Math 334: Final Exam

Northwestern University, Summer 2014

August 14, 2014

Name: ____________________

1. (15 points) Give an example of each of the following. No justification is needed.

(a) An inner product on $\mathbb{C}^2$ with respect to which $\left(\begin{smallmatrix} 4 & 2-i \\ 2+i & 1 \end{smallmatrix}\right)$ is self-adjoint.

(b) A nonzero polynomial in $\mathcal{P}_2(\mathbb{R})$ which is orthogonal to x with respect to the inner product

$$\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)\, dx.$$

(c) A nonzero generalized eigenvector of $\left(\begin{smallmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$ which is not an ordinary eigenvector.

(d) An operator on $\mathbb{C}^4$ with characteristic polynomial $(z-2i)^4$ and minimal polynoial $(z-2i)^2$.

2. (15 points) Suppose V is an inner-product space and that T is an operator on V. Show that a subpsace U of V is T-invariant if and only if its orthogonal complement $U^{\perp}$ is T^{*}-invariant.

3. (20 points) Suppose V is a complex inner-product space and that S is a self-adjoint operator on V with the property that $\|Sv\| = \|v\|$ for all $v \in V$. Show that if -1 is not an eigenvalue of S, then $Sv = v$ for all $v \in V$. Hint: First show that 1 is the only eigenvalue of S.

4. (20 points) Suppose that V is a complex vector space. If you get stuck in (a) below, assume it is true and use it in (b).

(a) Show that for any nilpotent operator N and complex scalar $a \neq -1/4$, there exists an operator S such that $S^2 + S = aI + N$. Hint: S will be of the form $S = a_0 I + a_1 N + \cdots + a_n N^n$.

(b) Show that for any $T \in \mathcal{L}(V)$ for which $-1/4$ is not an eigenvalue, there exists $K \in \mathcal{L}(V)$ such that $K^2 + K = T$.

5. (15 points) Suppose that an operator T on a complex vector space V has characteristic polynomial $(z+2)^3(z-4)^4(z+3)^5$ and minimal polynomial of the form

$$(z+2)^2(z-4)^k(z+3)^\ell, \text{ where } k \geq 2 \text{ and } \ell \geq 1.$$

Moreover, suppose that $\dim \operatorname{range}(T-4I) \leq 10$ and $\dim \operatorname{null}(T+3I) = 3$. Determine the possible Jordan forms which T could have.

6. (15 points) Find the Jordan form of the matrix

$$A = \begin{pmatrix} -1 & -1 & 4 & -5 & 9 \\ 0 & -2 & 1 & -4 & 5 \\ 0 & -1 & 0 & -7 & 15 \\ 0 & 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & -1 & 3 \end{pmatrix},$$

whose characteristic polynomial is $(z-1)^2(z+1)^3$.