Math 334: Final Exam Solutions
Northwestern University, Summer 2014

1. Give an example of each of the following. No justification is needed.

(a) An inner product on $\mathbb{C}^2$ with respect to which $\left(\begin{smallmatrix} 4 & 2-i \\ 2+i & 1 \end{smallmatrix}\right)$ is self-adjoint.

(b) A nonzero polynomial in $\mathcal{P}_2(\mathbb{R})$ which is orthogonal to x with respect to the inner product

$$\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)\,dx.$$

(c) A nonzero generalized eigenvector of $\left(\begin{smallmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$ which is not an ordinary eigenvector.

(d) An operator on $\mathbb{C}^4$ with characteristic polynomial $(z-2i)^4$ and minimal polynoial $(z-2i)^2$.

Solutions. (a) This matrix is Hermitian—meaning that it equals its conjugate transpose—and so is self adjoint with respect to the standard dot product $(z_1, w_1)\cdot(z_2, w_2) = z_1\overline{z_2} + w_1\overline{w_2}$.

(b) The constant polynomial 1 is orthogonal to x since $\int_{-1}^{1} 1\cdot x\,dx = \frac{1}{2}x^2\Big|_{-1}^{1} = 0$.

(c) The vector $v = \left(\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right)$ satisfies $(A-2I)^2 v = (A-2I)\left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 0 \\ 0 \end{smallmatrix}\right)$, so v is a generalized eigenvector corresponding to 2, but it is not an eigenvector since $(A-2I)v \neq 0$.

(d) The matrix $\left(\begin{smallmatrix} 2i & 1 & 0 & 0 \\ 0 & 2i & 0 & 0 \\ 0 & 0 & 2i & 1 \\ 0 & 0 & 0 & 2i \end{smallmatrix}\right)$ works since the largest Jordan block is of size 2. $\square$

2. Suppose V is an inner-product space and that T is an operator on V. Show that a subpsace U of V is T-invariant if and only if its orthogonal complement $U^\perp$ is T^*-invariant.

Proof. Suppose that U is T-invariant and let $v \in U^\perp$. Then for any $u \in U$, we have:

$$\langle u, T^*v\rangle = \langle Tu, v\rangle\,.$$

Since U is T-invariant, $Tu \in U$ so $\langle Tu, v\rangle = 0$ since $v \in U^\perp$. Thus $\langle u, T^*v\rangle = 0$ for all $u \in U$, which says that $T^*v \in U^\perp$. Hence $U^\perp$ is T^*-invariant as required.

Conversely, if $U^\perp$ is T^*-invariant then the claim just proved shows that $(U^\perp)^\perp$ is $(T^*)^*$-invariant. But $(U^\perp)^\perp = U$ and $(T^*)^* = T$, so U is T-invariant as claimed. $\square$

3. Suppose V is a complex inner-product space and that S is a self-adjoint operator on V with the property that $\|Sv\| = \|v\|$ for all $v \in V$. Show that if -1 is not an eigenvalue of S, then $Sv = v$ for all $v \in V$. Hint: First show that 1 is the only eigenvalue of S.

Proof. By the Spectral Theorem there exists an orthonormal basis $e_1, \ldots, e_n$ of V where each e_i is an eigenvector of S; denote the corresponding eigenvalues by $\lambda_1, \ldots, \lambda_n$. Since

$$\|e_i\| = \|Se_i\| = \|\lambda_i e_i\| = |\lambda_i|\,\|\mathbf{e}_i\|\,,$$

we get that $|lambda_i| = 1$, so each λ_i is either 1 or -1. We are given that -1 is not an eigenvalue of S, so every λ_i must equal 1 and hence $Se_i = e_i$ for all i.

Thus:

$$\begin{aligned} Sv &= S(\langle v, e_1\rangle e_1 + \cdots + \langle v, e_n\rangle e_n) \\ &= \langle v, e_1\rangle Se_1 + \cdots + \langle v, e_n\rangle Se_n \\ &= \langle v, e_1\rangle e_1 + \cdots + \langle v, e_n\rangle e_n \\ &= v \end{aligned}$$

for any $v \in V$ as claimed. $\square$

4. Suppose that V is a complex vector space. If you get stuck in (a) below, assume it is true and use it in (b).

(a) Show that for any nilpotent operator N and complex scalar $a \neq -1/4$, there exists an operator S such that $S^2 + S = aI + N$. Hint: S will be of the form $S = a_0 I + a_1 N + \cdots + a_n N^n$.

(b) Show that for any $T \in \mathcal{L}(V)$ for which $-1/4$ is not an eigenvalue, there exists $K \in \mathcal{L}(V)$ such that $K^2 + K = T$.

Proof. (a) Let n be the smallest positive integer such that $N^{n+1} = 0$, which exists since N is nilpotent. We show that there are scalars $a_0, \ldots, a_n$ such that $S = a_0 I + a_1 N + \cdots + a_n N^n$ satisfies $S^2 + S = aI + N$. Indeed, for this we would need to have:

$$(a_0 I + a_1 N + \cdots + a_n N^n)(a_0 I + a_1 N + \cdots + a_n N^n) + (a_0 I + a_1 N + \cdots + a_n N^n) = aI + N.$$

Expanding the left side and regrouping terms gives:

$$(a_0^2 + a_0)I + (2a_0 a_1 + a_1)N + (2a_0 a_2 + a_1^2 + a_2)N^2 + \cdots + (\text{something involving } a_0, \ldots, a_n)N^n = aI + N.$$

Comparing coefficients of I on both sides we see that the scalar a_0 we want must first satisfy

$$a_0^2 + a_0 = a, \text{ so we can take } a_0 = \frac{-1 + \sqrt{1 + 4a}}{2}.$$

Note that since $a \neq -1/4$, $a_0 \neq -1/2$ since the term under the square root is nonzero.

Then comparing coefficients of N on both sides we see that a_0, a_1 must satisfy

$$2a_0 a_1 + a_1 = 1, \text{ so } a_1 = \frac{1}{2a_0 + 1}$$

where a_0 is the value we found above. Note that this fraction makes sense: the denominator is nonzero since $a_0 \neq -1/2$, which is why the assumption that $a \neq -1/4$ is important.

Comparing coefficients of N^2 gives the requirement that

$$2a_0 a_2 + a_1^2 + a_2 = 0,$$

which we can then use to solve for a_2. Continuing on in this manner allows to determine $a_3, \ldots, a_n$, so we conclude that there are scalars $a_0, \ldots, a_n$ such that $S = a_0 I + a_1 N + \cdots + a_n N^n$ satisfies the requirement that $S^2 + S = aI + N$.

(b) Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of T and let $U_1, \ldots, U_m$ be the corresponding generalized eigenspaces. Then we have

$$V = U_1 \oplus \cdots \oplus U_m.$$

Now, for each i,

$$T|_{U_i} = \lambda_i I + (T - \lambda_i I)|_{U_i}.$$

The second piece on the right is nilpotent and $\lambda_i \neq -1/4$, so part (a) gives us an operator R_i on U_i such that

$$R_i^2 + R_i = T|_{U_i}.$$

Define an operator S on V by

$$Sv = R_1 v_1 + \cdots + R_m v_m$$

where $v = v_1 + \cdots + v_m$ is the expression for v according to the decomposition of V into the generalized eigenspaces. For any v, we then have

$$\begin{aligned}(S^2 + S)v &= S^2 v + Sv \\ &= (R_1^2 v_1 + \cdots + R_m^2 v_m) + (R_1 v_1 + \cdots + R_m v_m) \\ &= (R_1^2 + R_1)v_1 + \cdots + (R_m^2 + R_m)v_m \\ &= T|_{U_1} v_1 + \cdots + T|_{U_m} v_m \\ &= Tv.\end{aligned}$$

Hence $S^2 + S = T$ as required. □

5. Suppose that an operator T on a complex vector space V has characteristic polynomial $(z+2)^3(z-4)^4(z+3)^5$ and minimal polynomial of the form

$$(z+2)^2(z-4)^k(z+3)^\ell, \text{ where } k \geq 2 \text{ and } \ell \geq 1.$$

Moreover, suppose that $\dim \operatorname{range}(T - 4I) \leq 10$ and $\dim \operatorname{null}(T + 3I) = 3$. Determine the possible Jordan forms which T could have.

Solution. From the characteristic polynomial, the eigenvalues of T are $-2, 4, -3$ with multiplicities $3, 4, 5$ respectively. Note that V is thus $3 + 4 + 5 = 12$ dimensional. Now, -2 appears 3 times on the diagonal of the Jordan form of T and the largest Jordan block corresponding to 2 is of size 2 since the $(z + 2)$ term in the minimal polynomial is of degree 2. Hence there must be two Jordan blocks corresponding to 2: one of size 2 and one of size 1:

$$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \text{ and } \begin{pmatrix} 2 \end{pmatrix}.$$

Next, since

$$12 = \dim V = \dim \operatorname{null}(T - 4I) + \dim \operatorname{range}(T - 4I)$$

by rank-nullity and $\dim \operatorname{range}(T - 4I) \leq 10$, we get that

$$\dim \operatorname{null}(T - 4I) = 12 - \dim \operatorname{range}(T - 4I) \geq 12 - 10 = 2.$$

Thus the eigenspace corresponding to 4 is at least 2-dimensional, so there are at least two Jordan blocks corresponding to 4. Since 4 has multiplicity 4 and the largest block has size at least 2 because of the degree of the $(z - 4)$ term in the minimal polynomial, we see that the possibilities for Jordan blocks corresponding to 4 are: one block of size 3 and one of size 1, or two blocks of size 2, or one block of size 2 and two of size 1:

$$\begin{pmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix} \text{ and } \begin{pmatrix} 4 \end{pmatrix}, \text{ or } \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix} \text{ and } \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}, \text{ or } \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix} \text{ and } \begin{pmatrix} 4 \end{pmatrix} \text{ and } \begin{pmatrix} 4 \end{pmatrix}.$$

The possibilities of having a single block of size 4 or four blocks of size 1 are ruled out by the previous considerations.

Finally, since $\dim \operatorname{null}(T + 3I) = 3$, the eigenspace corresponding to -3 is 3 dimensional so there are three Jordan blocks corresponding to -3. Since 3 has multiplicity 5, the possibilities are thus: two blocks of size 2 and one of size 1, or one block of size 3 and two of size 1:

$$\begin{pmatrix} -3 & 1 \\ 0 & -3 \end{pmatrix} \text{ and } \begin{pmatrix} -3 & 1 \\ 0 & -3 \end{pmatrix} \text{ and } \begin{pmatrix} -3 \end{pmatrix}, \text{ or } \begin{pmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{pmatrix} \text{ and } \begin{pmatrix} -3 \end{pmatrix} \text{ and } \begin{pmatrix} -3 \end{pmatrix}.$$

The possible Jordan forms of T are then obtained by writing out 12×12 matrices with blocks of the above possibilities. We won't list them all here, but there are 6 possible Jordan forms in total: 1 possibility for the Jordan blocks corresponding to 2, times 3 possibilities for the blocks corresponding to 4, times 2 for the blocks corresponding to -3. $\square$

6. Find the Jordan form of the matrix

$$A = \begin{pmatrix} -1 & -1 & 4 & -5 & 9 \\ 0 & -2 & 1 & -4 & 5 \\ 0 & -1 & 0 & -7 & 15 \\ 0 & 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & -1 & 3 \end{pmatrix},$$

whose characteristic polynomial is $(z-1)^2(z+1)^3$.

Solution. From the characteristic polynomial, the eigenvalues of A are 1 and -1 with multiplicities 2 and 3 respectively. Since $A - I$ reduces to:

$$A - I = \begin{pmatrix} -2 & -1 & 4 & -5 & 9 \\ 0 & -3 & 1 & -4 & 5 \\ 0 & -1 & -1 & -7 & 15 \\ 0 & 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix} \to \begin{pmatrix} -2 & -1 & 4 & -5 & 9 \\ 0 & -3 & 1 & -4 & 5 \\ 0 & 0 & 4 & 17 & -40 \\ 0 & 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

the eigenspace $\text{null}(A - I)$ corresponding to 1 is 1-dimensional, so there is only one Jordan block corresponding to 1 and it must be of size 2.

Now, $A + I$ reduces to:

$$A + I = \begin{pmatrix} 0 & -1 & 4 & -5 & 9 \\ 0 & -1 & 1 & -4 & 5 \\ 0 & -1 & 1 & -7 & 15 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & -1 & 4 \end{pmatrix} \to \begin{pmatrix} 0 & -1 & 4 & -5 & 9 \\ 0 & 0 & 3 & -1 & 4 \\ 0 & 0 & 3 & 2 & -6 \\ 0 & 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix} \to \begin{pmatrix} 0 & -1 & 4 & -5 & 9 \\ 0 & 0 & 3 & -1 & 4 \\ 0 & 0 & 0 & -3 & 10 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus $\text{null}(A + I)$ is 1-dimensional, so there is only one Jordan block corresponding to -1 and it must be of size 3. The Jordan form of A is therefore:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

$\square$