1. Give an example of each of the following. No justification is needed.
   (a) A sum \( \mathbb{R}^5 = U + W \) which is not a direct sum.
   (b) A nonzero operator \( T \) and basis \((v_1, v_2)\) of \( \mathbb{R}^2 \) such that \((Tv_1, Tv_2)\) is not a basis of \( \mathbb{R}^2 \).
   (c) A direct sum \( \mathbb{R}^3 = U \oplus W \) and operator \( T \) such that \( U \) is \( T \)-invariant but \( W \) is not.
   (d) An operator on \( \mathbb{R}^4 \) with no real eigenvalues.

Solution. Here are some possible answers. Even though no justification was required, I’ll give some anyway.
   (a) Take \( U = \{(x_1, x_2, x_3, 0, 0) \mid x_i \in \mathbb{R}\} \) and \( W = \{(0, 0, y_3, y_4, y_5) \mid y_i \in \mathbb{R}\} \). Then \( \mathbb{R}^5 = U + W \) since
\[
(a, b, c, d, e) = (a, b, c, 0, 0) + (0, 0, 0, d, e)
\]
expresses an arbitrary element of \( \mathbb{R}^5 \) as a sum of an element of \( U \) and an element of \( W \), but the sum is not a direct sum since \((0, 0, 1, 0, 0)\) is in both \( U \) and \( W \), so \( U \cap W \neq \{0\} \).
   (b) Take \( v_1 = (1, 0) \) and \( v_2 = (0, 1) \) to be the standard basis and \( T(x, y) = (x, 0) \). Then \( Tv_1 = (1, 0) \) and \( Tv_2 = (0, 0) \) are linearly dependent, so they do not form a basis of \( \mathbb{R}^2 \).
   (c) Take \( U = \{(a, b, 0) \mid a, b \in \mathbb{R}\} \) and \( W = \{(0, 0, c) \mid c \in \mathbb{R}\} \). For the operator \( T(x, y, z) = (x, y + z, 0) \), we have \( T(a, b, 0) = (a, b, 0) \in U \), so \( U \) is \( T \)-invariant, but \( T(0, 0, 1) = (0, 1, 0) \notin W \) even though \((0, 0, 1) \in W \), so \( W \) is not \( T \)-invariant.
   (d) Take the operator defined by \( T(\vec{x}) = A\vec{x} \) where
\[
A = \begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]
This behaves as rotation by 90° on the \( x_1x_2 \)-plane and as rotation by 90° on the \( x_3x_4 \)-plane, and these rotations have no real eigenvalues. (\( A \) has eigenvalues ±i, each with multiplicity 2, when considered as an operator on \( \mathbb{C}^4 \).)

2. Recall that the trace of a square matrix is the sum of its diagonal entries:
\[
\text{tr} \begin{pmatrix}
a_{11} & \cdots & a_{1n} \\
\vdots & \ddots & \vdots \\
a_{n1} & \cdots & a_{nn}
\end{pmatrix} = a_{11} + a_{22} + \cdots + a_{nn}.
\]
Let \( U \) be the set of all 3 \times 3 matrices of trace zero:
\[
U = \{A \in M_3(\mathbb{R}) \mid \text{tr} A = 0\}.
\]
   (a) Show that \( U \) is a subspace of \( M_3(\mathbb{R}) \).
   (b) Find a basis of \( U \). Justify your answer.

Solution. (a) I only asked about \( M_3(\mathbb{R}) \) instead of \( M_n(\mathbb{R}) \) in general to keep the notation simpler, but I’ll give a solution which works for any \( n \).
First, the trace of the zero matrix is $0 + 0 + \cdots + 0 = 0$, so the zero matrix is in $U$. If $A, B \in U$, then $\text{tr} A = \text{tr} B = 0$, so (where we denote the entries of $A$ by $a_{ij}$ and those of $B$ by $b_{ij}$):

$$
\text{tr}(A + B) = \text{tr} \begin{pmatrix}
  a_{11}b_{11} & \cdots & a_{1n} + b_{1n} \\
  \vdots & & \ddots \\
  a_{n1} + b_{n1} & \cdots & a_{nn} + b_{nn}
\end{pmatrix}
= (a_{11} + b_{11}) + \cdots + (a_{nn} + b_{nn})
= (a_{11} + \cdots + a_{nn}) + (b_{11} + \cdots + b_{nn})
= \text{tr} A + \text{tr} B
= 0,
$$

so $A + B \in U$ and hence $U$ is closed under addition. If in addition $c \in \mathbb{R}$, then

$$
\text{tr}(cA) = \text{tr} \begin{pmatrix}
  ca_{11} & \cdots & ca_{1n} \\
  \vdots & & \ddots \\
  ca_{n1} & \cdots & ca_{nn}
\end{pmatrix}
= (ca_{11} + \cdots + ca_{nn})
= c(a_{11} + \cdots + a_{nn})
= c(0)
= 0,
$$

so $cA \in U$ and hence $U$ is closed under scalar multiplication. Thus $U$ is a subspace of $M_3(\mathbb{R})$.

(b) Note: The problem which was actually on the midterm asked for a basis of $M_3(\mathbb{R})$, but I meant to ask for a basis of $U$. I’ll give a solution to the problem I meant to ask instead.

For a $3 \times 3$ matrix

$$
\begin{pmatrix}
  a & b & c \\
  d & e & f \\
  g & h & i
\end{pmatrix}
$$

to be in $U$ means that $a + e + i = 0$, so $i = -a - e$. Thus a matrix in $U$ looks like

$$
\begin{pmatrix}
  a & b & c \\
  d & e & f \\
  g & h & -a - e
\end{pmatrix},
$$

which can be written as the following linear combination:

$$
a \begin{pmatrix}
  1 & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & -1
\end{pmatrix} + b \begin{pmatrix}
  0 & 1 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 0
\end{pmatrix} + c \begin{pmatrix}
  0 & 0 & 1 \\
  0 & 0 & 0 \\
  0 & 0 & 0
\end{pmatrix} + d \begin{pmatrix}
  0 & 0 & 0 \\
  1 & 0 & 0 \\
  0 & 0 & 0
\end{pmatrix}
\quad e \begin{pmatrix}
  0 & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & -1
\end{pmatrix} + f \begin{pmatrix}
  0 & 0 & 0 \\
  0 & 0 & 1 \\
  0 & 0 & 0
\end{pmatrix} + g \begin{pmatrix}
  0 & 0 & 0 \\
  0 & 0 & 0 \\
  1 & 0 & 0
\end{pmatrix} + e \begin{pmatrix}
  0 & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 1 & 0
\end{pmatrix}.
$$

These eight matrices thus span $U$. If such a linear combination equals the zero matrix we get

$$
\begin{pmatrix}
  a & b & c \\
  d & e & f \\
  g & h & -a - e
\end{pmatrix} = \begin{pmatrix}
  0 & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 0
\end{pmatrix},
$$
so we must have all entries be 0, which means
\[ a = b = c = d = e = f = g = h = 0. \]

Thus the eight matrices above are linearly independent and hence form a basis of \( U \). \( \square \)

3. Suppose that \( U \) is a subspace of \( \mathcal{P}_n(\mathbb{R}) \) with the property that for any \( p(x) \) in \( U \), its derivative \( p'(x) \) is also in \( U \). Show that if \( x^n \in U \), then \( U = \mathcal{P}_n(\mathbb{R}) \).

**Proof.** By the given property which \( U \) is supposed to satisfy, since if \( x^n \in U \) then \( nx^{n-1} \in U \). Since \( U \) is closed under scalar multiplication, we then have
\[
\frac{1}{n}(nx^{n-1}) = x^{n-1} \in U.
\]
By the same reasoning, since \( x^{n-1} \in U \) we have \((n-1)x^{n-2} \in U \) so
\[
\frac{1}{n-1}[(n-1)x^{n-2}] = x^{n-2} \in U.
\]
And so on, repeating this argument repeatedly shows that
\[ 1, x, x^2, \ldots, x^n \text{ are all in } U. \]
But these vectors form a basis for \( \mathcal{P}_n(\mathbb{R}) \), so we must have \( U = \mathcal{P}_n(\mathbb{R}) \) as claimed. To be clearer, these vectors are linearly independent so \( \dim U \geq n + 1 \), but \( \dim U \leq \mathcal{P}_n(\mathbb{R}) = n + 1 \), so \( \dim U = \mathcal{P}_n(\mathbb{R}) = n + 1 \) and thus \( U = \mathcal{P}_n(\mathbb{R}) \). \( \square \)

4. Let \( T \in \mathcal{L}(V) \) and suppose that \( v \in V \) is a vector such that
\[ T^3v = 0 \text{ but } T^2v \neq 0. \]
Show that \( (v, Tv, T^2v) \) is linearly independent.

**Proof.** Suppose that
\[ av + bTv + cT^2v = 0 \]
for some \( a, b, c \in \mathbb{F} \). Applying \( T \) to the left side gives
\[ T(av + bTv + cT^2v) = aTv + bT^2v + cT^3v = aTv + bT^2v, \]
and applying \( T \) to the right side gives \( T(0) = 0 \), so we must have
\[ aTv + bT^2v = 0. \]
Applying \( T \) to both sides again and using \( T^3v = 0 \) gives
\[ aT^2v = 0. \]
Since \( T^2v \neq 0 \), this means that \( a = 0 \). Then \( aTv + bT^2v = 0 \) becomes
\[ bT^2v = 0, \]
so \( b = 0 \) since \( T^2v \neq 0 \). Finally, the equation we started out with becomes

\[
av = 0,
\]

so \( a = 0 \) since \( v \neq 0 \) because otherwise \( T^2v \) would be zero. Hence \( a = b = c = 0 \), so \((v, Tv, T^2v)\) is linearly independent.

Notice that a similar reasoning works if we place the exponents 2 and 3 with \( n \) and \( n + 1 \): if

\[
T^{n+1}v = 0 \text{ but } T^n v \neq 0,
\]

then \((v, Tv, T^2v, \ldots, T^n v)\) is linearly independent. This is related to the notion of a Jordan chain, which we will see when talking about Jordan forms.

5. Suppose that \( V \) is an \( n \)-dimensional complex vector space and that \( T \in \mathcal{L}(V) \) only has 0 as an eigenvalue. Show that \( T^n v = 0 \) for all \( v \in V \).

Proof. Since \( V \) is a finite-dimensional complex vector space, there exists a basis relative to which the matrix of \( T \) is upper-triangular:

\[
\mathcal{M}(T) = \begin{pmatrix}
0 & * & \cdots & * \\
& 0 & \ddots & \vdots \\
& & \ddots & * \\
& & & 0
\end{pmatrix}.
\]

The blank spaces denote 0’s, and the diagonal terms are all zero since the only eigenvalue of \( T \) is 0. Then we compute that \( \mathcal{M}(T)^2 \) has the form:

\[
\mathcal{M}(T)^2 = \begin{pmatrix}
0 & * & \cdots & * \\
& 0 & \ddots & \vdots \\
& & \ddots & * \\
& & & 0
\end{pmatrix} \begin{pmatrix}
0 & * & \cdots & * \\
& 0 & \ddots & \vdots \\
& & \ddots & * \\
& & & 0
\end{pmatrix} = \begin{pmatrix}
0 & 0 & * & \cdots & * \\
& 0 & 0 & \ddots & \vdots \\
& & \ddots & \ddots & * \\
& & & \ddots & 0 \\
& & & & 0
\end{pmatrix},
\]

where we have an additional diagonal containing all zeroes. And so on, in general taking an additional power of \( \mathcal{M}(T) \) will produce one more diagonal consisting of all zeroes; in particular \( \mathcal{M}(T)^{n-1} \) consists of all zeroes except for possibly a single nonzero term in the upper-right entry. Then \( \mathcal{M}(T)^n \) consists of all zeroes, so

\[
\mathcal{M}(T^n) = \mathcal{M}(T)^n = 0.
\]

Hence \( T^n \) is the zero operator, so \( T^n v = 0 \) for all \( v \in V \). □

6. Suppose that \( U \) and \( W \) are subspaces of \( V \) such that \( V = U \oplus W \). Suppose further that \( U \) and \( W \) are both invariant under an operator \( T \in \mathcal{L}(V) \). Show that if the restrictions \( T|_U \) and \( T|_W \) are both injective, then \( T \) is injective on all of \( V \).

Proof. Suppose that \( Tv = 0 \). We want to show that \( v = 0 \). Since \( V = U \oplus W \) we can write \( v \) as

\[
v = u + w \text{ for some } u \in U \text{ and } w \in W.
\]
Then $0 = T v = T u + T w$. Since $U$ and $W$ are each $T$-invariant, $Tu \in U$ and $Tv \in W$, so

$$0 = Tu + Tw$$

expresses $0$ as a sum of an element of $U$ with an element of $W$. Since $U \oplus W$ is a direct sum, we must thus have $Tu = 0$ and $Tw = 0$, so $u = 0$ and $w = 0$ since the restrictions $T|_U$ and $T|_W$ are injective. Thus $v = 0 + 0 = 0$ as desired, so since null $T = \{0\}$, $T$ is injective on all of $V$. □