Math 334: Midterm

Northwestern University, Summer 2014

July 24, 2014

Name: ______________________________

1. (15 points) Give an example of each of the following. No justification is needed.

(a) A sum $\mathbb{R}^5 = U + W$ which is not a direct sum.

(b) A nonzero operator T and basis (v_1, v_2) of $\mathbb{R}^2$ such that (Tv_1, Tv_2) is not a basis of $\mathbb{R}^2$.

(c) A direct sum $\mathbb{R}^3 = U \oplus W$ and operator T such that U is T-invariant but W is not.

(d) An operator on $\mathbb{R}^4$ with no real eigenvalues.

2. (20 points) Recall that the trace of a square matrix is the sum of its diagonal entries:

$$\operatorname{tr}\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = a_{11} + a_{22} + \cdots + a_{nn}.$$

Let U be the set of all 3×3 matrices of trace zero:

$$U = \{A \in M_3(\mathbb{R}) \mid \operatorname{tr} A = 0\}.$$

(a) Show that U is a subspace of $M_3(\mathbb{R})$.
(b) Find a basis of U. Justify your answer.

3. (15 points) Suppose that U is a subspace of $\mathcal{P}_n(\mathbb{R})$ with the property that for any $p(x)$ in U, its derivative $p'(x)$ is also in U. Show that if $x^n \in U$, then $U = \mathcal{P}_n(\mathbb{R})$.

4. (15 points) Let $T \in \mathcal{L}(V)$ and suppose that $v \in V$ is a vector such that

$$T^3 v = 0 \text{ but } T^2 v \neq 0.$$

Show that $(v, Tv, T^2 v)$ is linearly independent.

5. (20 points) Suppose that V is an n-dimensional *complex* vector space and that $T \in \mathcal{L}(V)$ only has 0 as an eigenvalue. Show that $T^n v = 0$ for all $v \in V$.

6. (15 points) Suppose that U and W are subspaces of V such that $V = U \oplus W$. Suppose further that U and W are both invariant under an operator $T \in \mathcal{L}(V)$. Show that if the restrictions $T|_U$ and $T|_W$ are both injective, then T is injective on all of V.