MATH 110 - MIDTERM

LECTURE 1, SUMMER 2009

July 16, 2009

Name: ____

1. (10 points) Give two equivalent definitions (or characterizations) of each of the following:

(a) A list $(v_1, \ldots, v_n)$ is linearly dependent

(b) An operator $T \in \mathcal{L}(V)$ can be represented by an upper-triangular matrix

(c) A scalar $\lambda \in \mathbb{F}$ is an eigenvalue of an operator $T \in \mathcal{L}(V)$

(d) A vector space V is the direct sum of subspaces U and W

2. (10 points) Determine whether the following statements are true or false. You do not need to provide justifications for your answers. There will be 1 point subtracted for each incorrect response, so do not guess just for the sake of guessing.

(a) Any operator on a real vector space has a 1-dimensional invariant subspace.

(b) If $V = U \oplus W$ and $p(z) = z^2$, then $p(P_{U,W}) = P_{U,W}$.

(c) If $v_1, \ldots, v_n \in V$, then $\operatorname{span}(v_1, \ldots, v_n)$ is n-dimensional.

(d) Any operator on a 10-dimensional complex vector space has a 4-dimensional invariant subspace.

3. (15 points) Let n be a positive integer and let U be the subset of $\mathcal{P}_n(\mathbb{R})$ defined by

$$U = \{p(x) \in \mathcal{P}_n(\mathbb{R}) \mid p''(1) - p'(2) + p(3) = 0\}.$$

(a) Prove that U is a subspace of $\mathcal{P}_n(\mathbb{R})$.

(b) Find a linear map $T : \mathcal{P}_n(\mathbb{R}) \to \mathbb{R}$ such that $\operatorname{null} T = U$, and use this to compute $\dim U$. (You do not need to prove that the map you write down is in fact linear)

4. (15 points) Let V and W be vector spaces and let $T : V \to W$ be a map such that $T(au + v) = aTu + Tv$ for all $a \in \mathbb{F}$ and $u, v \in V$. (Each part below depends on the previous part; if you get stuck on one, assume it is true and move on)

(a) Prove that T preserves addition.

(b) Prove that $T(0_V) = 0_W$, where 0_V and 0_W are the zero vectors of V and W respectively.

(c) Prove that T is linear.

5. (20 points) Let V be a vector space with subspaces U and W such that $V = U \oplus W$. Suppose that T is an operator on V such that U and W are both T-invariant. Prove that T is invertible if and only if $T|_U$ and $T|_W$ are invertible. (Note that V is not assumed to be finite dimensional. If you're having trouble, at least try the case where V is finite dimensional)

6. (15 points) Give examples, with brief justification, of each of the following.

(a) A sum $\mathbb{R}^3 = U + V$, where neither U nor V is all of $\mathbb{R}^3$, which is not a direct sum.

(b) An operator T on a 4-dimensional space which has no 3-dimensional invariant subspace.

(c) A 2-dimensional subspace of $C(\mathbb{R})$ which contains no nonzero polynomials and is invariant under the differentiation operator.

7. (15 points) Let S and T be diagonalizable operators on a finite dimensional vector space V and suppose that $ST = TS$. Prove that there exists a basis of V consisting of eigenvectors common to both S and T. (We say that S and T are *simultaneously diagonalizable* if this is the case)

You may use the fact that if an operator D on V is diagonalizable and U is a nonzero D-invariant subspace of V, then $D|_U$ is a diagonalizable operator on U.

8. (0 points) Linear algebra: great subject, or the greatest subject?