MATH 110 - PRACTICE MIDTERM

LECTURE 1, SUMMER 2009

July 10, 2009

The midterm will of course consist of fewer problems than the number given here (maybe 7 or 8), but these all should give a good indication of what to expect. The starred problems are more difficult than the rest, and the midterm will only contain possibly one such problem. You should definitely make sure you understand the rest of the problems before worrying about the starred ones. For the record, I would consider problems 2, 4, 7, 8, 9, and 12 to be on the "easy"-ish side (meaning, these mostly follow from just playing around with definitions), and all the rest (except for the starred problems) to be in the middle of the difficulty scale (meaning they require a bit more thought than just working with definitions). Happy studying!

Unless otherwise stated, all vector spaces can be assumed to be nonzero and finite-dimensional over a field $\mathbb{F}$.

1. Given vector spaces V and W, $V \times W$ is the vector space given by

$$V \times W = \{(v, w) \mid v \in V \text{ and } w \in W\},$$

with addition and scalar multiplication defined componentwise; i.e.

$$(v, w) + (v', w') := (v + v', w + w') \text{ and } a(v, w) := (av, aw).$$

Prove that a map $T : V \to W$ is linear if and only if its graph, defined by

$$\text{graph } T := \{(v, Tv) \in V \times W \mid v \in V\},$$

is a subspace of $V \times W$.

2. Let $T \in \mathcal{L}(V)$ and suppose that U and W are T-invariant subspaces of V. Prove that $U \cap W$ and $U + W$ are also T-invariant.

3. Axler, 2.7

4. Prove that $C(\mathbb{R})$ is infinite-dimensional. (Hint: Use the previous problem)

5. Axler, 3.8

6*. Axler, 5.13 (Hint: By problem 5.12, it is enough to show that every vector in V is an eigenvector of T. Say $\dim V = n$. Let $v \in V$ be nonzero and show that $\text{span}(v)$ can be written as the intersection of some number of $(n-1)$-dimensional subspaces of V, then use the second problem)

7. Axler, 2.3

8. Axler, 2.10

9. Let $T : V \to W$ be a linear map and let $v_1, \ldots, v_k \in V$. Prove that if $(Tv_1, \ldots, Tv_k)$ is linearly independent in W, then $(v_1, \ldots, v_k)$ is linearly independent in V.

10. Axler, 3.12

11. Let $S : \mathcal{L}(V) \to \mathcal{L}(V)$ be the map defined by

$$S(T) = \begin{cases} T^{-1} & \text{if } T \text{ is invertible} \\ 0 & \text{if } T \text{ is not invertible.} \end{cases}$$

Prove or disprove that S is linear. (Hint: Try to figure out a simple case first, say $V = \mathbb{R}^2$)

12. Axler, 5.10

13. Recall that $T \in \mathcal{L}(V)$ is *nilpotent* if $T^k = 0$ for some positive integer k. Suppose that V is a complex vector space. Prove that $T \in \mathcal{L}(V)$ is nilpotent if and only if 0 is the only eigenvalue of T. (Hint for the backwards direction: There exists a basis of V with respect to which $\mathcal{M}(T)$ is upper- triangular. What can you say about the diagonal entries of $\mathcal{M}(T)$?)

14. Suppose that U, W, and W' are subspaces of V such that

$$U \oplus W = U \oplus W'.$$

For each $w \in W$, since w is then also in $U \oplus W$, this implies that we can write

$$w = u + w'$$

in a unique way with $u \in U$ and $w' \in W'$. Define a linear map $T : W \to W'$ by setting

$$Tw = w' \text{ as constructed above.}$$

Prove that T is an isomorphism. (Hint: First show that $\dim W = \dim W'$; then it suffices to show that T is injective)

15*. Recall that an operator $T \in \mathcal{L}(V)$ is *diagonalizable* if there exists a basis of V consisting of eigenvectors of T. Suppose that $T \in \mathcal{L}(V)$ is diagonalizable and that U is a nonzero, T-invariant subspace of V. Prove that $T|_U \in \mathcal{L}(U)$ is diagonalizable. [Hint: First show that if $v_1, \ldots, v_k$ are eigenvectors which correspond to distinct eigenvalues of T and $v_1 + \cdots + v_k \in U$, then $v_i \in U$ for each i. (Hint for the hint: first do the case $k = 2$)]

16. Let $p(z) \in P(\mathbb{F})$ be a polynomial and let $T \in \mathcal{L}(V)$. Prove that $\operatorname{null} T$ and $\operatorname{range} T$ are invariant under $p(T)$.