Math 334: Midterm Solutions
Northwestern University, Summer 2014

1. Give an example of each of the following. No justification is needed.

(a) A sum $\mathbb{R}^5 = U + W$ which is not a direct sum.
(b) A nonzero operator T and basis (v_1, v_2) of $\mathbb{R}^2$ such that (Tv_1, Tv_2) is not a basis of $\mathbb{R}^2$.
(c) A direct sum $\mathbb{R}^3 = U \oplus W$ and operator T such that U is T-invariant but W is not.
(d) An operator on $\mathbb{R}^4$ with no real eigenvalues.

Solution. Here are some possible answers. Even though no justification was required, I'll give some anyway.

(a) Take $U = \{(x_1, x_2, x_3, 0, 0) \mid x_i \in \mathbb{R}\}$ and $W = \{(0, 0, y_3, y_4, y_5) \mid y_i \in \mathbb{R}\}$. Then $\mathbb{R}^5 = U + W$ since

$$(a, b, c, d, e) = (a, b, c, 0, 0) + (0, 0, 0, d, e)$$

expresses an arbitrary element of $\mathbb{R}^5$ as a sum of an element of U and an element of W, but the sum is not a direct sum since $(0, 0, 1, 0, 0)$ is in both U and W, so $U \cap W \neq \{0\}$.

(b) Take $v_1 = (1, 0)$ and $v_2 = (0, 1)$ to be the standard basis and $T(x, y) = (x, 0)$. Then $Tv_1 = (1, 0)$ and $Tv_2 = (0, 0)$ are linearly dependent, so they do not form a basis of $\mathbb{R}^2$.

(c) Take $U = \{(a, b, 0) \mid a, b \in \mathbb{R}\}$ and $W = \{(0, 0, c) \mid c \in \mathbb{R}\}$. For the operator $T(x, y, z) = (x, y + z, 0)$, we have $T(a, b, 0) = (a, b, 0) \in U$, so U is T-invariant, but $T(0, 0, 1) = (0, 1, 0) \notin W$ even though $(0, 0, 1) \in W$, so W is not T-invariant.

(d) Take the operator defined by $T(\vec{x}) = A\vec{x}$ where

$$A = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

This behaves as rotation by 90° on the x_1x_2-plane and as rotation by 90° on the x_3x_4-plane, and these rotations have no real eigenvalues. (A has eigenvalues $\pm i$, each with multiplicity 2, when considered as an operator on $\mathbb{C}^4$.) $\square$

2. Recall that the trace of a square matrix is the sum of its diagonal entries:

$$\operatorname{tr} \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = a_{11} + a_{22} + \cdots + a_{nn}.$$

Let U be the set of all 3×3 matrices of trace zero:

$$U = \{A \in M_3(\mathbb{R}) \mid \operatorname{tr} A = 0\}.$$

(a) Show that U is a subspace of $M_3(\mathbb{R})$.
(b) Find a basis of U. Justify your answer.

Solution. (a) I only asked about $M_3(\mathbb{R})$ instead of $M_n(\mathbb{R})$ in general to keep the notation simpler, but I'll give a solution which works for any n.

First, the trace of the zero matrix is $0+0+\cdots+0=0$, so the zero matrix is in U. If $A, B \in U$, then $\operatorname{tr} A = \operatorname{tr} B = 0$, so (where we denote the entries of A by a_{ij} and those of B by b_{ij}):

$$\begin{aligned}
\operatorname{tr}(A+B) &= \operatorname{tr}\begin{pmatrix} a_{11}b_{11} & \cdots & a_{1n}+b_{1n} \\ & \ddots & \\ a_{n1}+b_{n1} & \cdots & a_{nn}+b_{nn} \end{pmatrix} \\
&= (a_{11}+b_{11}) + \cdots + (a_{nn}+b_{nn}) \\
&= (a_{11}+\cdots+a_{nn}) + (b_{11}+\cdots+b_{nn}) \\
&= \operatorname{tr} A + \operatorname{tr} B \\
&= 0,
\end{aligned}$$

so $A+B \in U$ and hence U is closed under addition. If in addition $c \in \mathbb{R}$, then

$$\begin{aligned}
\operatorname{tr}(cA) &= \begin{pmatrix} ca_{11} & \cdots & ca_{1n} \\ & \ddots & \\ ca_{n1} & \cdots & ca_{nn} \end{pmatrix} \\
&= (ca_{11}+\cdots+ca_{nn}) \\
&= c(a_{11}+\cdots+a_{nn}) \\
&= c(0) \\
&= 0,
\end{aligned}$$

so $cA \in U$ and hence U is closed under scalar multiplication. Thus U is a subspace of $M_3(\mathbb{R})$.

(b) Note: The problem which was actually on the midterm asked for a basis of $M_3(\mathbb{R})$, but I meant to ask for a basis of U. I'll give a solution to the problem I meant to ask instead.

For a 3×3 matrix

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

to be in U means that $a+e+i=0$, so $i=-a-e$. Thus a matrix in U looks like

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & -a-e \end{pmatrix},$$

which can be written as the following linear combination:

$$a\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$e\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + f\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + g\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + e\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

These eight matrices thus span U. If such a linear combination equals the zero matrix we get

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & -a-e \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so we must have all entries be 0, which means

$$a = b = c = d = e = f = g = h = 0.$$

Thus the eight matrices above are linearly independent and hence form a basis of U. $\square$

3. Suppose that U is a subspace of $\mathcal{P}_n(\mathbb{R})$ with the property that for any $p(x)$ in U, its derivative $p'(x)$ is also in U. Show that if $x^n \in U$, then $U = \mathcal{P}_n(\mathbb{R})$.

Proof. By the given property which U is supposed to satisfy, since if $x^n \in U$ then $nx^{n-1} \in U$. Since U is closed under scalar multiplication, we then have

$$\frac{1}{n}(nx^{n-1}) = x^{n-1} \in U.$$

By the same reasoning, since $x^{n-1} \in U$ we have $(n-1)x^{n-2} \in U$ so

$$\frac{1}{n-1}[(n-1)x^{n-2}] = x^{n-2} inU.$$

And so on, repeating this argument repeatedly shows that

$$1, x, x^2, \ldots, x^n \text{ are all in } U.$$

But these vectors form a basis for $\mathcal{P}_n(\mathbb{R})$, so we must have $U = \mathcal{P}_n(\mathbb{R})$ as claimed. To be clearer, these vectors are linearly independent so $\dim U \geq n+1$, but $\dim U \leq \mathcal{P}_n(\mathbb{R}) = n+1$, so $\dim U = \mathcal{P}_n(\mathbb{R}) = n+1$ and thus $U = \mathcal{P}_n(\mathbb{R})$. $\square$

4. Let $T \in \mathcal{L}(V)$ and suppose that $v \in V$ is a vector such that

$$T^3 v = 0 \text{ but } T^2 v \neq 0.$$

Show that $(v, Tv, T^2 v)$ is linearly independent.

Proof. Suppose that

$$av + bTv + cT^2 v = 0$$

for some $a, b, c \in \mathbb{F}$. Applying T to the left side gives

$$T(av + bTv + cT^2 v) = aTv + bT^2 v + cT^3 v = aTv + bT^2 v,$$

and applying T to the right side gives $T(0) = 0$, so we must have

$$aTv + bT^2 v = 0.$$

Applying T to both sides again and using $T^3 v = 0$ gives

$$aT^2 v = 0.$$

Since $T^2 v \neq 0$, this means that $a = 0$. Then $aTv + bT^2 v = 0$ becomes

$$bT^2 v = 0,$$

so $b = 0$ since $T^2 v \neq 0$. Finally, the equation we started out with becomes

$$av = 0,$$

so $a = 0$ since $v \neq 0$ because otherwise $T^2 v$ would be zero. Hence $a = b = c = 0$, so $(v, Tv, T^2 v)$ is linearly independent.

Notice that a similar reasoning works if we place the exponents 2 and 3 with n and $n+1$: if

$$T^{n+1} v = 0 \text{ but } T^n v \neq 0,$$

then $(v, Tv, T^2 v, \ldots, T^n v)$ is linearly independent. This is related to the notion of a *Jordan chain*, which we will see when talking about Jordan forms. □

5. Suppose that V is an n-dimensional *complex* vector space and that $T \in \mathcal{L}(V)$ only has 0 as an eigenvalue. Show that $T^n v = 0$ for all $v \in V$.

Proof. Since V is a finite-dimensional complex vector space, there exists a basis relative to which the matrix of T is upper-triangular:

$$\mathcal{M}(T) = \begin{pmatrix} 0 & * & \cdots & * \\ & 0 & \ddots & \vdots \\ & & \ddots & * \\ & & & 0 \end{pmatrix}.$$

The blank spaces denote 0's, and the diagonal terms are all zero since the only eigenvalue of T is 0. Then we compute that $\mathcal{M}(T)^2$ has the form:

$$\mathcal{M}(T)^2 = \begin{pmatrix} 0 & * & \cdots & * \\ & 0 & \ddots & \vdots \\ & & \ddots & * \\ & & & 0 \end{pmatrix} \begin{pmatrix} 0 & * & \cdots & * \\ & 0 & \ddots & \vdots \\ & & \ddots & * \\ & & & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & * & \cdots & * \\ & 0 & 0 & \ddots & \vdots \\ & & \ddots & \ddots & * \\ & & & \ddots & 0 \\ & & & & 0 \end{pmatrix},$$

where we have an additional diagonal containing all zeroes. And so on, in general taking an additional power of $\mathcal{M}(T)$ will produce one more diagonal consisting of all zeroes; in particular $\mathcal{M}(T)^{n-1}$ consists of all zeroes except for possibly a single nonzero term in the upper-right entry. Then $\mathcal{M}(T)^n$ consists of all zeroes, so

$$\mathcal{M}(T^n) = \mathcal{M}(T)^n = 0.$$

Hence T^n is the zero operator, so $T^n v = 0$ for all $v \in V$. □

6. Suppose that U and W are subspaces of V such that $V = U \oplus W$. Suppose further that U and W are both invariant under an operator $T \in \mathcal{L}(V)$. Show that if the restrictions $T|_U$ and $T|_W$ are both injective, then T is injective on all of V.

Proof. Suppose that $Tv = 0$. We want to show that $v = 0$. Since $V = U \oplus W$ we can write v as

$$v = u + w \text{ for some } u \in U \text{ and } w \in W.$$

Then $0 = Tv = Tu + Tw$. Since U and W are each T-invariant, $Tu \in U$ and $Tv \in W$, so

$$0 = Tu + Tw$$

expresses 0 as a sum of an element of U with an element of W. Since $U \oplus W$ is a direct sum, we must thus have $Tu = 0$ and $Tw = 0$, so $u = 0$ and $w = 0$ since the restrictions $T|_U$ and $T|_W$ are injective. Thus $v = 0 + 0 = 0$ as desired, so since $\operatorname{null} T = \{0\}$, T is injective on all of V. $\square$