1/2$ will work. Next, the coefficient of N^2 would have to be zero, so

$$a_1^2 + 2a_2 = 0.$$

Since we've already solved for a_1, this allows us to find a_2. In general, if we've found $a_1, \ldots, a_{k-1}$, then setting the coefficient of N^k equal to zero will let us solve for a_k. Hence there do exist scalars with the required property, so $N + I$ has a square root.

(b) This is in the book. For completeness sake, here is a proof.

Let $\lambda_1, \ldots, \lambda_m$ be the eigenvalues of T and let $U_1, \ldots, U_m$ be the corresponding generalized eigenspaces. Then

$$V = U_1 \oplus \cdots \oplus U_m.$$

Note that since T is invertible, no eigenvalue is zero.

Now, for each i, setting $N_i = (T - \lambda_i I)|_{U_i}$, we have

$$T|_{U_i} = N_i + \lambda_i I = \lambda_i \left(\frac{N}{\lambda_i} + I\right).$$

The operator N/λ_i is nilpotent, so by part (a) there exists an operator S_i on U_i so that

$$S_i^2 = \frac{N_i}{\lambda_i} + I.$$

Define an operator S on V by setting

$$Sv = \sqrt{\lambda_1} S_1 u_1 + \cdots + \sqrt{\lambda_m} S_m u_m,$$

where $v = u_1 + \cdots + u_m$ is the unique way to express v according to the decomposition $V = U_1 \oplus \cdots \oplus U_m$. Then

$$S^2 v = \lambda_1 S_1^2 u_1 + \cdots + \lambda_m S_m^2 u_m = T|_{U_1} u_1 + \cdots + T|_{U_m} u_m = Tv,$$

so S is a square root of T. □

6. (15 points) Suppose that an operator T on a complex vector space has characteristic polynomial $z^3(z-2)^5(z+1)^2$ and minimal polynomial of the form

$$z^2(z-2)^k(z+1)^\ell \text{ where } k > 2 \text{ and } \ell \geq 1.$$

Suppose further that $\dim \operatorname{range}(T - 2I) = 7$ and that the eigenspace corresponding to -1 is 1-dimensional. Find, with justification, the Jordan blocks which make up the Jordan form of T. You do not have to write out the full Jordan form itself.

Proof. First, the eigenvalues of T are $0, 2, -1$ with multiplicities $3, 5, 2$. From the minimal polynomial, we see that the largest Jordan block for 0 is of size 2, so there must be one Jordan block for 0 of size 2 and one of size 1. Since the dimension of the eigenspace corresponding to -1 is 1, there is one Jordan block corresponding to -1 and it must hence be of size 2.

Now, the dimension of the eigenspace corresponding to 2 is

$$\dim (T - 2I) = \dim V - \dim \operatorname{range}(T - 2I) = 10 - 7 = 3,$$

so there are three Jordan blocks corresponding to 2. From the minimal polynomial, the size of the largest Jordan block corresponding to 2 is larger than 2. Thus since 2 has multiplicity 5, to get three blocks we need one of size 3, and two of size 1. Hence the Jordan blocks in the Jordan form of T are

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 \end{pmatrix}, \begin{pmatrix} 2 \end{pmatrix}.$$

□