MATH 321-1: Real Analysis
Northwestern University, Lecture Notes

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These are notes which provide a basic summary of each lecture for MATH 321-1, the first quarter of "MENU Real Analysis", taught by the author at Northwestern University. The book used as a reference is the 3rd edition of *Principles of Mathematical Analysis* by Rudin. Watch out for typos! Comments and suggestions are welcome.

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Lecture 1: Real Numbers

Real analysis is the study of functions defined on the set of real numbers, or subsets thereof. The key concepts we care about this quarter are continuity and differentiability, which are properties a function may or may not have. No doubt you've seen these concepts developed to some extent in a single-variable calculus course, but our aim here is to focus on the underlying theory behind these notions, and the role they play in modern mathematics. Fundamentally, this is a course which comes down to grappling with the notion of *growth* and being able to control—using estimates—how large or small a given quantity can be.

Many of the concepts which make analysis on $\mathbb{R}$ possible apply more broadly to *metric spaces*, which are abstract type of spaces on which it makes sense to talk about "distance". Essentially, whenever we see a definition on $\mathbb{R}$ which uses the absolute value distance $|x-a|$ between two real numbers x and a, there will be an analogous definition for metric spaces. In particular, one of the most important concepts we will see throughout the entire course—*compactness*—makes sense in the setting of metric spaces and it is only through this more general point of view that we will be truly able to see just what it actually means intuitively.

We start by highlighting some properties of real numbers which will be crucial going forward, and are at the core of most everything we will do. The most important concepts we'll see these first few days are those of *supremums/infimums* and *completeness*, which are THE reasons why analysis on $\mathbb{R}$ is even possible.

Ordered fields. The set $\mathbb{R}$ of real numbers comes equipped with two basic structures. The first is its *algebraic* structure, by which we mean the properties it has under addition and multiplication of real numbers. The key concept here is that of a *field*, of which $\mathbb{R}$ is a basic example. We will not spell out the full definition of a field here since the precise definition will not be so important for us, but will instead highlight that one of the most important aspects is that division by nonzero numbers is possible.

In addition to its algebraic structure, $\mathbb{R}$ also comes equipped with an *order*, meaning that it makes sense to say that one real number is less than another. The term "order" also has a precise abstract definition we will not spell out, but at times later it will helpful to point out examples where the notion of "order" does not make sense. Together with the algebraic structure, the order thus turns $\mathbb{R}$ into what's called an *ordered field*. The set of rational numbers $\mathbb{Q}$ is also an example of an ordered field, and understanding what is different about this ordered field vs $\mathbb{R}$ will be important in seeing why $\mathbb{R}$ is so important in analysis. We will spell this out in a few days.

Facts about absolute values. We will use absolute values a lot, since they give us a way to measure "size", so we should make a few things clear. Recall that the *absolute value* of $x \in \mathbb{R}$ is defined to be the number

$$|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

A key observation is that $|x-y|$ is precisely the distance between x and y.

For $\epsilon > 0$, the inequality $|x-a| < \epsilon$ means that the distance (on a number line) between x and a is less than ϵ. This can be rephrased as

$$-\epsilon < x - a < \epsilon.$$

After adding a to both sides, we see that this is the same as

$$a - \epsilon < x < a + \epsilon.$$

Finally, this in turn means that x is in the open interval from $a-\epsilon$ to $a+\epsilon$:

$$x \in (a-\epsilon, a+\epsilon).$$

It will be incredibly useful to become comfortable manipulating inequalities involving absolute values in this manner, and to interpret them visually in terms of intervals.

Triangle Inequality. Oftentimes we will want to estimate how large the absolute value of a sum $|a+b|$ or difference $|a-b|$ can be. The key fact here is the following, known as the *triangle inequality*:

$$|a+b| \leq |a| + |b| \text{ for all } a, b \in \mathbb{R},$$

which says that taking the absolute value of each term in our sum individually can only result in a larger or equal value. This can be verified by working through all possible cases: a, b both positive; both negative; one positive and one negative; etc. You can see details in the book if interested. The name "triangle inequality" comes from the analogous statement in higher dimensions, where we can indeed interpret this as saying that the sum of the lengths of two sides of a triangle is greater than or equal to the length of the third side; we'll come back to this interpretation later.

Note that the addition used in $|a+b|$ is not important and the same inequality works for $|a-b|$:

$$|a-b| \leq |a| + |b|.$$

This just comes from writing $a-b$ as $a+(-b)$ and applying the previous triangle equality.

Archimedean Property of $\mathbb{R}$. The Archimedean Property of $\mathbb{R}$ says:

> For any $x \in \mathbb{R}$, there exists $N \in \mathbb{N}$ such that $x < N$.

In words, given any real number we can find a positive integer larger than it. This is just saying that positive integers can get larger and larger without restriction. But this phrasing of the Archimedean Property is not the one will most often use. Rather, we will make more use of the following:

> For any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. In words, given any positive real number (no matter how small), there is a positive fraction of the form $\frac{1}{N}$, where N is a positive integer, which is smaller than it.

To see that this follows from the first version, note that since $\epsilon \neq 0$, $\frac{1}{\epsilon}$ is a real number. By the Archimedean Property of $\mathbb{R}$, there exists $N \in \mathbb{R}$ such that $\frac{1}{\epsilon} < N$. Since ϵ and N are both positive, multiplying by ϵ and dividing by N does not alter the inequality, so we get $\frac{1}{N} < \epsilon$ as desired.

This version says that fractions of the form $\frac{1}{N}$ can be made arbitrarily small. Practically, the point will be that we can use this to "fit" expressions involving integers "in between" other ones we care about. We will often use the phrase "Archimedean Property" to refer to this second version as well.

$\mathbb{Q}$ is dense in $\mathbb{R}$. As a first use of the Archimedean Property, we prove the following theorem, which is what it means to say that $\mathbb{Q}$ is *dense* in $\mathbb{R}$. Visually, the point is that no matter what nonempty interval we look at, even ones which are incredibly small, we will always find rational numbers inside. This says that in some sense rationals are "spread" throughout all of $\mathbb{R}$, and will eventually give us a way to approximate arbitrary expressions via rational expressions. It turns out that the set of irrational numbers is also dense in $\mathbb{R}$ in the same sense. Again, no matter what nonempty interval we take, irrationals will be present inside.

Theorem. *Given $x, y \in \mathbb{R}$ with $x < y$, there exists $\frac{a}{b} \in \mathbb{Q}$ such that $x < \frac{a}{b} < y$. In words, given any two numbers (no matter how close they are to each other) we can always find a rational number strictly between them.*

Proof. Since $x < y$, $y - x$ is positive so $\frac{1}{y-x}$ is real. By the Archimedean Property, there exists $b \in \mathbb{N}$ such that

$$\frac{1}{y-x} < b.$$

Since $y - x > 0$, this implies that

$$1 < b(y-x), \text{ so } 1 < by - bx.$$

Since the distance between by and bx is greater than 1, there must be an integer between them so there exists $a \in \mathbb{Z}$ such that

$$bx < a < by.$$

Since b is positive, this gives

$$x < \frac{a}{b} < y,$$

so $\frac{a}{b}$ is the rational we desire. □

Lecture 2: Supremums

Warm-Up. Suppose $x > 0$ and $x^2 < 2$. We show there exists $n \in \mathbb{N}$ such that

$$\left(x + \tfrac{1}{n}\right)^2 < 2.$$

The point of this statement is that it says whenever x is a positive number whose square is less than 2, then we can always increase x by a small amount to still get a positive number whose square is less than 2. That is, we can always "fit in" a number of the form $(x + \frac{1}{n})^2$ between x^2 and 2.

Now, one approach is to note that $x^2 < (x + \frac{1}{n})^2 < 2$ is—since x is positive—the same as

$$x < x + \frac{1}{n} < \sqrt{2}.$$

Thus, all we really need to know is that if $x < \sqrt{2}$, we can increase x by a small amount $\frac{1}{n}$ and remain less than 2. This can be achieved by choosing $n \in \mathbb{N}$ such that $\frac{1}{n} < \sqrt{2} - x$, which is possible by the Archimedean Property of $\mathbb{R}$. But, this assumes beforehand that we know $\sqrt{2}$ exists! If $\sqrt{2}$ did not exist (i.e. if there was no real number x satisfying $x^2 = 2$), it would not make sense to turn $x^2 < 2$ into $x < \sqrt{2}$. This might seem like a silly point since *of course* $\sqrt{2}$ exists—after all it is something like

$$\sqrt{2} = 1.4142135624\ldots.$$

But, how do we know that this particular decimal expansion should in fact describe a number satisfying $x^2 = 2$? This is highly non-obvious, and starts to highlight some of the subtleties we face when working with $\mathbb{R}$: there is certainly no rational number whose square is 2, so why should there be such a real number?

We will come back to this next time where we will see how we can guarantee that a real number satisfying $x^2 = 2$ exists, but for now the point is that we should avoid using "$\sqrt{2}$" in our solution, so we should avoid the approach given above. Rather, we argue for the existence of $n \in \mathbb{N}$ satisfying

$(x+\frac{1}{n})^2 < 2$ in a way which does not involve taking square roots. After expanding $(x+\frac{1}{n})^2$, we see that the inequality we want to obtain is the same as

$$x^2 + \frac{2x}{n} + \frac{1}{n^2} < 2.$$

After rearranging, we see that this is the same as

$$\frac{2x}{n} + \frac{1}{n^2} < 2 - x^2.$$

Now, since

$$\frac{2x}{n} + \frac{1}{n^2} \leq \frac{2x}{n} + \frac{1}{n} = \frac{2x+1}{n}$$

because $\frac{1}{n^2} \leq \frac{1}{n}$ for $n \geq 1$, we can make $\frac{2x}{n} + \frac{1}{n^2}$ smaller than $2 - x^2$ as desired by instead making $\frac{2x+1}{n}$ smaller than $2 - x^2$. This we can do using the Archimedean Property: since $\frac{2-x^2}{2x+1} > 0$ (recall that $x^2 < 2$ and $x > 0$), there exists $n \in \mathbb{N}$ large enough so that

$$\frac{1}{n} < \frac{2-x^2}{2x+1},$$

and this is the n which then satisfies $\frac{2x+1}{n} < 2 - x^2$ as desired.

After all that scratch work, here then is our final proof. Suppose $x > 0$ and $x^2 < 2$. Pick $n \in \mathbb{N}$ large enough such that

$$\frac{1}{n} < \frac{2-x^2}{2x+1},$$

which is possible since the right side is a positive number. Then

$$\frac{2x}{n} + \frac{1}{n^2} \leq \frac{2x}{n} + \frac{1}{n} = \frac{2x+1}{n} \leq 2 - x^2,$$

so for this n we have

$$\left(x + \frac{1}{n}\right)^2 = x^2 + \frac{2x}{n} + \frac{1}{n^2} < 2$$

as desired. (We will come back to this result next time to view it in the proper context.)

Supremums. Suppose that $S \subseteq \mathbb{R}$ is a nonempty set of real numbers. An *upper bound* of S is a real number u such that $s \leq u$ for all $s \in S$. We say that S is *bounded above* if it has an upper bound. When S is bounded above, the *supremum* of S is its "least upper bound". To be precise, b is the supremum of S when it satisfies the conditions: (i) b is an upper bound of S, and (ii) for any other upper bound u of S, $b \leq u$. We use the notation $b = \sup S$ for supremums.

Remark. We referred above to "the" supremum of S, without actually justifying the fact that if a set has a supremum, it has only one. This is proved in the book, and the argument comes down to showing that if b and b' both satisfy the definition of $\sup S$, then $b \leq b'$ and $b' \leq b$ (using only characterizations of b and b' as supremums of S), so that $b = b'$.

Example. We claim that the supremum of $\{x \in \mathbb{R} \mid x^2 \leq 2\}$ is $\sqrt{2}$. First, note that after taking square roots of the inequality defining this set, we see that this set is just the closed interval $[-\sqrt{2}, \sqrt{2}]$. To show that $\sqrt{2}$ is indeed the supremum of this, we must show that it is an upper bound and that it is smaller than any other upper bound. First, any $x \in [-\sqrt{2}, \sqrt{2}]$ certainly satisfies $x \leq \sqrt{2}$ simply due to the definition of closed intervals. Thus $\sqrt{2}$ is an upper bound for this set.

To show that $\sqrt{2}$ is the least upper bound, suppose that u is another upper bound for $[-\sqrt{2}, \sqrt{2}]$. Then

$$s \leq u \text{ for any } s \in [-\sqrt{2}, \sqrt{2}].$$

But in particular, $\sqrt{2}$ itself is in $[-\sqrt{2}, \sqrt{2}]$ so u, being an upper bound, is larger than or equal to it: $\sqrt{2} \leq u$. This shows that $\sqrt{2}$ is an upper bound of $[-\sqrt{2}, \sqrt{2}]$ which is $\leq$ any other upper bound, so it is the supremum as claimed.

Alternative characterization of supremums. An upper bound b of $S \subseteq \mathbb{R}$ is the supremum of S if and only if for any $\epsilon > 0$, there exists $s \in S$ such that $b - \epsilon < s$.

The condition after the "if and only if" is a precise way of saying that nothing smaller than b can possibly be an upper bound of S: as ϵ varies through all positive numbers, $b - \epsilon$ varies through all possible numbers smaller than b, and no such number can be an upper bound of S since we can always find something in S larger than it. Since b is an upper bound of S with the property that nothing smaller than it can be an upper bound, b must be the least upper bound as claimed.

Example. We claim that the supremum of the open interval $(-\sqrt{2}, \sqrt{2})$ is also $\sqrt{2}$. This should hopefully be intuitively clear, but proving it is a little different than we did above for the closed interval. The difference is that in this case, the claimed supremum is no longer in the set in question, so the argument we gave before will no longer work. (Make sure you understand why not.)

Instead we use the alternate characterization of supremums given above. First, again it should be simple enough to see that $\sqrt{2}$ is an upper bound of $(-\sqrt{2}, \sqrt{2})$. To show that nothing smaller can be an upper bound, let $\epsilon > 0$. Our goal is to find some $s \in (-\sqrt{2}, \sqrt{2})$ such that $\sqrt{2} - \epsilon < s$. If you draw a picture of $\sqrt{2} - \epsilon$ and $\sqrt{2}$ on a number line, all we need is some number between; their midpoint, which is explicitly given by $\sqrt{2} - \frac{\epsilon}{2}$, works. This should be the number in $(-\sqrt{2}, \sqrt{2})$ which is larger than $\sqrt{2} - \epsilon$, showing that $\sqrt{2} - \epsilon$ cannot be an upper bound.

There is one slight issue with this, in that if ϵ is too large then $\sqrt{2} - \frac{\epsilon}{2}$ won't actually be in $(-\sqrt{2}, \sqrt{2})$. In particular, this happens if $\epsilon \geq 4\sqrt{2}$ since in this case

$$\sqrt{2} - \frac{\epsilon}{2} \leq \sqrt{2} - 2\sqrt{2} = -\sqrt{2}.$$

We get around this by restricting our values of ϵ to those which are $< 4\sqrt{2}$. We'll see why this is enough in the proof below.

Proof that $\sup(-\sqrt{2}, \sqrt{2}) = \sqrt{2}$. Anything in $(-\sqrt{2}, \sqrt{2})$ is stricly less than $\sqrt{2}$, so $\sqrt{2}$ is an upper bound for $(-\sqrt{2}, \sqrt{2})$. Now, let $0 < \epsilon < 4\sqrt{2}$. Then

$$-\sqrt{2} = \sqrt{2} - 2\sqrt{2} < \sqrt{2} - \frac{\epsilon}{2} < \sqrt{2}$$

so $s = \sqrt{2} - \frac{\epsilon}{2}$ is in $(-\sqrt{2}, \sqrt{2})$. This s also satisfies

$$\sqrt{2} - \epsilon < s$$

so for $0 < \epsilon < 4\sqrt{2}$ we have found an element of $(-\sqrt{2}, \sqrt{2})$ which is larger than $\sqrt{2} - \epsilon$. For $\epsilon \geq 4\sqrt{2}$, 0 is an element of $(-\sqrt{2}, \sqrt{2})$ which is larger than $\sqrt{2} - \epsilon$ since in this case $\sqrt{2} - \epsilon$ is negative. Thus, this shows that for any $\epsilon > 0$, $\sqrt{2} - \epsilon$ is not an upper bound of $(-\sqrt{2}, \sqrt{2})$, so $\sqrt{2}$ is the least upper bound as claimed. $\square$

Example. We claim that the supremum of $(-\sqrt{2},\sqrt{2})\cap\mathbb{Q}$ is also $\sqrt{2}$. Here, our set consists only of the *rational* numbers between $-\sqrt{2}$ and $\sqrt{2}$. We use the same idea as above, only now that the choice $s=\sqrt{2}-\frac{\epsilon}{2}$ no longer necessarily works since this might not be rational, as we need it to be in order to be in the set in question. We are saved this time by the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$, which says that for any $\epsilon>0$ we can for sure find a rational s such that

$$\sqrt{2}-\epsilon<s<\sqrt{2}.$$

There is the same issue as above that we have to be careful if $\epsilon\geq 4\sqrt{2}$, but the same way we got around that before works here. I'll leave it to you to write out a precise proof, mimicking the one for $(-\sqrt{2},\sqrt{2})$.

Example. Denote the set $\left\{\frac{n}{n+1}\;\middle|\; n\in\mathbb{N}\right\}$ by A. We claim that $\sup A=1$, which we can guess based on the fact that the fractions above appear to be getting closer and closer to 1 as we plug in larger and larger values of n, or by considering the limit of the given expression as $n\to\infty$. (We'll talk about the relation between limits and supremums soon enough.)

All fractions we are looking at are positive and the numerator is always smaller than the denominator, so all such fractions are certainly smaller than 1. To show that 1 is the least upper bound, again take $\epsilon>0$. We want a number of the form $\frac{n}{n+1}$ such that

$$1-\epsilon<\frac{n}{n+1}.$$

Moving some terms around, we can rewrite this as

$$1-\frac{n}{n+1}<\epsilon,\text{ or }\frac{1}{n+1}<\epsilon.$$

This last expression just comes from writing the left-hand side as a single fraction. Again, we are looking for a value n which makes this true, but now we see that the Archimedean Property of $\mathbb{R}$ precisely says that we can find such a value. Here is our proof.

Proof that $\sup A=1$. For any $n\in\mathbb{N}$, we have $n<n+1$, so $\frac{n}{n+1}<1$ and hence 1 is an upper bound of S. To show that 1 is the least upper bound, let $\epsilon>0$. By the Archimedean Property of $\mathbb{R}$ there exists $N\in\mathbb{N}$ such that $\frac{1}{N}<\epsilon$. But then also

$$\frac{1}{N+1}<\frac{1}{N}<\epsilon.$$

Since $\frac{1}{N+1}=1-\frac{N}{N+1}$, this gives

$$1-\frac{N}{N+1}<\epsilon,\text{ or }1-\epsilon<\frac{N}{N+1}.$$

Thus $\frac{N}{N+1}$ is an element of S which is larger than $1-\epsilon$, so we conclude that $\sup A=1$ as claimed. ☐

Lecture 3: Completeness of $\mathbb{R}$

Warm-Up. We claim that the supremum of

$$\left\{\frac{3n^2}{n^2+n-1}\;\middle|\; n\in\mathbb{N}\text{ and }n\geq 2\right\}$$

is 3. We come up with this value as a result of the fact that the given fraction is always smaller than or equal to 3 (as we will justify in a bit) and if you plug in larger and larger values of 3 the fraction appears to get closer and closer to 3, or by taking the limit of the given fraction as $n \to \infty$. (Again, we will talk about the relation between supremums and limits shortly.)

First, since $n^2 + n - 1 > n^2$ for $n \geq 2$, $\frac{1}{n^2+n-1} < \frac{1}{n^2}$ so

$$\frac{3n^2}{n^2 + n - 1} < \frac{3n^2}{n^2} = 3 \text{ for } n \geq 2.$$

Thus 3 is an upper bound of the given set. Now, let $\epsilon > 0$; we must show there is something in the given set which is larger than $3 - \epsilon$. That is, we want $N \geq 2$ such that

$$3 - \epsilon < \frac{3N^2}{N^2 + N - 1}.$$

Rearranging terms, this is the same as

$$3 - \frac{3N^2}{N^2 + N - 1} < \epsilon, \text{ or } \frac{3N - 3}{N^2 + N - 1} < \epsilon.$$

Since

$$\frac{3N - 3}{N^2 + N - 1} \leq \frac{3N}{N^2 + N - 1} \leq \frac{3N}{N^2} = \frac{3}{N},$$

choosing N such that $\frac{3}{N} < \epsilon$ (which we can do by the Archimedean Property) gives us what we want. That is, for N such that $\frac{1}{N} < \frac{\epsilon}{3}$, we have

$$3 - \frac{3N^2}{N^2 + N - 1} = \frac{3N - 3}{N^2 + N - 1} \leq \frac{3N}{N^2} = \frac{3}{N} < \epsilon,$$

so $3 - \epsilon < \frac{3N^2}{N^2+N-1}$ as required.

Example. For nonempty subsets A and B of $\mathbb{R}$, define $A + B$ to be the set

$$A + B = \{a + b \mid a \in A \text{ and } b \in B\}.$$

We claim that if A and B have supremums, $\sup(A + B) = \sup A + \sup B$. We give two proofs of this: one using the definition of supremum and one using the alternate characterization in terms of ϵ.

Proof 1. For any $a \in A$ and $b \in B$, we have $a \leq \sup A$ and $b \leq \sup B$ so

$$a + b \leq \sup A + \sup B.$$

This shows that anything in $A + B$ is $\leq \sup A + \sup B$, so $\sup A + \sup B$ is an upper bound of $A + B$. Now, suppose that u is any upper bound of $A + B$. Then

$$a + b \leq u \text{ for any } a \in A \text{ and } b \in B.$$

For a fixed $b \in B$, this means that

$$a \leq u - b \text{ for any } a \in A.$$

Hence for a fixed $b \in B$, $u - b$ is an upper bound of A and thus

$$\sup A \leq u - b \text{ for all } b \in B.$$

Rearranging terms, this gives

$$b \leq u - \sup A \text{ for any } b \in B.$$

Thus $u - \sup A$ is an upper bound of B so $\sup B \leq u - \sup A$. Therefore $\sup A + \sup B \leq u$ and we conclude that $\sup A + \sup B$ is the supremum of $A + B$ since it is an upper bound which is smaller than any other upper bound. □

Proof 2. As above, $\sup A + \sup B$ is an upper bound of $A + B$. Let $\epsilon > 0$. By the alternate characterization of supremums, there exists $a \in A$ such that

$$\sup A - \frac{\epsilon}{2} < a$$

and there exists $b \in B$ such that

$$\sup B - \frac{\epsilon}{2} < b.$$

Then

$$\sup A + \sup B - \epsilon = \left(\sup A - \frac{\epsilon}{2}\right) + \left(\sup B - \frac{\epsilon}{2}\right) < a + b,$$

so $a + b$ is an element of $A + B$ which is larger than $\sup A + \sup B$. Hence nothing smaller than $\sup A + \sup B$ can be an upper bound for $A + B$, so $\sup A + \sup B$ is $\sup(A + B)$ as claimed. □

Infimums. Suppose that A is a nonempty subset of $\mathbb{R}$. We say that a real number t is a *lower bound* of A if

$$t \leq a \text{ for all } a \in A.$$

We say that A is *bounded below* if it has a lower bound, and A is *bounded* if it is both bounded above and below.

The *infimum* (or "greatest lower bound") of A is a lower bound ℓ of A such that $t \leq \ell$ for any other lower bound t of A. We use the notation $\inf A$ for infimums. As with supremums, if a set has an infimum it only has one, so infimums are unique.

Also, a lower bound ℓ of A is the infimum of A if and only if for any $\epsilon > 0$ there exists $a \in A$ such that $a < \ell + \epsilon$. This is an analog of the alternative characterization of supremums. The part after "if and only if" says that nothing larger than ℓ can be a lower bound of A since we can find something in A smaller than it.

Completeness Axiom. The *completeness axiom* of $\mathbb{R}$, also called the *least upper bound property* of $\mathbb{R}$, says that any nonempty set of real numbers which is bounded above has a supremum. So, to show that a set of a real numbers has a supremum all we need to do is show that it is bounded above and not empty. We say that $\mathbb{R}$ is "complete". Similarly, any set which is bounded below will have an infimum—I encourage you to think about how this follows directly from the corresponding fact for supremums.

The fact that $\mathbb{R}$ is complete is a crucially important property, and will underlie many things we do in this course. (In fact, it turns out that $\mathbb{R}$ is the *only* complete ordered field.) In particular, it is this property which allows us to visualize $\mathbb{R}$ as a continuous line with no "gaps". We will take this property as a given, but it is natural to think about why it should be true. To justify it precisely, we would have to start with a precise definition (or construction) of $\mathbb{R}$. We will say something about this in a bit.

$\mathbb{Q}$ **is not complete.** Contrast the completeness property of $\mathbb{R}$ with the following example, which shows that $\mathbb{Q}$ is not complete. The set $S = \sup\{r \in \mathbb{Q} \mid r^2 < 2\}$ of rational numbers is bounded above by the rational number 2 but has no supremum in $\mathbb{Q}$. Of course, $\sqrt{2}$ is also an upper bound of S in $\mathbb{R}$ but it is not an upper bound of S "in" $\mathbb{Q}$ since $\sqrt{2}$ is not rational; this is why I used 2 as an upper bound above. Similarly, of course S has a supremum in $\mathbb{R}$ (which is $\sqrt{2}$), but the point is that this supremum does not exist in $\mathbb{Q}$ itself, which is what it means to say that $\mathbb{Q}$ is not complete. If you try to "draw" $\mathbb{Q}$ as a line you will find "gaps" all over the place, which must be filled in with non-rational *real* numbers.

Even if we didn't know ahead of time that the supremum of the set above is $\sqrt{2}$, or if we didn't assume beforehand that a number like $\sqrt{2}$ even exists, we can still show that this set does not have a rational supremum using the Warm-Up from last time, or something like it. Suppose $s \in \mathbb{Q}$ was a rational supremum of the set in question. Since there is no rational satisfying $r^2 = 2$ (i.e. since $\sqrt{2}$ is irrational), $s^2 \neq 2$. Thus either $s^2 < 2$ or $2 < s^2$. But the Warm-Up from last time shows that if $s^2 < 2$, then $(s + \frac{1}{n})^2 < 2$ for some $n \in \mathbb{N}$, meaning that $s + \frac{1}{n}$ would also be in the set in question and hence s could not have been an upper bound. While if instead $2 < s^2$, an argument very similar (which we omit) to the Warm-Up from last time shows that we can find $n \in \mathbb{N}$ satisfying $2 < (s - \frac{1}{n})^2$, which will imply that $s - \frac{1}{n}$ is larger than all things in the given set, and hence s would not have been the *least* upper bound of that set. Hence the given set cannot have a rational supremum, even if we didn't know anything about the existence of $\sqrt{2}$.

Dedekind cuts. As mentioned above, in order to give a reason as to why $\mathbb{R}$ indeed has the least upper bound property, we essentially need a definition of what a real number actually is. The tricky part is to formulate a definition which uses only rational numbers, and nothing like decimal expansions since the existence of decimal expansions for real numbers *depends* on the existence of $\mathbb{R}$ and the completeness property.

To get a sense for how we can characterize real numbers solely in terms of rational numbers, consider for any $a \in \mathbb{R}$ the set

$$\{r \in \mathbb{Q} \mid r < a\}.$$

This is a set consisting solely of rational numbers, and in fact has supremum (in $\mathbb{R}$) equal to a itself. In this way we can then associate to a real number a specific set of rational numbers, namely the set of those rationals which are smaller than it. Different real numbers will correspond to different sets of rationals, so we can say that this particular set of rationals completely characterizes that specific real number. The idea is then to *define* a real number as being a particular type of subset of $\mathbb{Q}$, and with this definition at hand to then *show* that the set $\mathbb{R}$ thus constructed has all the properties you expect the set of real numbers to have.

To see what types of subsets of $\mathbb{Q}$ we should consider, note that the set

$$\{r \in \mathbb{Q} \mid r < a\}$$

has the following three properties. First, it is not empty and is not all of $\mathbb{Q}$. Second, it has the property that if r is inside of it, then any rational smaller than r is also inside of it. (So this set is something like an "interval" of rational numbers extending to $-\infty$ on the left.) And finally, this set has no maximum element: if r is a rational in this set, we can always find another larger rational in this set. These three properties characterize what is called a *Dedekind cut* of $\mathbb{Q}$. So, the upshot is that we can try to *define* the set of real numbers as being the set of Dedeking cuts of $\mathbb{Q}$! This is a pretty abstract characterization of what a real number is, but is one built up only out of rationals without a predetermined notion of "real number" needed beforehand. One can then go on to define what it means to "add" two Dedekind cuts together in a way which mimics what it should mean to add two real numbers, and similarly we can define "multiplication" of Dedekind cuts and what it means for one Dedekind cut to be "smaller" than another. The set of Dedekind cuts constructed in this way can then be shown to be a complete ordered field, so it does provide a valid "construction" of the set of real numbers.

Although we will not return to this perspective of what a real number is—indeed, from now we will simply work with real numbers in the way we always have before—it is an important foundational core behind all that we will do. In particular, let us briefly say something about where the completeness property of $\mathbb{R}$ comes from from this perspective. Given a nonempty set of real numbers which is bounded above, we want to show that there is a real number which is its supremum. A set of real numbers here is a set of Dedekind cuts, so something like

$$S = \{x \subseteq \mathbb{Q} \mid x \text{ is a Dedekind cut of } \mathbb{Q}\}.$$

To say that one Dedekind cut is "less than" another simply means that the it is *contained* in the other (i.e. the notation $x < y$ when x and y are Dedekind cuts means that $x \subseteq y$). Thus, we define the Dedekind cut b which we claim will be the supremum of S to be the *union* of all the Dedekind cuts making up S:

$$b = \bigcup_{y \in S} y.$$

The fact that S has an upper bound guarantees that this b defined in this way is not all of $\mathbb{Q}$, and one can then show that this is b is indeed a Dedekind cut of $\mathbb{Q}$ itself, and satisfies the properties needed to be called $\sup S$. So, again, a fairly abstract point of view on what real numbers are, but one which can be made fully rigorous and from which all properties we will care about follow.

Lecture 4: Cardinality

Warm-Up. Given $a > 1$ and $n \in \mathbb{N}$, we show that $\sqrt[n]{a}$ exists: i.e., there is a positive $y \in \mathbb{R}$ such that $y^n = a$. This is not obvious, since again there is not necessarily going to be a rational number which has this property, so how do we know that if we enlarge our set to $\mathbb{R}$, we do get such a number? Again, we can't just simply plug in $\sqrt[n]{a}$ into our calculator and see what it equals, since this process assumes that such a number already exists. (This is the same issue we mentioned with $\sqrt{2}$ previously.) The goal is to show that such a y exists using only rationals and the *completeness* property of $\mathbb{R}$. (In more abstract language, the goal is to show this using only the "Dedekind cut" definition of $\mathbb{R}$.)

So, consider the set $A = \{r \in \mathbb{Q} \mid r^n < a\}$. This set is nonempty since $1 \in A$ (recall that $a > 1$), and it is bounded above since if $r^n < a$, then $r < a$ as well, again because $a > 1$, so that a is an upper bound of A. (This part of the argument requires some modification for $0 < a < 1$, but the claim is still true for such a as well.) Thus by completeness, A has a supremum in $\mathbb{R}$—call it y. We claim that this is the y we want, namely that it satisfies $y^n = a$. (This y is certainly positive since it is larger than $1 \in A$.) To show that $y^n = a$ we will show that neither $y^n < a$ nor $y^n > a$ are possible. To do so, we use the following fact, which we take for granted for the time being:

For all $\epsilon > 0$, there exists $\delta > 0$ such that if $|x - y| < \delta$, then $|x^n - y^n| < \epsilon$.

This is the precise definition of what it means for the function $f(x) = x^n$ to be continous at y, and we will *prove* that this is indeed true later when we discuss continuity, but we go ahead and use this fact without justification for now. (The book has an alternative way of showing that $y^n = a$ using the Archimedean Property, but here we are doing it differently in order to give a different perspective and not simply repeat what the book does.)

Suppose for a contradiction that $y^n < a$. Then $a - y^n > 0$, so for this positive choice of ϵ the fact we are taking for granted above says that there exists $\delta > 0$ such that

$$\text{if } |x - y| < \delta, \text{ then } |x^n - y^n| < a - y^n.$$

In particular, $|x - y| < \delta$ means that $x \in (y - \delta, y + \delta)$, and $x^n - y^n \leq |x^n - y^n|$, so we get that

$$\text{for } x \in (y - \delta, y + \delta),\ x^n - y^n < a - y^n.$$

Thus any $x \in (y - \delta, y + \delta)$ satisfies $x^n < a$. But in particular (by the denseness of $\mathbb{Q}$ in $\mathbb{R}$), we can take a rational r in $(y, y + \delta)$, which then satisfies $r^n < a$. This rational r is then an element of A which is larger than y, which contradicts the fact that y was supposed to be an upper bound of A. Thus $y^n < a$ is not possible.

Similarly, suppose $y^n > a$. Then $y^n - a > 0$, so there exists $\delta > 0$ such that

$$\text{if } |x - y| < \delta, \text{ then } |x^n - y^n| < y^n - a.$$

But $|x^n - y^n| = |y^n - x^n| \geq y^n - x^n$, so this implies that

$$\text{if } |x - y| < \delta, \text{ then } y^n - x^n < y^n - a.$$

Hence any $x \in (y - \delta, y + \delta)$ satisfies $a < x^n$. In particular, for a rational $s \in (y - \delta, y)$ we have $a < s^n$. For any $r \in A$ we then have

$$r^n < a < s^n,$$

so $r < s$. Thus s is an upper bound of A which is smaller than y (since $s \in (y - \delta, y)$), contradicting the fact that y was meant to be the *least* upper bound of A. Hence it is not possible that $y^n > a$, so we conclude that $y^n = a$ as desired. The point, again, is that we can *show* $\sqrt[n]{a}$ exists as a real number by constructing it as the supremum of an appropriate set.

Countable vs uncountable. There is one more important property the set of real numbers has: it is *uncountable*. This notion belongs to the topic of *cardinality*, which seeks to solve the problem of determining when two sets have the same "number" of elements. This is an easy question when dealing with finite sets (just count the number of elements in each!), but more subtle for infinite sets. The basic definition is that two sets A and B have the same cardinality if there exists a bijection $A \to B$ between them. The point is that such a bijection gives a way to "match" elements of A with elements of B so that no elements from either set are used twice, and no elements of either are left unused. Intuitively, this says that there should be as many things in A as there are in B, so that they should have the same "number" of elements, even if that number is infinite. The surprising part is that, according to this definition, it is possible for one infinite set to have "more" elements than another infinite set, as we will see.

We say that a set S is *countable* either if it is finite or if there exists a bijection $\mathbb{N} \to S$. (This latter condition says that $\mathbb{N}$ and S have the same cardinality, so that there are as many things in S has there are positive integers.) We say that S is *uncountable* if it is not countable. The basic intuition is that countable infinite sets are in some sense the "smallest" possible infinite sets, and uncountable ones have "more" elements than countably infinite ones.

$\mathbb{Z}$ and $\mathbb{Q}$ are countable. $\mathbb{N}$ is countable since there does exist a bijection $\mathbb{N} \to \mathbb{N}$, namely the identity function that sends every positive integer to itself. We also claim that $\mathbb{Z}$ is countable. (This might seem surprising, since after all $\mathbb{N}$ is a proper subset of $\mathbb{Z}$, so that at first glance you might think $\mathbb{Z}$ has "more" elements than $\mathbb{N}$ does, but this is not so!) To see this, write the elements of $\mathbb{Z}$ in the following way in list form:

$$0 \quad 1 \quad -1 \quad 2 \quad -2 \quad 3 \quad -3 \quad \ldots$$

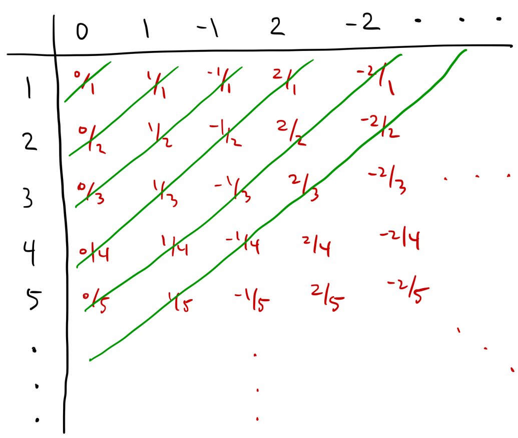
where after the initial 0, we alternate between a positive integer and its negative. This infinite list will contain every single integer exactly once. But from this we can define a bijection function $\mathbb{N} \to \mathbb{Z}$: send $n \in \mathbb{N}$ to the n-th integer in this list. So, 1 is sent to 0, 2 to 1, 3 to -1, 4 to 2, and so on. The function defined in this way will be bijective precisely because every integer appears exactly once in this list. So, as claimed, $\mathbb{Z}$ is countable.

This argument gives the basic intuition behind countable sets: they are precisely the sets whose elements we can "list", or "count", in either a finite or infinite list. Given an infinite listing of the elements of a set A as

$$a_1 \quad a_2 \quad a_3 \quad a_4 \quad \ldots,$$

the list gives a way to define bijection $\mathbb{N} \to A$, by sending $n \in \mathbb{N}$ to the n-th thing in the list. Uncountable sets, then, are intuitively sets with so many elements that it is not possible to give all of them in one single infinitely long list.

Thus to show that $\mathbb{Q}$ is countable, all we need to do is come up with a way of creating a list of all elements of $\mathbb{Q}$. The standard approach is to take the following grid:



where in each spot we give the rational number whose numerator is in the integer in the top row and whose deminator the positive integer in the lefthand column. This grid will contain every single rational, in fact more than once since there will be dupclitates such as $0/1 = 0/2 = 0/3$, $1/2 = 2/4 = 3/6$, and so on. Nonetheless, from this grid we can create a single list that will contain all rational numbers: start with listing the element in the upper-left corner of the grid, then move down to the next diagonal (as drawn), and list the elements along that diagonal, skipping over any duplicates we've already listed before. So, we first list $0/1$, then in the next diagonal we skip $0/2 = 0$ since it was already listed, so next we list $1/1$. Then we move down a a diagonal again and do the same thing: skip $0/3$, list $1/2$, and list $-1/1$. Continuing in this way will produce our desired list containing every rational exactly once:

$$0 \quad 1 \quad 1/2 \quad -1 \quad 1/3 \quad -1/2 \quad 2 \quad 1/4 \quad -1/3 \quad \ldots.$$

This shows that $\mathbb{Q}$ is countable. (The same type of grid can be used to show that union of countably many countable sets is itself countable: that, if $A_1, A_2, A_3, A_4, \ldots$ is a collection of a countable number of sets, each of which is countable, then $\bigcup_n A_n$ is countable as well. We will not give a proof of this here, but it is in the book or various other places you can find, such as my MATH 300 lecture notes.)

$\mathbb{R}$ **is uncountable.** And now we come to main example of an uncountable set: $\mathbb{R}$. Actually, we will instead show that the interval $(0,1)$ is uncountable, but this implies that $\mathbb{R}$ is uncountable too. One reason is that it can be shown that any subset of a countable set is also countable, so if $\mathbb{R}$ were countalbe, $(0,1)$ would be as well. Alternatively, we can take a bijection $(0,1) \to \mathbb{R}$, such as the one given by the function

$$f(x) = \tan(\pi x - \tfrac{\pi}{2}),$$

and argue that if there did exist a bijection $\mathbb{N} \to \mathbb{R}$, composing with the inverse of the function f above would give a bijection $\mathbb{N} \to (0,1)$, which we will show cannot exist. In generak, if $Y \subseteq X$ and Y is uncountable, X must be uncountable.

To show that $(0,1)$ is uncountable we show that there does not exist a bijection $\mathbb{N} \to (0,1)$. To this end, let $f : \mathbb{N} \to (0,1)$ be *any* function. We claim that f is not surjective, which immediately implies that f is not bijective, thereby showing that no function $\mathbb{N} \to (0,1)$ can be bijective as required. To show that f is not surjective we will come up with an explicit element of $(0,1)$ which is not in the image of f. The following argument is known as "Cantor's diagonalization argument", and is a key tool for showing that given sets are uncountable. List the elements in the image of f in terms of their decimal expansions:

$$\begin{aligned} f(1) &= 0.x_{11}x_{12}x_{13}\ldots \\ f(2) &= 0.x_{21}x_{22}x_{23}\ldots \\ f(3) &= 0.x_{31}x_{32}x_{33}\ldots \\ \vdots \quad & \quad \vdots \end{aligned}$$

Define the number $y = 0.y_1y_2y_3\ldots \in (0,1)$ by taking the digit y_i to be anything different from x_{ii}; to be concrete, take

$$y_i = \begin{cases} 3 & \text{if } x_{ii} \neq 3 \\ 7 & \text{if } x_{ii} = 3. \end{cases}$$

(Note that the use of 3 and 7 here is not important—all we need to do is guarantee that y_i and x_{ii} are different. The name "diagonalization argument" comes from the use of the "diagonal" terms x_{11}, x_{22}, x_{33}, etc.) Now, this number y differs from $f(1)$ in the first decimal digit (since $y_1 \neq x_{11}$), so $y \neq f(1)$. Also, y differs from $f(2)$ in the second decimal digit (since $y_2 \neq x_{22}$), so $y \neq f(2)$. In general, $y \neq f(n)$ since y and $f(n)$ differ in the n-th decimal digit. Thus y is not equal to any element in the image of f, so it is not in the image of f and hence f is not surjective. As explained above, this shows that $(0,1)$, and hence $\mathbb{R}$, is uncountable.

Lecture 5: Metric Spaces

Warm-Up. Set $C_0 = [0,1]$. Then take C_1 to be the set obtained by removing the "middle third" portion of C_0:

$$C_1 = \left[0, \tfrac{1}{3}\right] \cup \left[\tfrac{2}{3}, 1\right].$$

Define C_2 be the set obtained by removing the middle third portion of each interval making up C_1:

$$C_2 = \left[0, \tfrac{1}{9}\right] \cup \left[\tfrac{2}{9}, \tfrac{3}{9}\right] \cup \left[\tfrac{6}{9}, \tfrac{7}{9}\right] \cup \left[\tfrac{8}{9}, 1\right],$$

and continuing in this manner define C_n in general to be the set obtained by removing the middle portion of each interval making up C_{n-1}. The *Cantor set* is the set C consisting of what remains after we continue this process indefinitely, or equivalently the intersection of all the C_n's:

$$C = \bigcap_n C_n.$$

We show that the Cantor set is uncountable. This can seem surprising at first, since it seems to be challenging to specify precisely what real numbers belong to the Cantor set. For sure, all endpoints of all intervals in each step of the construction of C remain throughout the entire process, so all such endpoints belong to C. (So, for instance, $0, 1, \frac{1}{3}, \frac{2}{3}, \frac{1}{9}, \frac{2}{9}, \frac{7}{9}$, and $\frac{8}{9}$ are all in C.) However, all such endpoints are rational, so there are only countably many of them, and yet we are saying that C is uncountable, meaning that there are way more elements of C that aren't among these endpoints than there are endpoints. We'll take a second after the proof that C is uncountable to say more precisely what the Cantor set consists of.

Let $x \in C$. We construct an element of $\{0,2\}^\infty$ associated to this as follows, where $\{0,2\}^\infty$ is the set of infinite binary sequences, instead we use a 2 instead of a 1. (We'll see why I'm using $\{0,2\}$ instead of $\{0,1\}$ afterwards. The same proof that showed $\{0,1\}^\infty$ is uncountable in the discussion section problem also shows that $\{0,2\}^\infty$ is uncountable.) Since $C = \bigcap C_n$, $x \in C_n$ for all n. In particular, $x \in C_1$ so x is in one of the two intervals making up C_1; take the first element in our sequence to be 0 if x is in the "left" interval $[0, 1/3]$ and take the first element in our sequence to be 2 if x is in the "right" interval $[2/3, 1]$. Now, whichever of these intervals x is in will itself split into two smaller intervals in the construction of C_2. Since $x \in C_2$, x will be in one of these smaller intervals; take the next element in our sequence to be 0 if it is the "left" interval x is in and take it to be 2 if x is in the "right" interval. For instance, the interval $[0, 1/3]$ splits into $[0, 1/9]$ and $[2/9, 1/3]$. If $x \in [0, 1/9]$ the first two terms in the sequence we are constructing will be $0, 0$, while if $x \in [2/9, 1/3]$ we have $0, 2$ as the beginning of our sequence. Continuing in this manner, whichever interval making up C_2 that x is in will split into two smaller pieces; take 0 as the third term in our sequence if x is in the left piece and 2 if x is in the right piece, and so on. By keeping track of which interval x is in at each step in the construction of the Cantor set in this manner we get a sequence of 0's and 2's.

For instance, if we get the sequence $(0, 2, 2, 2, 0, 0, 0, \ldots)$, x is in the "left" interval of C_1, then in the "right" smaller interval which this interval splits into, then in the "right" smaller interval this splits into, then "right" again, then in the "left" smaller interval that this splits into, and so on. (This is easier to imagine if you draw a picture of this splitting into smaller and smaller intervals as we did in class. In general, a 0 means "go left" in the next step of the construction and 2 means "go right".)

This assignment of a sequence of 0's and 2's to an element $x \in C$ defines a function $C \to \{0,2\}^\infty$. It is injective since different elements in the Cantor set produces different sequences (at some point in the construction, two different numbers in the Cantor set will belong to two different "smaller" intervals since the lengths of these smaller intervals are getting closer and closer to zero), and it is surjective since given any sequence we can use it to single out an element of the Cantor set. (This actually requires more than we currently have available to prove formally—we'll come back to this after we discuss *compactness*.) Thus C and $\{0,2\}^\infty$ have the same cardinality, so we conclude that C is uncountable as well. (If. there was a bijection $\mathbb{N} \to C$, composing with the bijection $C \to \{0,2\}^\infty$ described above would give a bijection from $\mathbb{N}$ to the set of binary sequences, which is not possible.)

What's in the Cantor set? Just for fun, let's clarify what the Cantor set actually consists of.

Any real number in $[0,1]$ has a decimal expansion, where the notation

$$0.x_1x_2x_3\dots$$

really denotes the result of the infinite summation given by

$$\frac{x_1}{10}+\frac{x_2}{10^2}+\frac{x_3}{10^3}+\cdots.$$

By changing the "base" 10 used here, we can come up with decimal expansions with respect to other bases. In particular, any such number has a "base 3" decimal expansion

$$0.y_1y_2y_3\dots$$

where each digit y_i is $0,1$, or 2; this comes from expressing the given number as "base 3" infinite sum of the form:

$$\frac{y_1}{3}+\frac{y_2}{3^2}+\frac{y_3}{3^3}+\cdots.$$

If you think about how these digits relate to splitting an interval up into thirds, you can see that they precisely keep track of which third of an interval a given number belongs to when splitting it up further and further. For instance, a digit of 1 indicates that your given number should belong to the "middle third" portion of an interval. Since these middle thirds are removed in the construction of the Cantor set, we see that the Cantor set precisely consists of those numbers in $[0,1]$ whose base 3 decimal expansions contains only 0's and 2's. For instance, the base 3 decimal expansion of $\frac{1}{4}$ looks like

$$0.0202020202020\dots$$

with 0's and 2's alternating, so $\frac{1}{4}$ is a non-endpoint element of the Cantor set. Note, however, that $\frac{1}{4}$ is still rational, and yet it follows from what we showed before that the Cantor set contains uncountably many irrational numbers.

Metric spaces. A metric space is essentially a space where we have a notion of distance between points defined; no more, no less. The point is that many of the concepts we'll see this quarter—sequence convergence, continuity, limits—will work just fine in any setting where we have a notion of "distance". Thus we take the point of view that it is better to phrase things in as a general a way as possible at the start, and then specialize to specific examples when needed.

Let X be a set. A *metric* on X is a function $d : X \times X \to \mathbb{R}$ such that

- $d(p,q) \geq 0$ for all $p,q \in X$ and $d(p,q) = 0$ if and only if $p = q$,
- $d(p,q) = d(q,p)$ for all $p,q \in X$,
- $d(p,q) \leq d(p,r) + d(r,q)$ for all $p,q,r \in X$.

($X \times X$ denotes the set of all pairs (p,q) where $p,q \in X$.) A *metric space* is X together with a chosen metric d; we often use the notation (X,d) to denote a metric space, or simply X if the metric is clear from context. (But don't forget that the metric is part of required data.)

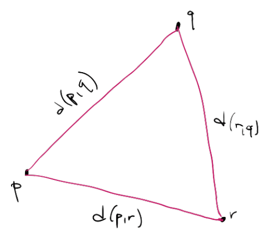
The intuition is simple: d should be thought of as a "distance function" which gives the distance $d(p,q)$ between two points $p,q \in X$. The first condition in the definition says that these "distances" are always nonnegative and equal 0 only when we are computing the distance from a point to itself and the second condition says that the distance from p to q should be the same as the distance from q to p, both of which are clearly properties which "distance" should satisfy.

The third property is called the *triangle inequality* and is the most important one: it says that the "distance" from p to q should give the shortest way of going from p to q in the sense that going through some "intermediate" point r can only increase the overall distance. Again, it makes intuitive sense that a notion of "distance" should satisfy this. We'll see what this looks like for $\mathbb{R}^2$ below, which will explain where the name "triangle inequality" comes from.

Example. The *Euclidean* metric on $\mathbb{R}^n$ is defined by the usual notion of distance in these spaces:

$$d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \sqrt{|x_1 - y_1|^2 + \cdots + |x_n - y_n|^2}.$$

In particular, on $\mathbb{R}^1$ this gives the distance $d(x, y) = |x - y|$ we've been using all along. Verifying that this is a metric is fairly straightforward, although the triangle inequality takes some work to establish. Instead of doing this thoroughly in general, let us point out why the triangle inequality is true in $\mathbb{R}^2$. In this case, we have a picture like:



in which case the triangle inequality says that the length of one side of this triangle is smaller than or equal to the sum of the lengths of the other two sides, which is clear from what we know about triangles. As alluded to earlier, this is where the name "triangle inequality" comes from.

The case of $\mathbb{R}^2$ with the Euclidean metric is probably the most important example of a metric space to keep in mind as far as intuition goes; in particular, whenever we draw a picture meant to illustrate some general property of metric spaces, it will be a picture in $\mathbb{R}^2$.

Other examples. To emphasize the role which the metric plays in all this, note that in addition to the Euclidean metric we have other possible metrics we can put on $\mathbb{R}^2$. (These definitions generalize to $\mathbb{R}^n$ but we'll state them only for $\mathbb{R}^2$ to keep things simpler.)

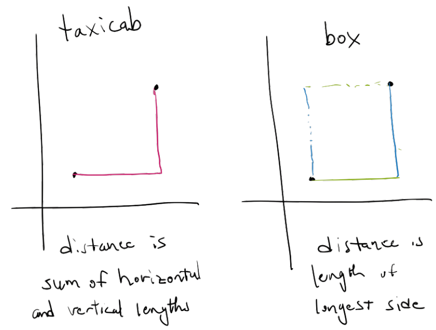
The *taxicab* metric on $\mathbb{R}^2$ is defined by adding together the distance between the x-coordinates of two points to the distance between their y-coordinates:

$$d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|,$$

and the *box* metric on $\mathbb{R}^2$ is defined by taking the maximum of the distance between the x-coordinates of two points and the distance between their y-coordinates:

$$d((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\}.$$

The names of these two metrics come from the following picture:



The distance between two points with respect to the taxicab metric is the distance you have to travel to get from point to the other if you can only move vertically and horizontally but not "diagonally" (as if you were driving a taxicab on grid-like streets), and the distance between two points with respect to the box metric is the length of the largest side of the rectangle (i.e. "box") with one corner at the first point and opposite corner at the other.

Discrete spaces. For any set X, the *discrete* metric on X is defined by setting the distance between distinct points to always be 1:

$$d(p,q) = \begin{cases} 1 & p \neq q \\ 0 & p = q. \end{cases}$$

The first two requirements in the definition of a metric are straightforward to check, and the triangle inequality comes from looking at the possible values the terms in the expression

$$d(p,q) \leq d(p,r) + d(r,q)$$

can have: if the left side is 0 then the inequality holds no matter what the right side is, while if the left side is 1, meaning $p \neq q$, then at least one of the term on the right is also 1 (since either $p \neq r$ or $q \neq r$ or both), so again the inequality holds.

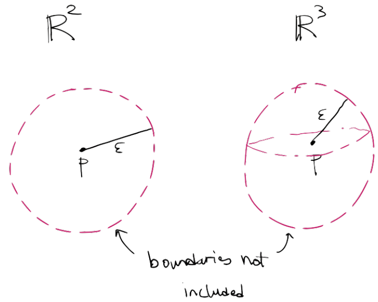
This will be a useful example to keep in mind, as it can give some insight as to what the various definitions we will be seeing mean. The name comes from the fact that distinct points are always separated by a minimum fixed positive distance, which is not true for "continuous" spaces like $\mathbb{R}$ with the Euclidean metric where distances can get arbitrarily small.

Subspaces and balls. Let (X,d) be a metric space and $Y \subseteq X$. Then restricting the metric on X to only allow ourselves to plug in points of Y gives a metric on Y, and in this case we call Y a *subspace* of X. So, a subspace of a metric space is nothing but a subset, but with the *same* metric as on the larger space. (For instance, $\mathbb{Q}$ with the usual Euclidean distance is a subspace of $\mathbb{R}$ with the usual Euclidean distance, but $\mathbb{Q}$ with the discrete metric is not.)

For $p \in X$ and $\epsilon > 0$, the *ball of radius* ϵ (or the ϵ*-ball*) in X centered at p is the set $B_\epsilon(p)$ of points of X whose distance to p is less than r:

$$B_\epsilon(p) = \{q \in X \mid d(p,q) < \epsilon\}.$$

(The book also calls this the ϵ-*neighborhood* of p, and denotes it by $N_\epsilon(p)$.) The name comes from the picture of what these sets look like in $\mathbb{R}^2$ or $\mathbb{R}^3$ with the standard Euclidean metrics: in $\mathbb{R}^2$ the ball of radius ϵ around a point is the (open) disk of radius ϵ centered at that point, and in $\mathbb{R}^3$ these ϵ-balls look like honest solid spheres (without boundary):



Examples. Consider $\mathbb{R}^2$ with the taxicab metric. The ball of radius 1 centered at the origin is defined by

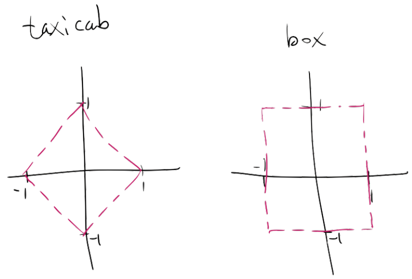
$$B_1((0,0)) = \{(x,y) \in \mathbb{R}^2 \mid d((0,0),(x,y)) < 1\} = \{(x,y) \in \mathbb{R}^2 \mid |x| + |y| < 1\}.$$

The inequality $|x| + |y| < 1$ in $\mathbb{R}^2$ describes a diamond-shaped region, which is thus the "ball" of radius 1 around the origin origin with respect to the taxicab metric. Each of the points on the diamond itself (the boundary) are at a distance 1 from $(0,0)$ and so are not in this ball.

With respect to the box metric, the ball of radius 1 centered at the origin is

$$B_1((0,0)) = \{(x,y) \in \mathbb{R}^2 \mid \max\{|x|,|y|\} < 1.$$

The condition $\max\{|x|,|y|\} < 1$ describes a square (i.e. box) centered at the origin of width and length equal to 2. The points on the boundary square are not included in the ball of radius 1 since they are at a distance 1 from the origin.



Back to discrete. Let (X,d) be a discrete metric space. For any $0 < r \leq 1$ and a fixed $p \in X$, the only possible distance $d(p,q)$ which satisfies $d(q,p) < r$ is $d(p,q) = 0$ since the only possible values for $d(p,q)$ are 0 or 1 in this case. Since $d(p,q) = 0$ if and only if $p = q$, the r-ball around p for $0 < r \leq 1$ consists only of p. For any $r > 1$, *any* $q \in X$ satisfies $d(q,p) < r$, so any ball of radius larger than 1 in this case consists of the entire space X. To summarize:

$$B_r(p) = \begin{cases} \{p\} & 0 < r \leq 1 \\ X & r > 1. \end{cases}$$

Lecture 6: Open Sets

Warm-Up. Let $E \subseteq \mathbb{R}$ and define $C_b(E)$ to be the set of *bounded* real-valued functions on E:

$$C_b(E) := \{f : E \to \mathbb{R} \mid f \text{ is bounded}\}.$$

(To say that f is bounded means that its image, which is the set of all values $f(x)$ as $x \in E$ varies, is a bounded subset of $\mathbb{R}$.) Define a potential metric d on $C_b(E)$ by setting

$$d(f,g) = \sup_{x \in E} |f(x) - g(x)|,$$

so $d(f,g)$ is the supremum of the set of all values $|f(x) - g(x)|$ as x ranges throughout E. (The fact that we are considering only bounded functions guarantees that this supremum exists.) We show that d is indeed a metric on $C_b(E)$, which we call the "sup metric".

First, since $|f(x) - g(x)| \geq 0$ for all $x \in E$, the supremum $d(f,g)$ of these values is also nonnegative. If $d(f,g) = 0$ then we must have

$$|f(x) - g(x)| = 0 \text{ so } f(x) = g(x) \text{ for all } x \in E.$$

This verifies the first property in the definition of a metric. Since $|f(x) - g(x)| = |g(x) - f(x)|$ for all $x \in E$,

$$d(f,g) = \sup_{x \in E} |f(x) - g(x)| = \sup_{x \in E} |g(x) - f(x)| = d(g,f),$$

which is the second property.

Finally we verify the triangle inequality. Let $f, g, h \in C_b(E)$. For any $x \in E$ we have

$$|f(x) - h(x) \leq d(f,h) \text{ and } |h(x) - g(x)| \leq d(h,g)$$

since the values on the left of each inequality are among the values the right side is the supremum of. Thus for all $x \in E$,

$$|f(x) - g(x)| = |f(x) - h(x) + h(x) - g(x)| \leq |f(x) - h(x)| + |h(x) - g(x)| \leq d(f,h) + d(h,g).$$

Hence $d(f,h) + d(h,g)$ is an upper bound for the set of all values $|f(x) - g(x)|$ for $x \in E$, and so is bigger than or equal to the least upper bound of such values, which is $d(f,g)$. Thus

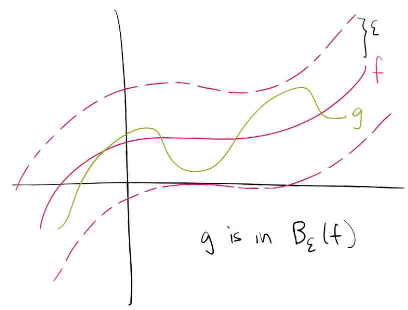
$$d(f,g) \leq d(f,h) + d(h,g)$$

as required, so we conclude that d is a metric on $C_b(E)$.

We can visualize the distance $d(f,g)$ as follows. Draw the graphs of f and g and consider, for all $x \in E$, the vertical distance between $f(x)$ and $g(x)$:

The supremum distance $d(f, g)$ is the "largest" of these vertical distance, and so measures the maximal extent to which f and g can differ. (I have "largest" in quotation marks since there might not actually be points which literally give this supremum vertical distance, since it is in general only a supremum and not a maximum.) Intuitively, the smaller $d(f, g)$ is, the "closer" the graph of f is to the graph of g.

Using this, we can also visualize what a ball looks like in this metric space. For $f \in C_b(E)$ and $\epsilon > 0$, draw the "ϵ-tube" around the graph of f, which is the "tube" made up by shifting the graph of f up by ϵ and down by ϵ. Then $B_\epsilon(f)$ almost consists of those functions g whose graphs lie fully within this tube:



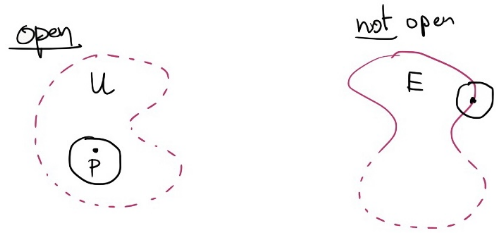
The point being that the requirement $d(f, g) < \epsilon$ says that all vertical distances $|f(x) - g(x)|$ as x ranges throughout E should be less than ϵ. However, note that a function g whose graph gets arbitrarily close to the boundary of the ϵ-tube will in fact have $d(f, g) = \epsilon$, and so such a function is not in the ϵ-ball centered at f. Thus $B_\epsilon(f)$ more precisely consists of those functions g whose graphs are contained within the ϵ-tube around the graph of f *and* do not come arbitrarily close the boundary of this tube.

Open sets. Most concepts we will see in this course are ones that can be phrased using the notion of an *open* subset of a metric space. This is because, as we'll seen, "openness" give a way to discuss the idea of things being "close", or more precisely "close enough". Compactness, convergence of sequences, limits of functions, and continuity are all concepts that can be phrased in this way.

Let X be a metric space and $U \subseteq X$ a subset. We say $p \in U$ is an *interior point* of U if there exists $r > 0$ such that $B_r(p) \subseteq U$. We say that U is *open* in X if every point of U is an interior point of U. The intuition is as follows: If U is open and $p \in U$, the definition says that we can surround p by an entire ball which remains fully contained in U; thus, this says that points which are "close enough" to an element of an open set are themselves also in that open set, so that an open set in a sense "surrounds" all of its points, or "absorbs" all elements "close enough" to one of its points. This notion will be useful for example when discussing limits of functions, since it will guarantee that points "close enough" to one in the domain of a function will also be in that same domain, so that it will make sense to evaluate the function on those points.

Example 1. An open interval (a, b) is an open subset of $\mathbb{R}$. Indeed, given $x \in (a, b)$ we can imagine visually that there is an open interval we can draw around x which is fully contained in (a, b). To be precise, take r to be whichever of $|x - a|$ and $|x - b|$ is smaller. Then the entire open interval $(x - r, x + r)$ will lie within (a, b), since $x - r$ and $x + r$ extend no further than one or both endpoints of (a, b). Thus (a, b) is open in $\mathbb{R}$ as claimed. (This fact is just a special case of the fact that any ball $B_r(p)$ in any metric space is always open, which we will prove later.)

More generally, it is easy to picture what open of $\mathbb{R}^2$ look like, which gives a lot of good intuition for this concept in general. Consider the following subsets of $\mathbb{R}^2$, where dotted curves indicate that those points are not in the subset in question while solid curves indicate that they are:



In the first picture, given a point $p \in U$ we have drawn a ball around it which is fully contained inside of U, showing that U is open in $\mathbb{R}^2$. In the second picture, for a point on the "boundary" of E we can see that any ball we draw around it will contain something not in E, so no such ball will be fully contained in E and hence E is not open in $\mathbb{R}^2$. In general, a subset of $\mathbb{R}^2$ is open if it contains none of its "boundary". (We'll make the notion of "boundary" precise later on.) The interior points in the second set drawn above are those in E but not on the "boundary" of E.

Example 2. We claim that the set E of rational numbers between $-\sqrt{2}$ and $\sqrt{2}$:

$$E := \{r \in \mathbb{Q} \mid -\sqrt{2} < r < \sqrt{2}\}$$

is open in $\mathbb{Q}$, even though it is not open $\mathbb{R}$. It is not open in $\mathbb{R}$ since it has no interior points at all, and thus it is certainly not true that every element of E is an interior point of E: given any rational r and any $s > 0$, $(r - s, r + s)$ contains irrationals since the irrationals are dense in $\mathbb{R}$, so no such interval can be fully contained in E.

Now, if $x \in E$, the open interval U in $\mathbb{R}$ of radius $r = \min\{\sqrt{2} - x, x - (-\sqrt{2})\}$ is fully contained in $(-\sqrt{2}, \sqrt{2})$. Then the ball in $\mathbb{Q}$ of radius r around x consists of only the *rational* numbers in U, and this ball is fully contained in E. (As in the previous example, the irrational numbers in U do not exist in the "larger" space $\mathbb{Q}$ we are considering, so they do not appear in the ball of radius r centered at x in the metric space $\mathbb{Q}$.) This shows that E is open in $\mathbb{Q}$. (The point is the notion of being "open" is a *relative* one, in that it matter what metric space we are considering our set to be a subset of. A certain set might be open in one metric space while not being open in another..)

Example 3. Consider $\mathbb{R}^2$ with the discrete metric, so the distance between two different points is declared to be zero. We claim that the set $\{(1,1)\}$ containing only the single point $(1,1)$ is open. Indeed, we have that $B_{1/2}((1,1)) = \{(1,1)\}$ since the only point whose distance to $(1,1)$ is smaller than $\frac{1}{2}$ is $(1,1)$ itself because every other point has distance 1 from $(1,1)$. Thus, $B_{1/2}((1,1))$ is a ball around $(1,1)$ that is fully contained in $\{(1,1)\}$, so $\{(1,1)\}$ is open in $\mathbb{R}^2$ with respect to the discrete metric.

In fact, the same reasoning shows that if X is *any* discrete metric space, then *every* subset of X is open in X. Indeed, let $U \subseteq X$ and pick $p \in U$. Then $B_{1/2}(p) = \{p\}$, which is contained in U, so every element of U is an interior point of U and thus U is open in X.

Lecture 7: Closed Sets

Warm-Up. We show that the interval $[-1,2)$ is not open in $\mathbb{R}$, but *is* open in $[-1,5)$. First $[-1,2)$ is not open in $\mathbb{R}$ since $-1 \in [-1,2)$ is not an interior point: given any $\epsilon > 0$, $B_\epsilon(-1) = (-1-\epsilon, -1+\epsilon)$

contains an element of $\mathbb{R}$, say $-1-\frac{\epsilon}{2}$, which is not in $[-1,2)$, so no such interval is fully contained in $[-1,2)$.

Now we show that $[-1,2)$ is open in the metric space $[1-,5)$ with the Euclidean metric. Surely, for $x \in (-1,2)$ we can draw an open interval around it which is fully contained in $[-1,2)$. So, the only point we have to worry about when asking if $[-1,2)$ is open in $[-1,5)$ is $x=-1$. We claim that the ball in $[-1,5)$ of radius 1 centered at -1 *is* contained in $[-1,2)$. The key point is that when we take a ball around a point, this ball by definition only contains points from our "larger" metric space, which is $[-1,5)$ in this scenario. The ball in the metric space $[-1,5)$ of radius 1 centered at -1 is

$$B_1(-1) = \{x \in [-1,5) \mid |x+1| < 1\} = [-1,0),$$

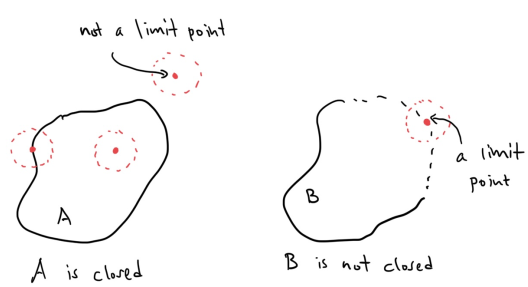
since the elements of $[-1,0)$ are the only numbers satisfying this inequality *among the points of* $[-1,5)$. (The ball in $\mathbb{R}$ of radius 1 centered at -1 is $(-2,0)$, but the points in $(-2,-1)$ do not exist in the "larger" metric space $[-1,5)$ we are considering here.) Thus the ball in $[-1,5)$ of radius 1 around -1 is indeed contained in $[-1,2)$, so $[-1,2)$ is open in $[-1,5)$.

Closed sets. For a subset E of a metric space X, we say that $p \in X$ is a *limit point* of E if for all $r > 0$, the ball $B_r(p)$ contains an element of E different from p. The intuition is that a limit point of E is a point for which we can find elements of E which come arbitrarily close to it: we can find an element of E within a distance 1 from p (pick an element of E in $B_1(p)$); we can find an element of E within a distance $1/2$ from p (pick an element of E in $B_{1/2}(p)$); we find an element of E within a distance $1/3$ from p, and so on and and so on for ever-shrinking radii. In this sense, a limit point of E can be obtained via some type of "limiting" process applied to elements of E. (We will make this point of view precise when we discuss sequences.)

We that $E \subseteq X$ is *closed* in X if it contains all of its limit points. Thus, intuitively, a closed set is one for which points which are "arbitrarily close" to that subset are actually in that subset. (Open and closed sets belong to the subject of *topology*, which is the study of concepts that can be characterized solely in terms of open and closed sets.)

Example 1. Closed intervals $[a,b]$ are closed in $\mathbb{R}$ with respect to the Euclidean metric. Indeed, the limits points of $[a,b]$ are the points of $[a,b]$ itself, since any interval drawn around an element of $[a,b]$ always contains other elements of $[a,b]$. No other element of $\mathbb{R}$ is a limit point of $[a,b]$: if $x < a$, then for the radius $r = |x-a|/2$ the ball $B_r(x) = (x-r, x+r)$ contains no element of $[a,b]$, so $x < a$ is not a limit point of $[a,b]$, and similar argument works for $b < x$.

More generally, it is easy to picture what closed subsets of $\mathbb{R}^2$ look like. Consider the following subsets of $\mathbb{R}^2$, where dotted curves indicate that those points are not in the subset in question while solid curves indicate that they are:



For the first picture, the limit points of $A \subseteq \mathbb{R}^2$ are precisely the elements of A: any ball around any element p of A contains an element of A (in fact infinitely many of them!) different from p.

The point drawn in the upper right is not a limit point of A since there is a ball around it (the one drawn) containing no element of A. Hence the set A in the first picture is closed in $\mathbb{R}^2$.

But the set in the second picture is not closed in $\mathbb{R}^2$. A point on the dotted "boundary", which is not in B, is a limit point of B since any ball around it contains an element of B, so B does not contain all of its limit points. In general, a subset of $\mathbb{R}^2$ is closed in $\mathbb{R}^2$ if it contains all of its "boundary".

Example 2. The set of integers $\mathbb{Z}$ is closed in $\mathbb{R}$ with respect to the Euclidean metric. Indeed, we claim that $\mathbb{Z}$ has no limit points in $\mathbb{R}$, so it is true that $\mathbb{Z}$ contains all of its limit points, simply because there are none! First, given any integer n, the interval of radius $1/2$ around n contains no other integer apart from n itself, so n is not a limit point of $\mathbb{Z}$. (The definition of a limit point p requires that every ball contains an element of the subset *different* from p.) And for any non-integer real number y, there is an interval around y containing no integers at all (take the radius to be half of the distance from y to the nearest integer), so no such $y \in \mathbb{R}$ is a limit point of $\mathbb{Z}$ either.

Example 3. As with the notion "open", the notion of being *closed* is also a relative one, in that it matters what our "larger" metric space is. For example, the interval $(1,2]$ is not closed in $\mathbb{R}$ since $1 \in \mathbb{R}$ is a limit point of $(1,2]$, so $(1,2]$ does not contain all of its limit points in $\mathbb{R}$.

But, we claim that $(1,2]$ *is* closed in $(1,5]$, viewed as a metric space with the Euclidean metric. The difference here is that $1 \in \mathbb{R}$ does not exist in the metric space $(1,5]$, so it is not a candidate limit point when asking whether $(1,2]$ is closed in $(1,5]$. The only thing that matters now is whether every limit point of $(1,2]$ *in the metric space* $(1,5]$ is in $(1,2]$, and this is in fact true.

Example 4. We showed last time that the set E of rationals between $-\sqrt{2}$ and $\sqrt{2}$ was open in $\mathbb{Q}$, and now we claim that is also closed in $\mathbb{Q}$. (Here's some great terminology: a subset of a metric space which is both closed and open is said to be *clopen*!) Note that E is not closed in $\mathbb{R}$ since, for example, $\sqrt{2} \in \mathbb{R}$ is a limit point of E and is not contained in E, but when asking if E is closed in $\mathbb{Q}$ only care about whether *rational* limits points of E are in E. And this is true: no rational smaller than $-\sqrt{2}$ nor larger than $\sqrt{2}$ can be a limit point of E since around any such rational we can find a small enough interval containing no element of E, so any possible rational limit point of E must come from E itself.

Example 5. Suppose X is a discrete metric space. We claim that *every* subset $E \subseteq X$ is closed. (We argued last time that every subset was open, so every subset is in fact clopen.) Indeed, the point is that no subset has any limit points at all, so any $E \subseteq X$ does contain all of its limit points. For any $p \in X$, the ball $B_{1/2}(p)$ consists only of p since p is the only element of X with distance from p less than 1, so $B_{1/2}(p) = \{p\}$ does not contain an element of E different from p. Hence p is not a limit point of $E \subseteq X$.

Lecture 8: More Topology

Warm-Up. Suppose X is a metric space. Let $p \in X$ and $r > 0$. Recall that the open ball of radius r around p in X is

$$B_r(p) = \{q \in X \mid d(q,p) < r\}.$$

Define the *closed ball* of radius r around p in X to be

$$M_r(p) = \{q \in X \mid d(q,p) \leq r\}.$$

(Note that the notation $M_r(p)$ for "closed balls" is not standard, and that there really is no widely common notation for this.) We show that $B_r(p)$ is open in X and $M_r(p)$ is closed, so that "open balls are always open" and "closed balls are always closed".

First, to show that $B_r(p)$ is open in X, let $q \in B_r(p)$ and set $s := r - d(p,q)$. Note that $s > 0$ since $d(p,q) < r$ because $q \in B_r(p)$. If $x \in B_s(q)$, then $d(x,q) < s$ and the triangle inequality gives

$$d(x,p) \leq d(x,q) + d(q,p)) < s + d(q,p) = (r - d(p,q)) + d(p,q) = r,$$

so $x \in B_r(p)$. Thus $B_s(q) \subseteq B_r(p)$, so that q is an interior point of $B_r(p)$ and hence $B_r(p)$ is open in X. (Draw a picture of what $B_s(q)$ looks like to see that it makes sense visually that $B_s(q)$ should be contained in $B_r(p)$!)

Now, to show that $M_r(p)$ is closed in X, we need to know that every limit point of $M_r(p)$ in X is in $M_r(p)$. But this is equivalent to saying that if $q \notin M_r(p)$, then q is not a limit point of $M_r(p)$. (Take the contrapositive of "if $q \in X$ is a limit point of $M_r(p)$, then $q \in M_r(p)$".) So, suppose $q \in X$ is not in $M_r(p)$. To show that q is not a limit point of $M_r(p)$ we need an open ball around q that does not contain any element of $M_r(p)$, or in other words an open ball around q that remains in the complement of $M_r(p)$. We claim that $B_s(q)$ for $s := d(q,p) - r$ works. (Note that $s > 0$ since $r < d(q,p)$ because $q \notin M_r(p)$.) Let $x \in B_s(q)$, so that $d(x,q) < s$. Then

$$d(x,p) \geq d(p,q) - d(q,x) > d(p,q) - s = d(p,q) - (d(q,p) - r) = r.$$

(The first inequality here is often called the *reverse triangle inequality*, and follows from rearranging terms in the usual triangle inequality $d(p,q) \leq d(p,x) + d(x,q)$.) Thus we have that $x \notin M_r(p)$, so $B_s(q) \subseteq M_r(p)^c$. (The ^{c} denotes the complement.) This shows that no element outside of $M_r(p)$ can be a limit point of $M_r(p)$, so $M_r(p)$ contains all of its limit points and is thus closed in X.

Open/closed and complements. The proof that $M_r(p)$ is closed above actually shows that $M_r(p)^c$ is open in X: for any $q \in M_r(p)^c$, there exists a ball $B_s(q)$ fully contained in $M_r(p)^c$, so any such q is an interior point of the complement. Along the same lines, the proof that $B_r(p)$ is open in X can be interpreted as showing that its complement $B_r(p)^c$ is closed in X. Indeed, the proof shows that any $q \in B_r(p)$ is an interior point of $B_r(p)$, so that no $q \in B_r(p)$ can be a limit point of $B_r(p)^c$, so $B_r(p)^c$ contains all of its limit points.

These facts generalize to all open and closed sets, with the result being that $E \subseteq X$ is open in X if and only if E^c is closed in X, and E is closed in X if and only if E^c is open in X. The proofs, as in the Warm-Up, just rely on taking the contrapositives of "if p is a limit point of E, then $p \in E$" and "if $p \in E$, then p is an interior point of E". Note that this matches the intuition we have of what "open" and "closed" look like in $\mathbb{R}^2$ with respect to the Euclidean metric: a set which contains none its "boundary" has a complement that contains all of its "boundary", and a set containing all of its boundary has a complement that contains none of its boundary.

Unions and intersections. The facts above gives a new way of producing open sets by taking the complement of closed sets, and a way of producing closed sets by taking complements of open sets. Another way to produce new open sets is by taking unions of other ones. The claim is that if $\{U_\alpha\}_\alpha$ is a collection of open subsets of X (indexed by α belonging to some indexing set), then the union

$$\bigcup_\alpha U_\alpha$$

is also open in X. (Note here that there is no restriction on how many open sets we are considering: it could be a finite number, an infinite number, an *uncountable* number, it doesn't matter.) To see this, let $p \in \bigcup_\alpha U_\alpha$. Then $p \in U_\beta$ for some β. Since U_β is open in X, there exists $r > 0$ such that $B_r(p) \subseteq U_\beta$. But $U_\beta \subseteq \bigcup_\alpha U_\alpha$, so we also have $B_r(p) \subseteq \bigcup_\alpha U_\alpha$. Hence p is an interior point of $\bigcup_\alpha U_\alpha$, so $\bigcup_\alpha U_\alpha$ is open in X.

But, care must be taken when taking *intersections* of open sets. It is *not* true that intersecting open sets necessarily produces open sets. For example, take the intervals $(-\frac{1}{n}, \frac{1}{n})$ in $\mathbb{R}$. Each of these is open in $\mathbb{R}$, but their intersection (as n varies among all positive integers) is $\{0\}$, which is not open in $\mathbb{R}$. We can only guarantee that we get an open set when intersecting *finitely* many open sets: if $U_1, \ldots, U_n$ are open in X, then $U_1 \cap \cdots \cap U_n$ is open in X. To see this, let $p \in U_1 \cap \cdots \cap U_n$. Then $p \in U_k$ for all $1 \leq k \leq n$. Since each U_k is open in X, there exists, for each $1 \leq k \leq n$, some $r_k > 0$ such that $B_{r_k}(p) \subseteq U_k$. Now take $r = \min\{r_1, \ldots, r_n\}$, which is positive since it is the minimum of finitely many positive numbers. For this minimum we have

$$B_r(p) \subseteq B_{r_k}(p) \subseteq U_k$$

for all $1 \leq k \leq n$, so $B_r(p) \subseteq U_1 \cap \cdots \cap U_n$. This shows that $U_1 \cap \cdots \cap U_n$ is open in X. (This argument does not necessarily work if we consider an infinite number of open sets, since then we get infinitely many radii r_i, which might not have a minimum; they will instead have an *infimum*, but this infimum might not be positive, and so does not give a valid radius for a ball.)

By taking complements, we get immediately that the intersection of an *arbitrary* number of closed sets is closed and that the union of a *finite* number of closed sets is closed. For example, if $\{A_\alpha\}_\alpha$ is a collection of closed sets, then

$$\left(\bigcap_\alpha A_\alpha\right)^c = \bigcup_\alpha A_\alpha^c$$

is a union of open sets since the complement of a closed set is open, and so is itself open. Thus the complement of $\bigcap_\alpha A_\alpha$ is open, so $\bigcap_\alpha A_\alpha$ is closed. A similar argument works for the union of finitely many closed sets.

Lecture 9: Compact Sets

Warm-Up. We say that $E \subseteq X$ is *dense* in X if $\overline{E}$, the *closure* of E (defined as the union of E and its set of limit points), is equal to all of X. For example, both $\mathbb{Q}$ and $\mathbb{Q}^c$ (the set of irrationals) is dense in $\mathbb{R}$ according to this definition, which is just a rephrasing in $\mathbb{R}$ of the previous notion of "dense" we considered. We show that $\mathbb{Q}^2$, the set of points in $\mathbb{R}^2$ with rational coordinates, is dense in $\mathbb{R}^2$. All we need to show to verify this is that every open ball in $\mathbb{R}^2$, no matter the radius and no matter the center, contains an element of $\mathbb{Q}^2$. (This characterization is true in general: E is dense in X if every $B_r(p)$ for $r > 0$ and $p \in X$ contains an element of E, since if $p \notin E$ and this property holds, then p is a limit point of E.)

Thus let $B_r((a, b))$ be any open ball in $\mathbb{R}^2$. We use a fact from discussion, that "open" with respect to the Euclidean metric means that same thing as "open" with respect to the box metric. Since $B_r((a, b))$ is open in $\mathbb{R}^2$ with respect to the Euclidean metric, it is also open with respect to the box metric so there exists $s > 0$ such that

$$(a - s, a + s) \times (b - s, b + s) \subseteq B_r((a, b)).$$

(The left side is the open ball of radius s centered at (a, b) with respect to the box metric.) Since $\mathbb{Q}$ is dense in $\mathbb{R}$, $(a - s, a + s)$ contains a rational $p \in \mathbb{Q}$, and $(b - s, b + s)$ contains a rational q. Then

$$(p, q) \in (a - s, a + s) \times (b - s, b + s) \subseteq B_r((a, b)),$$

so $B_r((a,b))$ contains an element of $\mathbb{Q}^2$, and hence $\mathbb{Q}^2$ is dense in $\mathbb{R}^2$.

Compact sets. We now come to one of the most important concepts in all of analysis and topology, that of *compactness*. This is a tough concept to grasp at first, and it will likely not be clear until later when we discuss continuity why such a notion is even useful. We will give a first glimpse as to why this might be a useful concept shortly, but the main intuition is that compact sets are ones which are not too "large", in a certain sense.

Let K be a subset of a metric space X. By an *open cover* of K we mean a collection $\{U_\alpha\}_\alpha$ of open subsets of X which cover K in the sense that K is contained in their union:

$$K \subseteq \bigcup_\alpha U_\alpha.$$

We say that K is *compact* if every open cover of K has a finite subcover, where by *finite subcover* we mean finitely many of the U_α's which still cover K:

$$K \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}.$$

Thus, compactness of K means that every open cover can be reduced to a finite subcover. Intuitively, any possibly "infinite amount of data" $\{U_\alpha\}_\alpha$ on K can be replaced by a "finite amount of data" $U_{\alpha_1}, \dots, U_{\alpha_n}$. (If K was too "large", something like this would not be possible.)

One point of possible confusion: the definition does not say that K has *a* finite open cover, but rather that *any* open cover and be reduced to a finite one. Indeed, we can always view X itself as a one-element open cover of any of its subsets so that any possible subset always has a finite cover; the key is that whether or not we can always find such finite subcovers no matter what arbitrary open cover we start with.

Examples. The collection of open intervals $\{(-n,n)\}_{n\in\mathbb{N}}$ is an open cover of $\mathbb{R}$ with no finite subcover, so $\mathbb{R}$ is not compact. Indeed, any finite number of these open intervals

$$(-n_1, n_1), \dots, (-n_\ell, n_\ell)$$

will have as their union the interval $(-N, N)$ where $N = \max\{n_1, \dots, n_\ell\}$, so no finite number of these intervals can cover all of $\mathbb{R}$. Also, the interval $(0,1)$ is not compact since the intervals $(\frac{1}{n}, 1)$ together form an open cover with no finite subcover; again, the union of any finite number $(\frac{1}{n_1}, 1), \dots, (\frac{1}{n_k}, 1)$ of such intervals is the one with the largest value of n_i, so no such union can cover all of $(0,1)$.

The main and most important examples of compact sets are the closed intervals $[a,b]$ in $\mathbb{R}$. The proof that this is compact will be left to discussion section, and uses very nicely the properties of a supremums. Many of the special properties that continuous functions defined on closed intervals have come from the fact that these sets are compact, as we will see.

Compact implies bounded. As a first step towards gaining a better understanding of compactness, we show that compact sets are always bounded. (Recall that $K \subseteq X$ is *bounded* if there exists some ball $B_r(p)$ of finite radius containing all of K.) Thus, immediately, any unbounded subset of $\mathbb{R}^n$ for example is not compact.

Suppose $K \subseteq X$ is compact. Let $p \in K$ and consider the collection $\{B_r(p)\}_{r>0}$ of all opens balls centered at p of any radii. These together cover all of K since eventually any point of K will be in one of these balls once r is large enough, so since K is compact this open cover has a finite subcover, say

$$B_{r_1}(p), \dots, B_{r_n}(p).$$

Setting $s = \max\{r_1, \ldots, r_n\} > 0$, we then have that $B_{r_i}(p) \subseteq B_s(p)$ for each $1 \leq i \leq n$, so

$$K \subseteq B_{r_1}(p) \cup \cdots \cup B_{r_n}(p) = B_s(p),$$

and thus K is bounded as claimed. (Note: replacing an infinite number of radii with finitely many allows us to take their maximum. This is not necessarily possible if have an infinite number of radii to work with—in this case we would need to take their supremum instead, but this supremum could be "infinite".)

This proof gives an example of the idea of "turning an infinite amount of data"—in this case radii—into a finite amount", which we alluded to previously. This truly is the key intuition to have in mind when considering compactness.

Lecture 10: More on Compactness

Warm-Up. Suppose X is a discrete metric space. We show that $K \subseteq X$ is compact if and only if K is finite. The backwards direction is actually true in any metric space: finite always implies compact. Indeed, suppose $K = \{p_1, \ldots, p_n\}$ is finite and let $\{U_\alpha\}_\alpha$ be an open cover of K. For each $1 \leq i \leq n$ pick α_i such that $p_i in U_{\alpha_i}$, which is possible since K is contained in the union of the U_α. Then $U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$ contains all p_i, and so is finite subcover of the original open cover. Hence K is compact.

Now, for the forward direction suppose $K \subseteq X$ is compact. (Recall that X is now discrete.) For each $p \in K$, the open ball $B_{1/2}(p) = \{p\}$ has p as its only element, since all other elements are a distance of 1 away from p. Thus the collection of all such open balls of radius $1/2$ cover K, so since K is compact there are a finite number of these open balls that still cover K:

$$K \subseteq \{p_1\} \cup \cdots \cup \{p_n\}.$$

But then $K = \{p_1, \ldots, p_n\}$ is finite, as claimed.

Compact implies closed. We continue to build up more properties of compact sets, now by showing that a compact subset K of a metric space X is always closed. Note in the proof how compactness allows us to replace an infinite amount of data with a finite amount of data.

We show that K is closed by showing that K^c is open. Let $p \in K^c$. Our goal is to show there is a ball around p which is contained in K^c. For any $x \in K$, we can find open balls U_x and V_x around x and p respectively which do not intersect each other, say by taking their common radius to be $r = \frac{d(x,p)}{2}$. (Draw a picture!) As $x \in K$ varies through all possible points, we then get an open cover $\{U_x\}_x$ of K. Since K is compact, this has a finite subcover, say:

$$K \subseteq U_{x_1} \cup \cdots \cup U_{x_n}.$$

The corresponding V's then all contain p and we claim that their intersection:

$$V_{x_1} \cap \cdots \cap V_{x_n}$$

is fully contained in K^c. Indeed, V_{x_i} is contained in the complement of U_{x_i}, so

$$V_{x_1} \cap \cdots \cap V_{x_n} \subseteq (U_{x_1})^c \cap \cdots \cap (U_{x_n})^c = (U_{x_1} \cup \cdots \cup U_{x_n})^c \subseteq K^c$$

as desired. But $V_{x_1} \cap \cdots \cap V_{x_n}$ is the intersection of finitely many open sets, so is itself and open and hence since $p \in V_{x_1} \cap \cdots \cap V_{x_n}$, there exists a ball $B_s(p)$ contained in this intersection, and hence contained in K^c. (In fact, this intersection is simply the ball among the V_{x_i} of smallest radius.) Thus K^c is open in X, so K is closed.

This is a tricky proof to follow at first, but drawing a picture helps to see what the rationale behind the U's and V's are. Again, note the fact that compactness of K allowed us to replace an infinite amount of data (the U_x's in general) by a finite amount (the U_{x_i}'s), and hence we were able to take their intersection and guarantee that the resulting set is still open.

So far we thus know that compact sets are always closed and bounded in any metric space in which they sit inside. In fact, in $\mathbb{R}^n$ (Euclidean metric) it turns out that the converse is true: a closed and bounded subset of $\mathbb{R}^n$ is compact! This is called the *Heine-Borel theorem*, which we will prove next time. Compact subsets of $\mathbb{R}^n$ are thus simple to describe, but take care that this converse is *not* true in an arbitrary metric space, where "closed and bounded" does not necessarily imply compact. For example, $(-\sqrt{2}, \sqrt{2}) \cap \mathbb{Q}$ is closed and bounded in $\mathbb{Q}$, but is actually not compact since the open cover consisting of sets of the form $(-\sqrt{2} + \frac{1}{n}, \sqrt{2}) \cap \mathbb{Q}$ has no finite subcover.

Closed in compact is compact. We finish by giving one more way to justify that certain sets are compact. We show if K is closed in X and X is compact, then K is compact as well. Take an arbitrary open over $\{U_\alpha\}_\alpha$ of K. Since K is closed in X, K^c is open so then

$$\{U_\alpha\}_\alpha \cup \{K^c\}$$

is an open cover of X. (The first collection covers all of K and K^c covers the rest of X.) Since X is compact, this has a finite subcover, say

$$U_1 \cup \ldots \cup U_n \cup K^c.$$

Then the sets $U_1, \ldots, U_n$ cover K since any element in K is also in X and hence is in one of the sets $U_1, \ldots, U_n, K^c$, but certainly does not belong to K^c. Thus $U_1, \ldots, U_n$ is a finite subcover of the open cover $\{U_\alpha\}_\alpha$ of K, so K is compact.

Lecture 11: Yet More Compactness

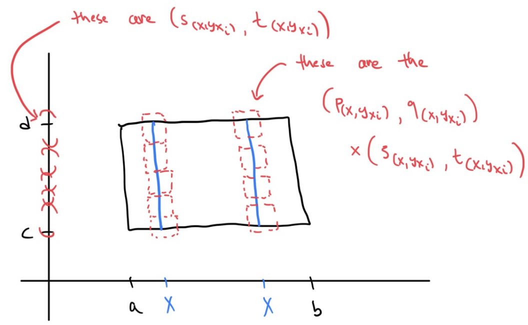
Warm-Up. The fact that closed intervals $[a, b]$ are compact was proved in discussion section. We now use this to show that closed rectangles $[a, b] \times [c, d]$ in $\mathbb{R}^2$ are also compact. To this end, suppose we cover $[a, b] \times [c, d]$ by some open sets U_α. Since each U_α is open in $\mathbb{R}^2$, for any $(x, y) \in [a, b] \times [c, d]$ we can find an open ball around (x, y) contained in some U_α. Even better: since we know that "open" with respect to the Euclidean metric means the same thing as open with respect to the box metric, for any $(x, y) \in [a, b] \times [c, d]$ there exist open intervals $(p_{(x,y)}, q_{(x,y)}) \subseteq \mathbb{R}$ containing x and $(s_{(x,y)}, t_{(x,y)}) \subseteq \mathbb{R}$ containing y such that

$$(x, y) \in (p_{(x,y)}, q_{(x,y)}) \times (s_{(x,y)}, t_{(x,y)}) \subseteq \text{ some } U_\alpha.$$

Now, for fixed $x \in [a, b]$, the intervals $(s_{(x,y)}, t_{(x,y)})$ with $y \in [c, d]$ varying form an open cover of $[c, d]$. Since $[c, d]$ is compact, there are finitely many intervals among these, say

$$(s_{(x,y_{x1})}, t_{(x,y_{x1})}), \ldots, (s_{(x,y_{xn_x})}, t_{(x,y_{xn_x})})$$

that still cover $[c, d]$. (The notation on the y-coordinates emphasizes that they depend on the x we pick, and the number n_x of them also depends on x.) The picture to have in mind is the following, where we end up with finitely many open rectangles that cover the vertical segment in $[a, b] \times [c, d]$ occurring at a fixed x:



We have such a picture no matter which x we take, so that in any "vertical direction" we have finitely many open rectangles.

Now we seek to vary $x \in [a, b]$. Consider all the intervals $(p_{(x,y)}, q_{(x,y)})$ corresponding to the specific vertical intervals constructed above, so in other words consider the intervals of the form

$$(p_{(x,y_{xi})}, q_{(x,y_{xi})}) \text{ where } 1 \leq i \leq n_x.$$

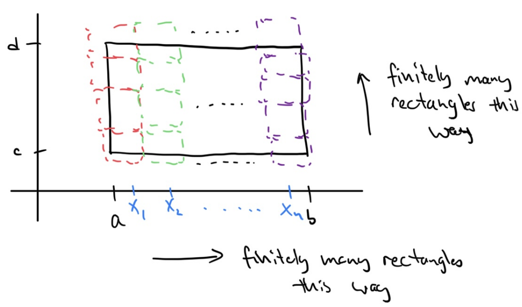
The collection of all such intervals as $x \in [a, b]$ varies form an open cover of $[a, b]$, since each x in particular belongs to any $(p_{(x,y_{xi})}, q_{(x,y_{xi})})$ corresponding to that x. Since $[a, b]$ is compact, there are finitely intervals among these, say

$$(p_{(x_1,y_{x_1 i})}, q_{(x_1,y_{x_1 i})}), \ldots, (p_{(x_m,y_{x_m i})}, q_{(x_m,y_{x_m i})})$$

where $1 \leq i \leq n_{x_j}$, that still cover $[a, b]$. Taking the product of these with the corresponding vertical intervals gives a finite number of open rectangles

$$(p_{(x_j,y_{x_j i})}, q_{(x_j,y_{x_j i})}) \times (s_{(x_j,y_{x_j i})}, t_{(x_j,y_{x_j i})})$$

that cover all of $[a, b] \times [c, d]$, since the vertical portions where chosen, for each x_j, to cover $[c, d]$, and the horizontal portions to cover $[a, b]$. (The notation is quite cumbersome since we have to keep track of x-coordinates and also y-coordinates that depend on the choice of x-coordinates, but the idea is simply that we get a finite number of rectangles in any "vertical" direction, and then a "finite number of a finite number" as we move "horizontally":



producing a finite number covering all of $[a, b] \times [c, d]$.)

But each of these open rectangles was initially chosen to be contained in some U_α, so if we take such a U_α for each of this finite number of rectangles, we get a finite number of U_α covering $[a, b] \times [c, d]$, so this is the finite subcover of our original cover, as required.

Boxes/n-cells. A similar argument, only applied to more directions, shows that the product of any n closed intervals

$$[a_1, b_1] \times \cdots \times [a_n, b_n]$$

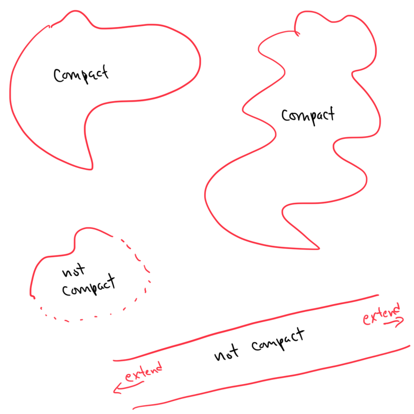
is compact in $\mathbb{R}^n$. Such a product looks like an n-dimensional "rectangular box", and is what the book calls a *n-cell.* (I like the term "box" better.) When $n = 3$, this is literally a box in $\mathbb{R}^3$ with rectangular sides, hence the name.

Heine-Borel. With the compactness of boxes (or n-cells) at hand, we can now give a complete description of all compact subsets of $\mathbb{R}^n$ with respect to the Euclidean metric. The *Heine-Borel* theorem states that $K \subseteq \mathbb{R}^n$ is compact if and only if K is closed and bounded. The forward direction is, as we've said previously, true in any metric space: compact always implies closed and bounded in any metric space. The backwards direction is the new one, and the proof is quick: Suppose K is closed and bounded. Since K is bounded, it is contained in some rectangular box:

$$K \subseteq [a_1, b_1] \times \cdots \times [a_n, b_n].$$

But $[a_1, b_1] \times \cdots \times [a_n, b_n]$ is compact, so K is a closed subset of a compact space and is thus compact itself. Boom!

The upshot is that compact subsets of Euclidean space are easy to visualize:



We emphasize once again, however, that this characterization of compactness so far only holds in $\mathbb{R}^n$, and that in general a closed and bounded space does not have to be compact.

Back to Cantor. We now revisit the Cantor set, and justify some claims we made earlier. Recall that the construction of the Cantor set begins with the set $C_0 = [0, 1]$, then $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, and so on removing the middle third of each interval making up C_{n-1} to produce the intervals making up C_n. The Cantor set is the intersection $C = \bigcap_n C_n$ of all C_n produced in this way. Note that the Cantor set is in fact compact. Indeed, it is bounded since it sits inside of $[0, 1]$, and it is closed since it is the intersection of closed intervals. Hence by the Heine-Borel theorem (which applies since the Cantor set is a subset of $\mathbb{R}$), it is compact.

Now, back when discussing the Cardinality of the Cantor set, we argued that elements of the Cantor set correspond bijectively to infinite binary sequences. That is, we have a map

$$C \to \{0, 1\}^\infty$$

which we hand-waived our way through showing was bijective. This map associated to each element of the Cantor set the string of 0's and 1's which characterized whether that element went into the "left" or "right" interval at each step in the construction. We can now prove that this map is in fact bijective. Actually, injectivity requires nothing new and is something we could have justified earlier: If $x, y \in C$ produce the same binary sequence, then x, y belong to the same interval at each step of the construction of C. For example, either both belong to $[0, 1/3]$ or both to $[2/3, 1]$, and then either both to $[0, 1/9]$, or both to $[2/9, 1/3]$, or both to $[2/3, 7/9]$, or both to $[8/9, 1]$, and so on. But the intervals at the C_1 state each of length $\frac{1}{3}$, those at the C_2 state have length $\frac{1}{3^2}$, and in general those at the C_n stage have length $\frac{1}{3^n}$. If x, y belong to the same interval at each stage, then the distance between them is at most $\frac{1}{3^n}$ for all n:

$$|x - y| < \frac{1}{3^n} \text{ for all } n \in \mathbb{N}.$$

Since the numbers $\frac{1}{3^n}$ get arbitrarily small, we must have $|x - y| = 0$, so $x = y$ as desired.

To see that the map $C \to \{0, 1\}^\infty$ is surjective, take an arbitrary binary sequence. The claim is that there is an element of the Cantor set which follows this given pattern of left/right movements. Denote by I_n the closed interval corresponding to the 0's and 1's in the binary sequence up to the n-th stage; so I_1 is $[0, 1/3]$ or $[2/3, 1]$ depending on whether the first term in our sequence is 0; if the first two terms in our sequence are $0, 0$ then $I_2 = [0, 1/3]$ (i.e. move left and then left again), while if the first two terms are $0, 1$ then $I_2 = [2/3, 1]$ (move left then right); etc. An element in the Cantor set corresponding to the specific binary sequence we take should be an element common to all the I_n. But each of these intervals is compact, and a problem from discussion showed that the intersection of nested (meaning $I_n \supseteq I_{n+1}$ for all n) compact sets is not empty, meaning that there is indeed an element of the Cantor set which corresponds to our given sequence. Hence the map $C \to \{0, 1\}^\infty$ is surjective, so it is a valid bijection. Huzzah!

Lecture 12: Sequences

Warm-Up. We show that an infinite subset E of a compact set $K \subseteq X$ always has a limit point. By way of contrapositive, suppose that E is a subset of K which does not have a limit point. Then for any $p \in K$, p is not a limit point of E so there exists a ball $B_{r_p}(p)$ around p which either contains no element of E or in which the only element of E is p if p happened to belong to E. Either way the key takeaway is that $B_{r_p}(p)$ contains at most one element of E. The collection of all such open balls as $p \in K$ varies is then an open cover of K, so by compactness there are finitely many balls among these which also cover K, say

$$B_{r_{p_1}}(p_1), \ldots, B_{r_{p_n}}(p_n).$$

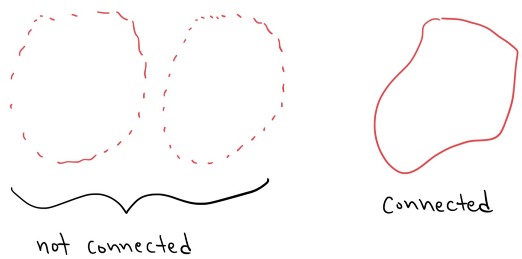
But each of these balls only contain at most one element of E, so their union contains only finitely many elements of E. Since $E \subseteq K$ is supposed to be fully contained in this union, we thus get that E is finite as desired. (The fact that infinite subsets of a compact set always have a limit point will lead to a way to characterize compactness in terms of *sequences*, as we will see.)

Here is the practical point of this. Imagine we have a closed and bounded subset of $\mathbb{R}^2$, for example, such as one of the ones we drew last time. Take any random collection of infinite points in that set. Since our set is compact (Heine-Borel), this random collection of infinite points will have a limit point, with the idea being that this is a point around which points of our infinite set "cluster" near. Thus, even if we begin with a completely random sample of points, among them there will be some that behave in a "controlled" way since they must "cluser" near something. This is essentially how compactness will be used later on, to extract "order" from "chaos".

Connected sets. There is one more basic topological concept which the book mentions at this point, that of *connectedness*. This notion will not be very useful until we discuss continuity, so we will postpone going into the details for now. But, here is the basic definition:

A metric space X is *disconnected* if it can be written as the union of two disjoint nonempty open subsets: there exist nonempty open sets $U, V \subseteq X$ such that $U \cap V = \emptyset$ and $X = U \cup V$. A space which is not disconnected is called *connected*. In other words, X is connected if whenever $X = U \cup V$ with U and V disjoint and open in X, it must be true that one of U or V is empty.

(This version of the definition of connected is slightly different than the one the book gives, but the two versions are in fact equivalent.) The intuition behind this definition can be seen by drawing some pictures:



In the first picture, the space X is the one consisting of the union of the two open balls drawn, and is disconnected. The point is that connected spaces are ones which consist of a single "piece", while disconnected ones consist of multiple "pieces". Essentially, in a disconnected space what happens in one "piece" has no effect on what happens in another.

The most basic examples of connected spaces are intervals in $\mathbb{R}$ and (open or closed) balls in $\mathbb{R}^n$. We will prove all this later.

Sequences. Let X be a metric space. A *sequence* in X is an infinite list (p_n) of elements of X:

$$p_1,\ p_2,\ p_3,\ p_4,\ \ldots.$$

We say that the sequence (p_n) *converges* to $p \in X$ if for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$d(p_n, p) < \epsilon \text{ for } n \geq N.$$

We call p the *limit* of the sequence (p_n), and often denote it by $p = \lim_{n\to\infty} p_n$. (Note that we need to know limits are unique in order for the use of word "the" here to make sense; we will prove that limits—when they exist—are indeed unique next time.)

The intuition is as follows: the condition $d(p_n, p) < \epsilon$ says that $p_n \in B_\epsilon(p)$, so the definition says that given any ball around p, no matter how small its radius, eventually all terms in the sequence (p_n) are in that ball. This captures the idea that the terms p_n are getting closer and closer to p: give me any small bound on how close we want to end up to p, and we can guarantee that we do in fact eventually end up that close. If a sequence converges in X we say it is *convergent*, and if not we say it is *divergent*, or that it *diverges* in X.

Sequences give an essential way to capture the idea of things being "close" to one another. We will see that all of the topological notions we have seen before—open, closed, dense, compact, etc—all have equivalent characterizations in terms of sequences. Later on, the notion of a *continuous function* will also have a characterization in terms of sequences, which will be crucial to understanding the intuition behind continuity.

Remark. Note that the specific metric space in question matters. For instance, take a sequence (r_n) of rationals converging to $\sqrt{2}$ with respect to the Euclidean metric; for example

$$r_1 = 1,\ r_2 = 1.4,\ r_3 = 1.41,\ r_4 = 1.414, \ldots$$

and so on where we take one more digit at a time in the decimal expansion of $\sqrt{2}$. This sequence is convergent in $\mathbb{R}$ but it is considered to be *divergent* in $\mathbb{Q}$ since the thing to which it converges does not exist in the metric space $\mathbb{Q}$. (We'll soon see that this is an example of a *Cauchy sequence* in $\mathbb{Q}$ that does not converge in $\mathbb{Q}$.)

Example. Consider the sequence (x_n) in $\mathbb{R}$ defined by

$$x_n = \frac{2n^2}{n^2+1}.$$

So, the first few terms are

$$x_1 = 1,\ x_2 = \frac{8}{5},\ x_3 = \frac{9}{5}, \ldots.$$

We claim that this sequence converges to 2 in $\mathbb{R}$, assuming the Euclidean metric. This is nothing but the precise statement of the fact that

$$\lim_{n\to\infty} \frac{2n^2}{n^2+1} = 2$$

you would have learned how to compute in a calculus course, but here we will prove this using only the definition of convergence.

First some scratch work. Let $\epsilon > 0$. We need to come up with some index $N \in \mathbb{N}$ beyond which

$$|x_n - 2| = \left| \frac{2n^2}{n^2+1} - 2 \right| < \epsilon$$

holds. We can compute the expression inside the absolute value directly to see that the inequality we need is

$$\left| \frac{-2}{n^2+1} \right| < \epsilon.$$

To make this happen, we seek to bound the term on the left by intermediate expressions in terms of things whose growth as n increases we already know how to control. By controlling how large these bounds are, we can force our original expression to indeed be smaller than ϵ. (This is precisely how *all* such "ϵ-arguments" will work in analysis, and is something we already saw glimpses of in some supremum examples.) In this case, we can note that

$$\left| \frac{-2}{n^2+1} \right| \leq \frac{2}{n^2} \leq \frac{2}{n}$$

by making the denominator smaller at each step. Thus, if we can force the final $\frac{2}{n}$ to be smaller than ϵ (which we can do in this case using the Archimedean Property), *that* in turn will make our original $|x_n - 2|$ smaller than ϵ as well.

Here then is our actual proof. Let $\epsilon > 0$ and pick $N \in \mathbb{N}$ large enough so that

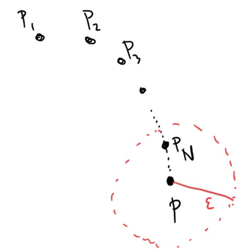
$$\frac{1}{N} < \frac{\epsilon}{2}, \text{ or equivalently } \frac{2}{N} < \epsilon.$$

Then if $n \geq N$, we have

$$|x_n - 2| = \left| \frac{2n^2}{n^2+1} - 2 \right| = \left| \frac{-2}{n^2+1} \right| \leq \frac{2}{n^2} \leq \frac{2}{n} \leq \frac{2}{N} < \epsilon.$$

This shows that (x_n) converges to 2 as claimed.

Another example. In $\mathbb{R}^2$ with the Euclidean metric, a convergent sequence looks like:



Indeed, as we described earlier when giving the intuition behind the definition of convergence, any ball we draw around p, no matter how small, has the property that all terms p_n past some index are in it. (This picture also illustrates something we mentioned earlier, that pictures drawn in $\mathbb{R}^2$ will help to clarify many metric concepts we'll come across.)

Here is a concrete example. Take the sequence (p_n, q_n) in $\mathbb{R}^2$ defined by

$$(p_n, q_n) = \left(\tfrac{1}{2^n}, \tfrac{1}{3^n}\right).$$

So the terms of our sequence are

$$\left(\tfrac{1}{2}, \tfrac{1}{3}\right), \ \left(\tfrac{1}{4}, \tfrac{1}{9}\right), \ \left(\tfrac{1}{8}, \tfrac{1}{27}\right), \ldots$$

which appear to be converging to $(0,0)$. To prove this we need to make the Euclidean distance

$$d((p_n, q_n), (0,0)) = \sqrt{\left(\tfrac{1}{2^n}\right)^2 + \left(\tfrac{1}{3^n}\right)^2} = \sqrt{\tfrac{1}{4^n} + \tfrac{1}{9^n}}$$

however small we need. To do so, we use the fact that we can make each of $\frac{1}{4^n}$ and $\frac{1}{9^n}$ however small we need, since each of these sequences converge to 0 in $\mathbb{R}$. Indeed, let $\epsilon > 0$ and pick $N_1, N_2 \in \mathbb{N}$ such that

$$\left|\frac{1}{4^n} - 0\right| < \frac{\epsilon^2}{2} \text{ for } n \geq N_1, \text{ and } \left|\frac{1}{9^n} - 0\right| < \frac{\epsilon^2}{2} \text{ for } n \geq N_2.$$

Then for $n \geq \max\{N_1, N_2\}$, we have

$$d((p_n, q_n), (0,0)) = \sqrt{\tfrac{1}{4^n} + \tfrac{1}{9^n}} < \sqrt{\tfrac{\epsilon^2}{2} + \tfrac{\epsilon^2}{2}} = \epsilon.$$

Thus $(\frac{1}{2^n}, \frac{1}{3^n})$ converges to $(0,0)$ in $\mathbb{R}^2$ (with respect to the Euclidean metric) as claimed.

(In fact, the exact same reasoning applies to all sequences in $\mathbb{R}^2$: if $p_n \to p$ and $q_n \to q$ in $\mathbb{R}$, then $(p_n, q_n) \to (p, q)$ in $\mathbb{R}^2$. You will prove a general version of this on the homework—the converge is also true!—and the upshot is that convergence in $\mathbb{R}^2$ is equivalent to "componentwise" convergence in $\mathbb{R}$. The same is true of sequences in $\mathbb{R}^n$.)

Discrete sequences. Now consider a discrete space (X, d). If $p_n \to p$ in X, we must have

$$d(p_n, p) < \frac{1}{2}$$

for all n past some index. However, the only way a distance can be smaller than $1/2$ in a discrete space is for it to zero, so the above condition gives

$$d(p_n, p) = 0 \text{ for all } n \text{ past some index},$$

which in turn says that $p_n = p$ for all n past some index. Thus, a sequence in a discrete space is convergent if and only if it is *eventually constant*, meaning that all terms past some index are the same. Note in particular that although the sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$

converges to 0 in $\mathbb{R}$ with respect to the usual Euclidean metric, it does *not* converge to 0, nor too anything, in $\mathbb{R}$ equipped with discrete metric. Convergence depends heavily on the metric being used!

Lecture 13: More on Sequences

Warm-Up. We show that the sequence (f_n) in $C_b(\mathbb{R})$, where $f_n : \mathbb{R} \to \mathbb{R}$ is defined by $f_n(x) = \frac{1}{n}\sin(nx)$, converges to the constant function 0 with respect to the sup metric. Recall that in this case the distance between f_n and the constant function 0 is

$$d(f_n, 0) = \sup_{x\in\mathbb{R}} |f_n(x) - 0| = \sup_{x\in\mathbb{R}} \left|\frac{1}{n}\sin(nx)\right|.$$

For each $x \in \mathbb{R}$, we have

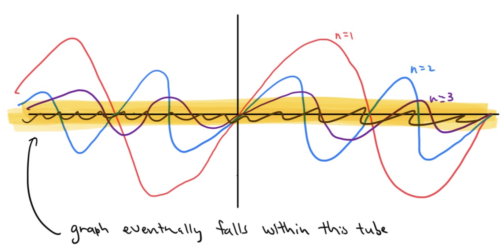
$$\left|\frac{1}{n}\sin(nx)\right| \leq \frac{1}{n}$$

since $|\sin(nx)| \leq 1$ for all n. Thus, $\frac{1}{n}$ is an upper bound on the value of numbers $|\frac{1}{n}\sin(nx)|$ as $x \in \mathbb{R}$ varies, so it is greater than or equal to the supremum of this set:

$$d(f_n, 0) = \sup_{x\in\mathbb{R}} \left|\frac{1}{n}\sin(nx)\right| \leq \frac{1}{n}.$$

Hence for $\epsilon < 0$, we can pick N such that $\frac{1}{N} < \epsilon$, and then $d(f_n, 0) \leq \frac{1}{n} \leq \frac{1}{N} < \epsilon$ for $n \geq N$, which says that $f_n \to 0$ in $C_b(\mathbb{R})$ as claimed.

The intuition behind this convergence statement comes from looking at the graphs of the f_n. These graphs are all sine curves, but whose amplitude decreases as n gets larger:



Given any "tube of radius ϵ" around the graph of 0, eventually the graphs of f_n will fall within this tube, which is what it means to say that $d(f_n, 0) < \epsilon$ for large enough n. This example also illustrates a general process for demonstrating convergence with respect to the sup metric: find a bound on $|f_n(x) - f(x)|$ which depends on n but is *independent* of x, so that you get a bound on the supremum of such values. (This is all related to the notion of *uniform convergence*, which you will see in the second quarter of real analysis.)

Sequences and limit points. With the notion of sequence convergence at hand we can now revisit and recast many of the topological notions we previously considered. First and foremost,

$p \in X$ is a limit point of $E \subseteq X$ if and only if there is a sequence of *distinct* elements x_n of E converging to p. This makes intuitive sense: limit points should be points that are "close" to E, and this new characterization says that indeed such points arise as *limits* of sequences in E which come arbitrarily close to it.

For the forward direction, suppose p is a limit point of E. Then any ball around p contains an element of E which is not equal to p. In particular, for any $n \in \mathbb{N}$ there exists $x_n \in E$ in $B_{1/n}(p)$ with $x_n \neq p$. The resulting x_n satisfy $d(x_n, p) < \frac{1}{n}$, which implies that the x_n converge to p, so that there is indeed a sequence in E converging to p. However, we have to be careful: we claimed that there was a sequence of *distinct* points in E converging to p, and so far we don't know that the elements x_n we've constructed are distinct. Indeed, they don't have to be: perhaps the element x_2 we chose to be within a distance of $\frac{1}{2}$ away from p was also already within $\frac{1}{3}$ away from p, so that it would been a plausible choice for x_3 as well. To guarantee that we get distinct points, we should, at each step, consider a radius small enough so as to exclude all points we've chosen up to that point.

So, for $n = 1$ pick any $x_1 \in B_1(p)$ in E different from p. But now for $n = 2$, pick x_2 to be within not only $\frac{1}{2}$ away from p but also within $d(x_1, p)$ away from p; that is, pick $x_2 \neq p$ in E in the ball of radius

$$\min\left\{\frac{1}{2}, d(x_1, p)\right\} > 0$$

around p. This point then in particular satisfies $d(x_2, p) < d(x_1, p)$, so x_2 cannot be x_1. Then for $n = 3$, pick $x_3 \neq p$ in E in the ball of radius

$$\min\left\{\frac{1}{3}, d(x_1, p), d(x_2, p)\right\} > 0$$

around p. This point is different from both x_1 and x_2 since its distance to p is smaller than that of either x_1 or x_2. And so on, pick at the n-step a point $x_n \neq p$ in E within $\frac{1}{n}$ from p and closer to p than any other point constructed up to that point. The resulting sequence x_n then does consist of distinct points, and converges to p as desired.

For the backwards direction, suppose there is a sequence (x_n) of distinct points of E converging to p. By definition of convergence, any ball around p contains all x_n past some x_N, so in particular it contains at least one such element that is different from p since the x_n's are distinct. Thus p satisfies the definition of being a limit point of E.

Sequences and closures. If we drop the requirement that the sequence (x_n) of E above consist of distinct points, then we get elements of the *closure* of E, which includes more than just the limit points. That is, $p \in \overline{E}$ if and only if there is a sequence (x_n) of elements of E (no requirement that they be distinct) converging to p. Recall that the closure is E union its set of limit points—the relation between limit points and sequences was clarified above, so the only remaining issue is how elements of E relate to convergence of sequences. But this is easy: if $p \in E$, then for sure there is a sequence of points in E converging to p, since if nothing else we can simply take the *constant* sequence

$$p,\ p,\ p,\ p,\ \ldots.$$

This is indeed a sequence of elements of E which converge to p, so elements of $E \subseteq \overline{E}$ also satisfy the sequence characterization of the closure given above.

Therefore, to say that E is *dense* in X (i.e. the closure of E in X is all of X) is to say that for *every* $p \in X$, there exists a sequence in E converging to p. Thus for example, every element x of $\mathbb{R}$

has a sequence of rationals r_n converging to it (pick $r_n \in \mathbb{Q}$ to be in $(x - \frac{1}{n}, x + \frac{1}{n})$) and also has a sequence of *irrationals* converging to it, precisely because both $\mathbb{Q}$ and $\mathbb{Q}^c$ are dense in $\mathbb{R}$.

Sequences and open/closed. We now immediately get a characterization of what it means for E to be closed in X: E is closed in X if and only if whenever there is a sequence in E converging to $p \in X$, we must have $p \in E$. That is, closed sets are those which contain the limits of all convergent sequences within it. In other words, we cannot escape a closed set by taking the limit of a sequence, so that a closed set is one which "attracts" limits of sequences within.

By considering complements, we then also get a characterization of open: E is open in X if and only if whenever we have a sequence of elements outside of E converging to $p \in X$, p must also be outside of E. In other words, an open set is one which "repels" limits of sequences that are outside of it, which can also phrase as saying that limiting process that take place outside of E can never bring us inside of E.

Limits are unique. We will look at characterizing other topological notions—namely compactness—in terms of sequences soon. For now we finish with two more basic properties of convergent sequences. First is the fact that if a sequences converges, then it converges to only one thing, or in other words that limits of convergent sequences are *unique*. Indeed, suppose $(x_n) \in X$ converges to both p and q. Then for any $\epsilon > 0$, there exists N_1 such that

$$d(x_n, p) < \frac{\epsilon}{2} \text{ for } n \geq N_1,$$

and there exists N_2 such that

$$d(x_n, q) < \frac{\epsilon}{2} \text{ for } n \geq N_2.$$

Thus for any x_n beyond x_{N_1} and x_{N_2}, we have

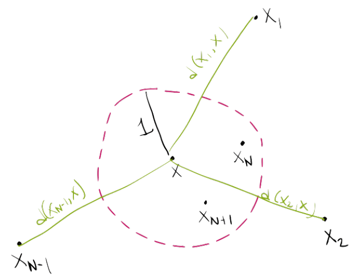
$$d(p, q) \leq d(p, x_n) + d(x_n, q) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

The nonnegative number $d(p, q)$ is thus smaller than any positive ϵ, so we must have $d(p, q) = 0$, so that $p = q$ and hence the limit of (x_n) is unique as claimed. (The intuition is that we can compare the the distance between p and q to the distances between each of p, q and x_n via the triangle inequality, so since we can make the latter distances arbitrarily small it must be that the distance from p and q is also arbitrarily small, so it must be zero since it is a fixed distance.)

Convergent sequences are bounded. Finally, we show that a convergent sequence in an arbitrary metric space (X, d) is bounded. (Recall that $S \subseteq X$ is bounded if there exists $x \in X$ and $r > 0$ such that $S \subseteq B_r(p)$. So, here we mean that the set whose elements are the terms in our sequence should be contained in a ball of finite radius.) Suppose that $x_n \to x$ in X. Then there exists $N \in \mathbb{N}$ such that $d(x_n, x) < 1$ for $n \geq N$, which can be rephrased as saying

$$x_n \in B_1(x) \text{ for } n \geq N.$$

Thus we have a ball of finite radius containing at least all terms in our sequence starting with the N-th one. The idea is now to make this radius large enough so that the corresponding ball includes *all* terms of (x_n). The picture (drawn in $\mathbb{R}^2$ to get some intuition) to have in mind is the following:



We can get a ball which includes x_1 by increasing our current radius of 1 to $1 + d(x_1, x)$, then we can make the ball include x_2 by increasing our radius if need be to make it at least as large as $1 + d(x_2, x)$, and so on. Thus if we define

$$r := \max\{d(x_1, x), d(x_2, x), \dots, d(x_{N-1}, x)\} + 1 > 0,$$

we claim that $B_r(x)$ will contain all terms of (x_n). Indeed, for $1 \leq k \leq N-1$ we have

$$d(x_k, x) < d(x_k, x) + 1 \leq r$$

so $x_k \in B_r(x)$ for $1 \leq k \leq N-1$, and for $n \geq N$ we have

$$d(x_n, x) < 1 \leq r$$

so $x_n \in B_r(x)$ for $n \geq N$. Thus $x_n \in B_r(x)$ for all n, so (x_n) is bounded as claimed.

Lecture 14: Numerical Sequences

Warm-Up. A sequence (x_n) in $\mathbb{R}$ is said to be *increasing* if $x_n \leq x_{n+1}$ for all n (each term is at least as the large as the term before), and is *decreasing* if $x_n \geq x_{n+1}$ for all n (each term is at most as large as the term before). We say that (x_n) *monotone* if it is either increasing or decreasing. (Note that "increasing" and "decreasing" allow for the possibility that terms are repeated, so constant sequences are both increasing and decreasing for example.) We prove the *Monotone Convergence Theorem*: any monotone and bounded sequence in $\mathbb{R}$ converges in $\mathbb{R}$. This is our first result that can guarantee a sequence converges without knowing what the limit will be ahead of time.

We suppose that (x_n) is increasing and bounded above. (The proof in the case that (x_n) is decreasing and bounded below will be very similar.) The set $\{x_n \mid n \in \mathbb{N}\}$ containing the terms of our sequence is nonempty and bounded above, so by the completeness property of $\mathbb{R}$ it has a supremum, call it b. We claim that $x_n \to b$. Indeed, let $\epsilon > 0$. By the alternative characterization of supremums there exists $N \in \mathbb{N}$ such that

$$b - \epsilon < x_N \leq b.$$

Since (x_n) is increasing, we know that $x_N \leq x_n$ for $n \geq N$, and thus

$$b - \epsilon < x_n \leq b \text{ for } n \geq N.$$

Hence if $n \geq N$, $|x_n - b| < \epsilon$ so we conclude that (x_n) converges to b. (In the decreasing case the sequence will converge to the infimum of $\{x_n \mid n \in \mathbb{N}\}$, with almost the same proof.)

Example. Define the sequence (x_n) recursively by setting

$$x_1 = \sqrt{2},\ x_{n+1} = \sqrt{2 + x_n} \text{ for } n \geq 1.$$

Computing a few terms of this sequence:

$$\sqrt{2},\ \sqrt{2+\sqrt{2}},\ \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$$

suggests that it is increasing, which we can prove using induction; that is, we show that $x_n \leq x_{n+1}$ for all n. First, $x_1 = \sqrt{2} \leq \sqrt{2+\sqrt{2}} = x_2$ so our claim is true for $n = 1$. Suppose that $x_k \leq x_{k+1}$ for some k. Then

$$x_{k+1} = \sqrt{2 + x_k} \leq \sqrt{2 + x_{k+1}} = x_{k+2}$$

so $x_k \leq x_{k+1}$ implies $x_{k+1} \leq x_{k+2}$. We conclude by induction that $x_n \leq x_{n+1}$ for all n as claimed, so (x_n) is increasing and hence monotone.

We claim also that this sequence is bounded above by 2, and again we use induction to show this. First, $x_1 = \sqrt{2} \leq 2$ so our claim holds for $n = 1$. Suppose that $x_k \leq 2$ for some k. Then

$$x_{k+1} = \sqrt{2 + x_n} \leq \sqrt{2+2} = 2.$$

Thus $x_k \leq 2$ implies $x_{k+1} \leq 2$ so by induction we conclude that $x_n \leq 2$ for all n.

Since (x_n) is bounded and monotone, it converges—let x denote its limit. To determine the exact value of x we can proceed as follows. First, from the recursive definition of x_n we get

$$x_{n+1}^2 = 2 + x_n.$$

The sequence (x_{n+1}) is the *subsequence* of (x_n) consisting of all terms except for the first (we will discuss subsequences more carefully next time), so (x_{n+1}) also converges to x since (x_n) does. Thus using some limit laws we will soon prove, we have

$$x_{n+1}^2 \to x \text{ and } 2 + x_n \to 2 + x.$$

However, $x_{n+1}^2 = 2 + x_n$ so since limits of a sequence are unique we must have $x^2 = 2 + x$. Solving for x gives $x = -1, 2$. We can't have the limit of (x_n) equal -1 since all terms x_n are positive, so we must have $x = 2$. Thus (x_n) converges to 2, a fact which is more challenging to show directly using only the definition of convergence.

Sums of convergent sequences. The fact we used above that $x_n \to x$ implies $2 + x_n \to 2 + x$ reflects a more fact about sequences in $\mathbb{R}$, namely that sums of convergent sequences are convergent. The claim is the following:

> Suppose we have sequences $a_n \to a$ and $b_n \to b$ in $\mathbb{R}$. Then the sequence $(a_n + b_n)$ converges to $a + b$. In other words, we have
>
> $$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n,$$
>
> assuming both limits on the right exist.

The idea of the proof, as usual, is to find a way to bound $|(a_n + b_n) - (a + b)|$ (the expression we have to make smaller than ϵ in order to show $a_n + b_n \to a + b$) in terms of $|a_n - a|$ and $|b_n - b|$, which we know we can control. This is simple: according to the triangle inequality we have

$$|(a_n + b_n) - (a + b)| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b|.$$

Now we see that if we make the two terms on the right smaller than $\epsilon/2$ (which we know we can do past some indeces), the expression on the left will be smaller than ϵ. This is an example of what's called an "$\frac{\epsilon}{2}$-trick", and again "picking the maximum of indices" makes an appearance.

Here, then, is our proof. Let $\epsilon > 0$. Since $a_n \to a$ and $b_n \to b$ there exist $N_1, N_2 \in \mathbb{N}$ such that

$$|a_n - a| < \frac{\epsilon}{2} \text{ for } n \geq N_1$$

and

$$|b_n - b| < \frac{\epsilon}{2} \text{ for } n \geq N_2.$$

Then if $n \geq \max\{N_1, N_2\}$ we have

$$|(a_n + b_n) - (a + b)| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

We conclude that $a_n + b_n \to a + b$ as claimed.

Products of convergent sequences. The fact that $x_n \to x$ implies $x_n^2 \to x^2$ in the "nested square roots of 2" example is a special case of the general fact that products of convergent sequences in $\mathbb{R}$ are convergent. That is,

> Suppose we have sequences $x_n \to x$ and $y_n \to y$ in $\mathbb{R}$. Then the sequence $(x_n y_n)$ converges to xy. In other words, we have
>
> $$\lim_{n\to\infty} x_n y_n = \left(\lim_{n\to\infty} x_n\right)\left(\lim_{n\to\infty} y_n\right),$$
>
> assuming both limits on the right exist.

The proof again works by making $|x_n y_n - xy|$ small enough past some index, by bounding it in terms of the two expressions $|x_n - x|$ and $|y_n - y|$ we have some control over. The tricky part is in coming up with such a bound.

So, given $\epsilon > 0$, we want to find an index N large enough so that for $n \geq N$, we have

$$|x_n y_n - xy| < \epsilon.$$

Note that the triangle inequality implies

$$|x_n y_n - xy| \leq |x_n y_n - x_n y| + |x_n y - xy|.$$

(We can also see this by adding and subtracting $x_n y$ inside $|x_n y_n - xy|$ and then using the usual triangle inequality for the absolute value.) Thus we have

$$|x_n y_n - xy| \leq |x_n||y_n - y| + |x_n - x||y|.$$

Now we are in business, and since we have two terms to work with we try an "$\epsilon/2$-trick".

The second term is easy to bound: since $(x_n) \to x$, we know there exists $N_1 \in \mathbb{N}$ such that

$$|x_n - x| < \frac{\epsilon}{2|y|} \text{ for } n \geq N_1,$$

and this will give us $|x_n - x||y| < \epsilon/2$. However, note this doesn't quite work if $y = 0$, since then we would be trying to divide by zero. To get around this, we can simply make $|y|$ larger and consider $|y| + 1$ instead: pick $N_1 \in \mathbb{N}$ such that

$$|x_n - x| < \frac{\epsilon}{2(|y| + 1)} \text{ for } n \geq N_1,$$

and this will still give us $|x_n - x||y| < |x_n - x|(|y| + 1) < \epsilon/2$. (We don't need the sharpest possible bound, just a bound that works!)

Now we have that for $n \geq N_1$,

$$|x_n y_n - xy| \leq |x_n||y_n - y| + |x_n - x||y| < |x_n||y_n - y| + \frac{\epsilon}{2}.$$

The first term looks almost as easy to bound, and a first guess may be to use the fact that $(y_n) \to y$ to pick $N_2 \in \mathbb{N}$ so that for $n \geq N_2$,

$$|y_n - y| < \frac{\epsilon}{2|x_n|}.$$

However, this is bad since the right hand side is changing as n does because of the x_n term. We need to find a way to bound $|x_n||y_n - y|$ by something which does not depend on n, since we don't know what n's to consider until after we've specified the constant bound on $|y_n - y|$ we want. To do this, note that since (x_n) converges, it is bounded, so we can find some $M > 0$ such that $|x_n| \leq M$ for all n. This gives us

$$|x_n||y_n - y| < M|y_n - y|$$

for $n \geq N_1$, and now we can apply our $\epsilon/2$-trick as we did before since the only thing depending on n now is $|y_n - y|$. This will give us a natural number N_2, and to make sure that all our bounds hold we need to guarantee that the n's we consider are larger than both N_1 and N_2.

Here then is our final proof. Let $\epsilon > 0$. Pick a positive bound $M > 0$ on the $|x_n|$, which exists since (x_n) converges. Choose $N_1 \in \mathbb{N}$ such that

$$|x_n - x| < \frac{\epsilon}{2(|y| + 1)} \text{ if } n \geq N_1.$$

Next, choose $N_2 \in \mathbb{N}$ such that

$$|y_n - y| < \frac{\epsilon}{2M} \text{ if } n \geq N_2.$$

If $n \geq \max\{N_1, N_2\}$, we then have:

$$\begin{aligned}
|x_n y_n - xy| &= |x_n y_n - x_n y + x_n y - xy| \\
&\leq |x_n||y_n - y| + |x_n - x||y| \\
&\leq M|y_n - y| + |x_n - x|(|y| + 1) \\
&< M\frac{\epsilon}{2M} + \frac{\epsilon}{2(|y| + 1)}(|y| + 1) \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

We conclude that $(x_n y_n)$ converges to xy as claimed.

This is an absolutely crucial argument to understand, since it is reflective of so much of what analysis is about. The entire goal is to find a way to bound some expression of interest ($|x_n y_n - xy|$ in this case) by things ($|x_n - x|$ and $|y_n - y|$) we know we can control, and in doing so get a way to control the growth our original expression. Controlling "growth" in this way *is*, in many ways, the entire point of analysis.

Lecture 15: Subsequences

Warm-Up. Suppose (x_n) is a sequence of nonzero real numbers that converges to x. If $x \neq 0$, we show that $\frac{1}{x_n} \to \frac{1}{x}$. This is yet another basic "arithmetic" property of sequences we expect to be true from calculus:

$$\lim_{n\to\infty} \frac{1}{x_n} = \frac{1}{\lim\limits_{n\to\infty} x_n},$$

provided that the limit on the right exists and that at no point are we dividing by zero. We want to make the quantity $\left|\frac{1}{x_n} - \frac{1}{x}\right|$ smaller than ϵ. We can first rewrite this as

$$\left|\frac{1}{x_n} - \frac{1}{x}\right| = \frac{|x_n - x|}{|x_n||x|}.$$

Now, the numerator is good because this is something we know we can control given that (x_n) converges to x. But, in order to make the quantity above smaller than ϵ, we cannot jump to requiring that

$$|x_n - x| < \epsilon |x_n||x|$$

since the right side is not a fixed positive number, but rather varies as n does. So, we need to find a way to bound $\frac{|x_n - x|}{|x_n||x|}$ by something whose only dependence on n comes from $|x_n - x|$, meaning that we need to find a way to bound $|x_n|$ from *below* since it occurs in the denominator:

$$\frac{|x_n - x|}{|x_n||x|} \leq \frac{|x_n - x|}{(\text{some positive quantity smaller than } |x_n|)|x|}.$$

After we have a bound like this, we can choose make $|x_n - x|$ appropriately small in order to make everything smaller than ϵ.

In order to find a positive *lower* bound on $|x_n|$, we use the fact that $x_n \to x$: since the terms x_n are converging to x, they must eventually be bounded away from 0 by at least, say, $\frac{|x|}{2}$. In other words, by the definition of convergence, there must exist a point in our sequence beyond which

$$x_n \text{ lies in } (x - \tfrac{|x|}{2}, x + \tfrac{|x|}{2}),$$

at which point $|x_n| \geq \frac{|x|}{2}$ is true. We can also derive this using the reverse triangle inequality: pick N beyond which $|x_n - x| < \frac{|x|}{2}$ is true, and then we get

$$|x| - |x_n| \leq |x - x_n| < \frac{|x|}{2}, \text{ so } |x| - \frac{|x|}{2} \leq |x_n| \text{ for such } n.$$

The point is that we can, for large enough n, bound $|x_n|$ from below by $\frac{|x|}{2}$, and for these terms we thus have:

$$\frac{|x_n - x|}{|x_n||x|} \leq \frac{|x_n - x|}{\frac{|x|}{2}|x|}.$$

Now that we have a constant in the denominator, we can proceed to make the fraction smaller than ϵ by forcing $|x_n - x|$ to be smaller than $\frac{\epsilon|x|^2}{2}$. Note that we used two indices throughout all this: one beyond which $|x_n - x| < \frac{|x|}{2}$ is true, and one beyond which $|x_n - x| < \frac{\epsilon|x|^2}{2}$ is true. So, in order to make both of these apply we need to take the maximum of these indices.

Here is our final proof. Let $\epsilon > 0$. Since $x_n \to x$, there exists $N_1 \in \mathbb{N}$ such that

$$|x_n - x| < \frac{\epsilon|x|^2}{2} \text{ for } n \geq N_1.$$

There also exists $N_2 \in \mathbb{N}$ such that $|x_n - x| < \frac{|x|}{2}$ for $n \geq N_2$, which implies that

$$|x| - |x_n| \leq |x - x_n| < \frac{|x|}{2}, \text{ so } \frac{|x|}{2} < |x_n| \text{ for } n \geq N_2.$$

Thus for $n \geq \max\{N_1, N_2\}$, we have:

$$\left|\frac{1}{x_n} - \frac{1}{x}\right| = \frac{|x_n - x|}{|x_n||x|} \leq \frac{|x_n - x|}{\frac{|x|}{2}|x|} < \frac{\epsilon|x|^2/2}{|x|^2/2} = \epsilon,$$

which shows that $\frac{1}{x_n} \to \frac{1}{x}$ as claimed.

Subsequences. We are now working towards a sequential characterization of compactness. The correct phrasing depends on the notion of a subsequence of a sequence, which is simple to grasp: a *subsequence* of a sequence (p_n) in X is a sequence (p_{n_k}) of terms coming from among the p_n such that if $k_1 \geq k_2$, then $n_{k_1} \geq n_{k_2}$. This property says precisely that the order in which terms occur the subsequence (as given by the index k) matches the relative order in which the terms occur in the original sequence (as given by the index n_k). If one term comes after another in the subsequence, that must also have been true in the original sequence. So, for example,

$$p_4, \ p_1, \ p_6, \ p_7, \ p_{20}, \ p_{10}, \ \ldots$$

is not considered to be a subsequence of

$$p_1, \ p_2, \ p_3, \ p_4, \ p_5, \ \ldots$$

since ordering isn't preserved: p_4 comes before p_1 in the subsequence, even though p_1 came before p_4 in the original sequence, and similarly for p_{20} vs p_{11}.

The first basic fact about subsequences is that if the original sequence converges, so does any subsequence and to the same limit. That is, suppose $p_n \to p$ in X. Then for any $\epsilon > 0$ there exists N such that

$$d(p_n, p) < \epsilon \text{ for } n \geq N.$$

The point is that all terms in a subsequence (p_{n_k}) are among these p_n once the subsequence index n_k is large enough: pick $K \in \mathbb{N}$ such that $n_K \geq N$ and then we have

$$d(p_{n_k}, p) < \epsilon \text{ for } k \geq K,$$

since $n_k \geq n_K \geq N$. This shows that (p_{n_k}) also converges to p as claimed. (In particular then, a sequence for which there exists two subsequences converging to different points, or for which there exists a subsequence that does not converge, cannot be convergent itself.)

Bolzano-Weierstrass. The most important fact related to sequences and subsequences in $\mathbb{R}$ is the following result, known as the *Bolzano-Weierstrass theorem*: every bounded sequence in $\mathbb{R}$ has a convergent subsequence. The point is that even if our original sequence behaves in a completely random way, as long as it is bounded there will be terms among them that behave in a very controlled way, by converging to *something*. (So, our original sequence wasn't behaving so randomly after all!) It is precisely this fact which will allow for many of the nice properties a continuous function can have to hold.

We will give two proofs of this theorem in the next few days, but for now we emphasize that this is ultimately a result about *compactness*. (Boom!!!) Indeed, here, finally, is the characterization of compactness in terms of sequences we've been alluding to for a while: a subset K of a metric space X is compact if and only if every sequence in K has a convergent subsequence in K. (Saying "convergence subsequence in K" requires that the limit of the subsequence be in K as well.) To distinguish between this characterization of compactness and the previous ones in terms of open covers, we often refer to this version as *sequential compactness* and the previous one as *covering compactness*. The two, however, are indeed equivalent.

We will prove that covering compactness implies sequential compactness next time, and you will show on the homework (with the help of a discussion problem) that sequential compactness implies covering compactness. (This latter direction is the harder one.) But, for now we see that we can restate the Bolzano-Weierstrass theorem as essentially the claim that closed intervals $[a, b]$ are sequentially compact. Indeed, suppose (x_n) is a sequence in $[a, b]$. Then in particular $[a, b]$ is bounded, so the Bolzano-Weierstrass theorem says that (x_n) has a convergent subsequence (x_{n_k}). If $p \in \mathbb{R}$ denotes the limit of this subsequence, then since $[a, b]$ is closed in $\mathbb{R}$ and each x_{n_k} is in $[a, b]$, we have that the limit p must be in $[a, b]$ as well, so that (x_n) does have a convergent subsequence *in* $[a, b]$, and so $[a, b]$ is sequentially compact. Going the other way, if we know that closed intervals are always sequentially compact, then given a bounded sequence (x_n) in $\mathbb{R}$, take a closed interval containing all x_n. By sequential compactness, this sequence (x_n) then has a convergent subsequence (whose limit is in the same closed interval), which gives the Bolzano-Weierstrass Theorem.

Lecture 16: Sequential Compactness

Warm-Up. Suppose (p_n, q_n) is a sequence in a rectangle $[a, b] \times [c, d]$ in $\mathbb{R}^2$. We show that (p_n, q_n) has a convergent subsequence in $[a, b] \times [c, d]$. Here is a first attempt, that does not work. Since each (p_n, q_n) is in $[a, b] \times [c, d]$, each p_n is in $[a, b]$ and each q_n is in $[c, d]$. Thus, by the Bolzano-Weierstrass theorem, the sequence (p_n) has a convergent subsequence (p_{n_k}) in $[a, b]$, and the sequence (q_n) has a convergent subsequence (q_{n_ℓ}) in $[c, d]$. The problem is that these two do *not* necessarily produce a convergent subsequence of (p_n, q_n): terms like (p_{n_k}, q_{n_ℓ}) are not necessarily among the (p_n, q_n) since the indices n_k and n_ℓ might be different. For example, it might be that the subsequence (p_{n_k}) actually consists of the odd-indexed terms (p_{2n-1}), and the subsequence (q_{n_ℓ}) might be the one with even-indexed terms (q_{2n}), but then

$$(p_1, q_2),\ (p_3, q_4),\ (p_5, q_6),\ \ldots$$

are not among

$$(p_1, q_1),\ (p_2, q_2),\ (p_3, q_3),\ \ldots.$$

We need a way to force the indices of the p's and q's to match up, since this is what is required in our original sequence (p_n, q_n).

Here is the fix. As above, take a convergent subsequence (p_{n_k}) of (p_n), converging to, say, $p \in [a, b]$. Now consider the corresponding sequence (q_{n_k}) of y-coordinates in $[c, d]$. This sequence is bounded, so *this* sequence has a convergent subsequence $(q_{n_{k_\ell}})$ in $[c,d]$, converging to, say, $q \in [c,d]$. (So, the $q_{n_{k_\ell}}$ form a "sub-subsequence" of the original q_n's.) Now go back and take the corresponding x-coordinates $(p_{n_{k_\ell}})$, which still converges to $p \in [a,b]$ since it is a subsequence of the convergent sequence (p_{n_k}). We thus get a subsequence $(p_{n_{k_\ell}}, q_{n_{k_\ell}})$ of the original (p_n, q_n) converging to $(p,q) \in [a,b] \times [c,d]$, as desired.

Heine-Borel via sequences. The Warm-Up shows that $[a,b] \times [c,d]$ is sequentially compact, which we recall means that any sequence in that space contains a subsequence converging in that same space. The same reasoning as in the Warm-Up shows more generally that boxes (or k-cells) in $\mathbb{R}^k$ are compact: given a sequence $(x_{1n}, x_{2n}, \dots, x_{kn}$ in

$$[a_1,b_1] \times [a_2,b_2] \times \cdots \times [a_k,b_k],$$

we take first a convergent subsequence of the x_{1n}'s in $[a_1,b_1]$, then look at the corresponding second coordinates and take a convergent "sub-subsequence" of the x_{2n}'s, then look at the corresponding third coordinates and take a convergent "sub-sub-subsequence" of the x_{3n}'s, and so on. (Said another way, this shows that the Bolzano-Weierstrass theorem holds in $\mathbb{R}^k$: any bounded sequence in $\mathbb{R}^k$ has a convergent subsequence.)

This in turns give a sequence-based proof of the Heine-Borel theorem, under the assumption that covering and sequential compactness are equivalent. Indeed, suppose $K \subseteq \mathbb{R}^k$ is closed and bounded. Let (p_n) be a sequence in K. Since K is bounded, there is a k-cell which contains it, and which thus contains (p_n) as well. By the generalization of the Warm-Up, (p_n) has a convergent subsequence in the k-cell, and since K is closed the limit of this convergent subsequence is in K. Hence K is sequentially compact, and thus covering compact too.

Covering implies sequential. We now show that covering compactness implies sequential compactness. (As we said last time, the other direction is harder, but you will work through it on the homework.) So, suppose $K \subseteq X$ is covering compact. The key is a result we proved in a Warm-Up a while back: every infinite subset of K has a limit point. (This property is often called *limit point compactness*, and the upshot is all of these notions of compactness—covering, limit point, and sequential—are all equivalent.)

Let (p_n) be a sequence in K. If the set $\{p_n \mid n \in \mathbb{N}\}$ of terms of the sequence is finite (take note: the sequence contains infinitely many terms, but the *elements* which make up those terms do not have to be distinct), then there is at least one element p which occurs infinitely often in our sequence. (This is an application of the pigeonhole principle!) The subsequence consisting of these repeating p's is then constant, so it converges and hence (p_n) has a convergent subsequence in K in this case.

Otherwise assume $\{p_n \mid n \in \mathbb{N}\}$ is infinite. By limit point compactness, this set has a limit point p. Our goal is to get a subsequence of the original (p_n) which converges to this p. If we have this, then since K is closed in X (compact always implies closed), $p \in K$ and (p_n) will have a convergent subsequence in K as desired. Now, to get the subsequence we want we mimic the proof from a few days ago that limit points are characterized by having sequences of distinct elements converging to them. Pick some p_{n_1} from our sequence in $B_1(p)$, which exists by the definition of limit point. Next pick p_{n_2} from our sequence in the ball of radius

$$\min\left\{\frac{1}{2}, d(p_1,p), d(p_2,p), \dots, d(p_{n_1},p)\right\} > 0$$

around p. The reason for including the distances $d(p_1,p), \dots, d(p_{n_1},p)$ here is to guarantee that p_{n_2} is not among $p_1, \dots, p_{n_1}$—since it is closer to p than any of these points—so that p_{n_2} really does come after p_{n_1} in our original sequence, which we need in order to get a *subsequence* in the end. Then similarly pick p_{n_3} from our original sequence in the ball of radius

$$\min\left\{\frac{1}{3}, d(p_1,p), d(p_2,p), \dots, d(p_{n_2},p)\right\} > 0,$$

which is guaranteed to be further along in our original sequence than p_{n_2}. And so on, at the k-th stage pick p_{n_k} from our original sequence which is within $\frac{1}{k}$ away from p and also closer to p than any term from p_1 through $p_{n_{k-1}}$. The resulting p_{n_k}'s form a subsequence of (p_n) satisfying

$$d(p_{n_k},p) < \frac{1}{k},$$

which implies that $p_{n_k} \to p \in K$. This thus shows that K is sequentially compact.

Lecture 17: Cauchy Sequences

Warm-Up. We show that every sequence in $\mathbb{R}$ has a monotone subsequence. Consider the collection of all indices n such that x_n is greater than or equal to everything coming after it. There are two possibilities, either there are infinitely many such indices or finitely many.

If there are infinitely many such indices we can list them in increasing order:

$$n_1 < n_2 < n_3 < \dots.$$

Then by the property these indices satisfy we have

$$x_{n_1} \geq x_{n_2} \geq x_{n_3} \geq \dots,$$

since each x_{n_i} is greater than or equal to all terms coming after it. This gives a decreasing subsequence of (x_n).

If there are finitely many such indices pick $m_1 \in \mathbb{N}$ larger than them all. Then x_{m_1} is not greater than or equal to everything coming after it, so there is some x_{m_2} beyond x_{m_1} (so with $m_1 < m_2$) such that $x_{m_1} < x_{m_2}$. Similarly, m_2 is not among the indices considered above, so x_{m_2} is not greater than or equal to everything coming after it, and hence there is some $m_2 < m_3$ such that $x_{m_2} < x_{m_3}$. Continuing in this manner gives a sequence of indices $m_1 < m_2 < m_3 < \dots$ such that

$$x_{m_1} < x_{m_2} < x_{m_3} < \dots,$$

which gives an increasing sequence of (x_n) in this case. Thus either way, (x_n) has a monotone subsequence.

Bolzano-Weierstrass revisited. We can now give another (quick!) proof of the Bolzano-Weierstrass theorem. (We previously gave a proof using compactness of $[a,b]$.) Take a bounded (x_n) sequence in $\mathbb{R}$. By the Warm-Up, this sequence has a monotone subsequence. But this monotone subsequence is still bounded, and hence converges by the Monotone Convergence Theorem. Hence (x_n) has a convergent subsequence as desired.

Cauchy sequences. The monotone convergence theorem is the first result we've seen that can guarantee a sequence converges without knowing what the limit will be ahead-of-time. But, this only applies to monotone (and bounded!) sequences. More generally, we can consider the following type of sequence: a sequence (p_n) in a metric space (X,d) is *Cauchy* if for any $\epsilon > 0$ there exists

$N \in \mathbb{N}$ such that $d(p_n, p_m) < \epsilon$ for $m, n \geq N$. Note that there is no mention a potential limit here: the definition uses only distances between two terms of (p_n).

The intuition is that a Cauchy sequence is one whose terms are getting "bunched" up closer and closer to one another: no matter how small an ϵ we take, eventually we come across terms in our sequence all of which are within ϵ of each other. Informally this suggests that the sequence appears as if it *should* converge, but whether or not it actually does is a bit more subtle. (In fact, in $\mathbb{R}$ it is indeed true that Cauchy sequences always converge, which highlights an important property of $\mathbb{R}$. We'll elaborate more on this next time.)

Example. The sequence $\left(\frac{1}{n}\right)$ in $\mathbb{R}$ is Cauchy. This sequence in fact converges, and we will show next time that convergent sequences are always Cauchy (intuitively, if the terms of a sequence are all approaching some fixed definite limit, then for sure they should be approaching one another as well), but let us give a direct proof that $(\frac{1}{n})$ is Cauchy here.

Let $\epsilon > 0$ and pick $N \in \mathbb{N}$ such that $\frac{2}{N} < \epsilon$. For any $m, n \geq N$, we have

$$\left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{1}{n} + \frac{1}{m} \leq \frac{1}{N} + \frac{1}{N} = \frac{2}{N} < \epsilon,$$

so $(\frac{1}{n})$ is Cauchy as claimed.

Another example. Let (x_n) be the sequence in $\mathbb{R}$ defined by

$$x_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n+1}}{n}.$$

We claim (x_n) is also Cauchy. (This sequence is actually related to the *alternating harmonic series*:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots .$$

Namely, (x_n) is the *sequence of partial sums* of this series. We will discuss series in detail later.)

To show (x_n) is Cauchy we must show that we can make $|x_m - x_n|$ smaller than ϵ past some index. Suppose $m \geq n$. To make the notation simpler, we set $m = n + k$ for some $k \geq 0$, so that what we want is some index N beyond which

$$|x_{n+k} - x_n| < \epsilon$$

holds for all $k \geq 0$. (Requiring this for all $k \geq 0$ is what guarantees that $m = n + k$ takes on all values beyond n, as needed in the definition of Cauchy.) The key point in this case is that the expression for x_{n+k} contains the expression for x_n plus some more terms. Indeed, we have

$$x_{n+k} = \underbrace{1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{n+1}}{n}}_{x_n} + \frac{(-1)^{n+2}}{n+1} + \frac{(-1)^{n+3}}{n+2} + \cdots + \frac{(-1)^{n+k}}{n+k-1} + \frac{(-1)^{n+k+1}}{n+k},$$

so that

$$x_{n+k} - x_n = \frac{(-1)^{n+2}}{n+1} + \frac{(-1)^{n+3}}{n+2} + \cdots + \frac{(-1)^{n+k}}{n+k-1} + \frac{(-1)^{n+k+1}}{n+k}$$

is the quantity whose absolute value we want to make smaller than ϵ. As a first attempt to do so, we can use the triangle inequality to get

$$|x_{n+k} - x_n| \leq \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+k-1} + \frac{1}{n+k} \leq \frac{k}{n+1},$$

where in the final step we use the fact that each of the k terms we had are all smaller than or equal to $\frac{1}{n+1}$. The problem is that k is suppose to take on all possible positive integer values here, in order to make $n+k$ take on all integer values beyond n, and there is no way we can make this final quantity smaller than ϵ if its numerator k is unbounded. What we need is a way to bound $|x_{n+k}-x_n|$ in a way which depends *only* on n but not k, so that our bound works for all k.

We can give the exact expression for $|x_{n+k}-x_n|$ as follows. Note that in

$$\frac{(-1)^{n+2}}{n+1}+\frac{(-1)^{n+3}}{n+2}+\cdots+\frac{(-1)^{n+k}}{n+k-1}+\frac{(-1)^{n+k+1}}{n+k},$$

the first term will be positive or negative depending on the value of $(-1)^{n+2}$, but whatever it is the subsequent terms will alternate in sign: if the first term is positive, then the second is negative, third positive, and so on; while if the first term is negative, the second is positive, third negative, and so on. If the first term were positive, then we subtract a *smaller* quantity, and then add back a positive value, but then subtract a quantity *smaller* than the one we added on, and so on. This shows that any expression of the form

$$\frac{1}{n+1}-\frac{1}{n+2}+\frac{1}{n+3}-\frac{1}{n+4}+\cdots+(-1)^{k+1}\frac{1}{n+k}$$

is positive, since we never subtract more than the positive value we had previously. The sum

$$\frac{(-1)^{n+2}}{n+1}+\frac{(-1)^{n+3}}{n+2}+\cdots+\frac{(-1)^{n+k}}{n+k-1}+\frac{(-1)^{n+k+1}}{n+k}$$

either equals this positive expression above or is its negative, so either way we get that the absolute value of $x_{n+k}-x_n$ is exactly this positive expression:

$$|x_{n+k}-x_n|=\frac{1}{n+1}-\frac{1}{n+2}+\frac{1}{n+3}-\frac{1}{n+4}+\cdots+(-1)^{k+1}\frac{1}{n+k}.$$

Now, this entire expression is smaller or equal to its firm term $\frac{1}{n+1}$, since as described above, as we begin to subtract and add subsequent terms we never subtract more than what we had previously. (So, we get a bound independent of k as desired!) Thus, for $\epsilon>0$, pick $N\in\mathbb{N}$ such that $\frac{1}{N}<\epsilon$. Then if $n\geq N$ and $k\geq 0$ we have

$$\begin{aligned}|x_{n+k}-x_n|&=\frac{1}{n+1}-\frac{1}{n+2}+\frac{1}{n+3}-\frac{1}{n+4}+\cdots+(-1)^{k+1}\frac{1}{n+k}\\&\leq\frac{1}{n+1}\\&\leq\frac{1}{n}\\&\leq\frac{1}{N}<\epsilon,\end{aligned}$$

which shows that (x_n) is Cauchy as claimed. (Once we know that Cauchy sequences in $\mathbb{R}$ always converge, we will know that this sequence (x_n) converges. In fact, it converges to $-\ln 2$, but proving this requires the theory of power series, which will be developed next quarter.)

Lecture 18: More on Cauchy

Warm-Up. Suppose that (p_n) is a Cauchy sequence in a metric space X and that (p_{n_k}) is a convergent subsequence. We show that (p_n) converges as well. It is not true that if an arbitrary sequence has a convergent subsequence then the original sequence must converge as well, so the assumption here that our original sequence is *Cauchy* is crucial. The intuition is that the Cauchy condition gives a way to compare terms in (p_n) to terms in (p_{n_k}), and the convergence condition gives a way to compare terms in (p_{n_k}) to its limit, so the triangle inequality gives a way to compare terms in our original sequence to this limit.

Say that $p_{n_k} \to p \in X$ and let $\epsilon > 0$. Since (p_n) is Cauchy there exists N such that

$$d(p_n, p_m) < \frac{\epsilon}{2} \text{ for } n \geq N.$$

Since $p_{n_k} \to p$ there exists K such that

$$d(p_{n_K}, p) < \frac{\epsilon}{2}.$$

By making K larger if necessary we may also assume that $n_k \geq N$, so that p_{n_K} is within the range of terms where the first inequality above holds. Then for $n \geq N$ we have

$$d(p_n, p) \leq d(p_n, p_{n_K}) + d(p_{n_K}, p) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

so $p_n \to p$ as claimed. (Again, informally: if the terms of (p_n) are bunching up near each other, and *some* terms among those actually approach some definite p, then all of (p_n) must approach p.)

Convergent implies Cauchy. We now show that convergent sequences are always Cauchy. Suppose that (x_n) converges to x in X and let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that

$$d(x_n, x) < \frac{\epsilon}{2} \text{ for } n \geq N.$$

Hence if $m, n \geq N$ we have

$$d(x_m, x_n) \leq d(x_m, x) + d(x, x_n) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

so (x_n) is Cauchy as was to be shown.

It is crucial to understand, however, that the converse of this result is not true: Cauchy does not necessarily imply convergent. The issue, essentially, is that the candidate "limit" of the sequence might be missing. For example, take the metric space $\mathbb{Q}$ and a sequence of *rationals* r_n that converges to $\sqrt{2}$ in $\mathbb{R}$. (That such a sequence exists is the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$.) This sequence (r_n), viewed as a sequence in $\mathbb{R}$, is convergent and hence Cauchy. But the Cauchy definition does not care if we are working in $\mathbb{Q}$ or $\mathbb{R}$ as long as we use the same metric, so (r_n) is also Cauchy in $\mathbb{Q}$. But, (r_n) does not converge in $\mathbb{Q}$, since the thing $\sqrt{2}$ to which it would have to converge is not in $\mathbb{Q}$. So (r_n) is a Cauchy sequence in $\mathbb{Q}$ that does not converge.

Completeness. We say that a metric space X is *complete* if every Cauchy sequence in X converges in X. (The convergence "in" X is the important part.) So, $\mathbb{Q}$ is not complete, but we will show in a bit that $\mathbb{R}$ is complete. (In fact, this use of the word "complete" when it comes to $\mathbb{R}$ is actually equivalent to the previous usage in terms of existence of supremums. The homework problem asking to prove the equivalence between the monotone convergence theorem and the least upper bound property is essentially the reason why.)

The intuition is that $\mathbb{Q}$ fails to be complete because it is "missing" elements which would arise as "limits" of certain Cauchy sequences, but once we throw those limits in to obtain $\mathbb{R}$, we now have something complete. In fact, this same type of idea can be used to show that *any* metric space can be "completed" in a similar way: if X is a metric space, there always exists a complete metric space containing X as a subspace. The "smallest" complete metric space with this property is called the *completion* of X, so for example $\mathbb{R}$ is the completion of $\mathbb{Q}$. (We don't define what "smallest" means in this context, but one way to state what the completion should be is that it is the complete metric space containing X as a *dense* subset.)

Proving that every metric space has a completion is not something we'll get into, but there are some notes on my website which go into details if you'd really like to see how it works. We essentially construct the completion of X as the space whose *points* are the Cauchy sequences of X, and then consider "Cauchy sequences of Cauchy sequences" to show that the result is complete. Good stuff, but not essential for our purposes so we'll skip it.

Cauchy implies bounded. Before proving that $\mathbb{R}$ is complete, we need one more property of Cauchy sequences: the fact that they are always bounded. This is would certainly be a consequence of knowing that Cauchy sequences are convergent, since convergent sequences are always bounded, but here we prove this first on the way towards proving completeness of $\mathbb{R}$.

The proof is essentially the same as the one we gave for why convergent sequences are always bounded, only stated without using a limit in mind. Suppose (p_n) is Cauchy in X. Then there exists $N \in \mathbb{N}$ such that

$$d(p_n, p_m) < 1 \text{ for all } m, n \geq N.$$

In particular then, $p_n \in B_1(p_N)$ for all $n \geq N$. Then we simply enlarge the radius of this ball if necessary to ensure it includes all terms of our sequence before p_N: for

$$r = \min\{d(p_1, p_N), d(p_2, p_N), \ldots, d(p_{N-1}, p_N)\} + 1 > 0,$$

we indeed have $p_n \in B_r(p_N)$ for all n. Hence all terms in (p_n) belong to some ball of finite radius, so (p_n) is bounded as claimed.

$\mathbb{R}^n$ is complete. We now prove that $\mathbb{R}$ is complete. The proof is quick, after we put various things we've done together! Suppose (x_n) is a Cauchy sequence in $\mathbb{R}$. Then (x_n) is bounded, so it has a convergent subsequence by Bolzano-Weierstrass. But the Warm-UP from today show's that any Cauchy sequence with a convergent subsequence must itself converge, so (x_n) converges in $\mathbb{R}$ and hence $\mathbb{R}$ is complete. Don't let the shortness of this proof fool you, however: the fact that $\mathbb{R}$ is a complete is a deep result that depends on other deep results, most importantly the Bolzano-Weierstrass theorem.

Going up a dimension, we can now show that $\mathbb{R}^2$ is complete. Indeed, suppose (p_n, q_n) is a Cauchy sequence in $\mathbb{R}^2$. The key point is that then both (p_n) and (q_n) are Cauchy in $\mathbb{R}$. This comes from the fact that

$$|p_n - p_m| \leq \sqrt{(p_n - p_m)^2 + (q_n - q_m)^2}$$

and similarly for $|q_n - q_m|$; once we make the square root above smaller than ϵ, that will make the distance between the x- or y-coordinates smaller than ϵ as well. Since $\mathbb{R}$ is complete, both (p_n) and (q_n) converge in $\mathbb{R}$, say to p and q respectively, and then (p_n, q_n) converges to (p, q), so that every Cauchy sequence in $\mathbb{R}^2$ converges. In fact, the same reasoning applied to more coordinates, shows that $\mathbb{R}^n$ is complete in general.

Compact implies complete. We finish with one more general observation, namely the fact that compact spaces are always complete. This is really just the Warm-Up again. Suppose X is compact and that (p_n) is a Cauchy sequence in X. By (sequantial) compactness, X has a convergent subsequence in X, but then by the Warm-Up this means that (p_n) must converge as well. Hence every Cauchy sequence in X converges in X, so X is complete.

Lecture 19: Limits of Functions

Warm-Up. A *series* is an infinite sum

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

of real numbers a_n. (We will discuss series in more detail at the end of this quarter, so here we are only giving a flavor of this topic.) To say that a series *converges* to $S \in \mathbb{R}$ means that the sequence (s_n) formed by taking the *partial sums*

$$s_n = a_1 + \cdots + a_n$$

of the series converges to S in the usual sense of sequence convergence. (The intuition is that in order for the infinite sum $a_1 + a_2 + a_3 + \cdots$ to "equal" S it had better be the case that the sums obtained by adding one more a_n at each step should be getting closer and closer to S.) We show that the series $\sum a_n$ above converges if and only if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|a_n + a_{n+1} + \cdots + a_m| < \epsilon \text{ for all } m \geq n \geq N.$$

(This says that portion of our sum occurring between the m-th and n-th terms should be getting smaller and smaller the further and further we go in $a_1 + a_2 + a_3 + \cdots$. That is, in order for this infinite sum to exist as a finite value, the contribution from any number of terms we are adding on should get more and more negligible as we go.)

But the condition we want above is precisely a Cauchy condition! In other words, if we take

$$s_m = a_1 + a_2 + \cdots + a_m \quad \text{and} \quad s_{n-1} = a_1 + a_2 + \cdots + a_{n-1}$$

for $m \geq n$, with all terms making up s_{n-1} occurring among the terms making up s_m, then the sum $a_n + \cdots + a_m$ is precisely their difference:

$$s_m - s_{n-1} = a_n + a_{n+1} + \cdots + a_m.$$

The condition that we can make $|s_m - s_{n-1}|$ smaller than epsilon for all $m \geq n$ past some index just says that the sequence (s_n) of partial sums is Cauchy, so that this problem is just a consequence of the fact that $\mathbb{R}$ is complete when applied to the Cauchy sequence of partial sums of the series.

This result is often called the *Cauchy criterion* for convergence of a series, and is useful because it gives a way to show a series converges without knowing the value of its sum, which can be difficult to determine in general. In this setting, the example we looked at previously

$$x_n = 1 - \frac{1}{2} + \frac{1}{3} + \cdots + \frac{(-1)^{n+1}}{n}$$

of a Cauchy sequence in $\mathbb{R}$ is just the sequence of partial sums of the series $\sum \frac{(-1)^{n+1}}{n}$, so that example amounts to showing that this series converges. Another example you've seen before which fits into this framework is that of the sequence

$$1 + x + x^2 + \cdots + x^n$$

where $|x| < 1$. A homework problem, phrased in terms of convergence in the metric space C_b of bounded functions, showed essentially that this sequence converges to $\frac{1}{1-x}$. But this sequence is precisely the sequence of partial sums of the *geometric series* $\sum x^n$, so we find that this series converges to $\frac{1}{1-x}$ for $|x| < 1$. Again, we'll come back to this when we discuss series later.

Limits of functions. Most of the remaining time this quarter will focus on properties of *functions* between metric spaces. The study of functions will take up the bulk of MATH 321-2, and so here we begin to lay the groundwork. The main properties we care about will be *continuity* and *differentiability*, both of which depend on the notion of a *limit* of a function.

Suppose $f : E \subseteq X \to Y$ is a function defined on a subset E of a metric space X, taking values in a metric space Y. If $p \in X$ is a limit point of E, we say that the *limit of f as x approaches p* is $q \in Y$ if for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$\text{if } 0 < d_X(x,p) < \delta, \text{ then } d_Y(f(x),q) < \epsilon.$$

(Here we use d_X to denote the metric on X and d_Y the metric on Y.) If such $q \in Y$ exists, we use the notation

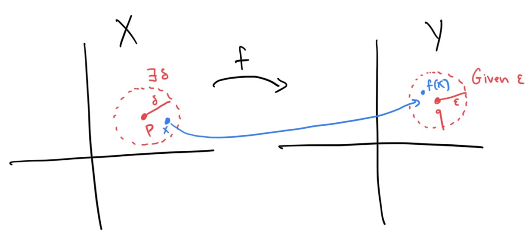
$$\lim_{x \to p} f(x) = q$$

to denote it. The basic intuition is the same as in calculus: as the input x gets closer and closer to $p \in X$, the output $f(x)$ should be getting closer and closer to $q \in Y$.

Let us digest the definition. First, we require that p be a limit point of E simply so that it makes sense to approach p using elements of E. Second, by saying that $0 < d_X(x,p)$, we exclude $x = p$ from consideration, so that the limit as we approach p never depends on the behavior of f *at* p (indeed, f might not even be defined at p if $p \notin E$), only on the behavior of f *near* p. Now, using open balls we can rephrase the "ϵ-δ" condition in the definition as

$$\text{if } x \in B_\delta(p) \text{ and } x \neq p, \text{ then } f(x) \in B_\epsilon(q).$$

Thus the definition says the following: given any open ball around the candidate limit value q, there exists an open ball around the point p we are approaching so that all x in this ball—apart from possibly p itself—are sent into the given ball around q:



That is, given any measure for how close we want to end up to q, we can find a measure for how close we need to be to p in order to guarantee that we do end up within that measure of "closeness" away from q. As the measure ϵ gets smaller perhaps we have to come in closer (via a smaller δ) to p to end up within ϵ from q, but we are guaranteed that we can always do so.

Example. Let us look at a key example which illustrates well how to work with the ϵ-δ definition of a limit in general. We claim that for any $a \in \mathbb{R}$,

$$\lim_{x \to a} x^3 = a^3.$$

(So, we are considering the limit of the function $f : \mathbb{R} \to \mathbb{R}, f(x) = x^3$ as x approaches a.) This is the type of thing you might expect to be true from a calculus course, but let us actually prove it.

Let $\epsilon > 0$. We want $\delta > 0$ such that

$$\text{if } 0 < |x-a| < \delta, \text{ then } |x^3 - a^3| < \epsilon.$$

First, we need to somehow bound the expression we want to make smaller than ϵ by something in terms of $|x-a|$ since this is the only quantity we can control, by controlling δ. In this case, we have

$$|x^3 - a^3| = |x^2 + ax + a^2||x-a|.$$

The naive choice of

$$\delta = \frac{\epsilon}{|x^2 + ax + a^2|}$$

does not work since δ should not depend on x, since, after all, the x we consider in the definition are themselves determined by the choice of δ. In

$$|x^3 - a^3| = |x^2 + ax + a^2||x-a|,$$

the first factor on the right will be smaller than δ, so what we need then is to find a *constant* bound on the second factor.

Let us assume for the time being that $|x-a| < 1$. (Later we will shrink δ if necessary in order to guarantee that this holds.) Then $|x| - |a| \leq |x-a| < 1$, so

$$|x| < 1 + |a|, \text{ and thus } |x^2 + ax + a^2| \leq |x|^2 + |a||x| + |a|^2 < (1+|a|)^2 + |a|(1+|a|) + |a|^2.$$

Thus when $|x-a| < 1$, we get

$$|x^3 - a^3| = |x^2 + ax + a^2||x-a| \leq [(1+|a|)^2 + |a|(1+|a|) + |a|^2]|x-a|.$$

The fact that we now have our bound of the form "constant times $|x-a|$" is good, since we can now make this smaller than ϵ by picking δ to be

$$\delta = \frac{\epsilon}{(1+|a|)^2 + |a|(1+|a|) + |a|^2}.$$

(Note that this expression is defined since the denominator is strictly positive.) These is a somewhat messy looking expression, but the exact expression for δ does not really matter—what matters is that such a value exists.

However, there is one final wrinkle. If we then pick x satisfying $|x-a| < \delta$ for this δ, we are not guaranteed to also have $|x-a| < 1$ as a consequence. This is bad since $|x-a| < 1$ is an assumption we used in deriving our bounds in the first place, so that these bounds will not apply without knowing that $|x-a| < 1$. The fix is to make δ smaller if need be, and actually pick δ to be the *minimum* of 1 (so that the bounds we use work) and the expression above in terms of ϵ (so that the final quantity will be smaller than ϵ). Picking a minimum of such δ's is the "ϵ-δ" analog of picking a maximum of indices in the previous sequence convergence proofs we looked at. (Moreover, using 1 as the constant in $|x-a| < 1$ is not important, in that we could have used $|x-a| < 2$ or $|x-a| < 1000$ instead and still made it work. We would get different constant bounds in the end, and hence different expressions for δ, but that's fine. Choosing to use 1 is just a standard convenient choice.)

Here is our final proof. Let $\epsilon > 0$ and set

$$\delta = \min\left\{1, \frac{\epsilon}{(1+|a|)^2 + |a|(1+|a|) + |a|^2}\right\},$$

which is positive. Suppose that $0 < |x - a| < \delta$. Since $\delta \leq 1$ we have $|x - a| < 1$ so

$$|x| < 1 + |a|, \text{ and thus } |x^2 + ax + a^2| \leq |x|^2 + |a||x| + |a|^2 < (1 + |a|)^2 + |a|(1 + |a|) + |a|^2.$$

Hence

$$|x^3 - a^2| = |x - a||x^2 + ax + a^2| < \delta[(1 + |a|)^2 + |a|(1 + |a|) + |a|^2].$$

By the choice of δ, this expression is smaller than or equal to ϵ, so $|x^3 - a^3| < \epsilon$ and we conclude that $\lim_{x \to a} x^3 = a^3$ as claimed.

Towards continuity. The same type of argument as that above can be used to show more generally that

$$\lim_{x \to a} x^n = a^n$$

for $n \in \mathbb{N}$. (Here you use the identity $|x^n - a^n| = |x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + xa^{n-2} + a^{n-1}||x - a|$.) Here is a definition we look at more carefully next time: a function f is *continuous* at a if

$$\lim_{x \to a} f(x) = f(a).$$

That is, for a continuous function the value of a limit as we approach a point should just be the value of the function at that point. In this language, what we showed in our example is that the function $f(x) = x^3$ is continuous at all $a \in \mathbb{R}$, and the remark above says that $f(x) = x^n$ is in general continuous at all points as well.

Lecture 20: Continuous Functions

Warm-Up. Suppose $f, g, h : X \to \mathbb{R}$ are real-valued functions on a metric space X such that $f(x) \leq g(x) \leq h(x)$ for all $x \in X$. We prove the *squeeze theorem*: If $\lim_{x \to p} f(x)$ and $\lim_{x \to p} h(x)$ both exist and are equal, then $\lim_{x \to p} g(x)$ exists and has this same value. Let us denote the limit of f and h by $L \in \mathbb{R}$. Note that since $f(x) \leq g(x) \leq h(x)$ for all $x \in X$, we also have

$$f(x) - L \leq g(x) - L \leq h(x) - L,$$

so

$$|g(x) - L| \leq \max\{|f(x) - L|, |h(x) - L|\} \text{ for all } x \in X.$$

Thus let $\epsilon > 0$ and pick $\delta_1, \delta_2 > 0$ such that

$$|f(x) - L| < \epsilon \text{ whenever } 0 < d_X(x, p) < \delta_1$$

and

$$|h(x) - L| < \epsilon \text{ whenever } 0 < d_X(x, p) < \delta_2.$$

If $0 < d_X(x, p) < \min\{\delta_1, \delta_2\}$, then

$$|g(x) - L| \leq \max\{|f(x) - L|, |h(x) - L|\} < \epsilon$$

since both $|f(x) - L|$ and $|h(x) - L|$ are smaller than ϵ for such x. This shows that $\lim_{x \to p} g(x) = L$ as claimed. (The intuition is that $g(x)$ is "squeezed" between two values which are each approaching L, so $g(x)$ should approach L as well.)

Limits via sequences. The notion of the limit of a function can be characterized via sequences as follows. Recall our notation: $f : E \subseteq X \to Y$ is a function defined on a subset E of X and

$p \in X$ is a limit point of E. We claim that $\lim_{x\to p} f(x) = q$ in the ϵ-δ sense if and only if whenever (x_n) is a sequence of points in E, none of which equal p, converging to p, then $(f(x_n))$ converges to q in Y. This matches our intuition: as x_n gets closer and closer to p, the values $f(x_n)$ of the function should get closer and closer to the limit q. Excluding p from being one of the x_n's just says, as usual, that the behavior of f at p should play no role in the definition of a limit.

For the forward direction, suppose $\lim_{x\to p} f(x) = q$ and let $\epsilon > 0$. Then there exists $\delta > 0$ such that

$$\text{if } 0 < d_X(x,p) < \delta, \text{ then } d_Y(f(x),q) < \epsilon.$$

Suppose (x_n) is a sequence of points in E not equal to p and converging to p. Then in particular, there exists $N \in \mathbb{N}$ such that

$$d_X(x_n,p) < \delta \text{ for } n \geq N.$$

But these points also satisfy $0 < d_X(x_n,p)$ since $x_n \neq p$, so by the choice of δ we have that as a consequence

$$d_Y(f(x_n),q) < \epsilon \text{ for } n \geq N.$$

This means that $(f(x_n))$ converges to q in Y as desired.

For the backwards direction, we argue by contrapositive. Suppose $\lim_{x\to p} f(x) \neq q$. Then there exists $\epsilon > 0$ such that for all $\delta > 0$, there exists $x \in E$ satisfying

$$0 < d_X(x,p) < \delta \text{ but } d_Y(f(x),q) \geq \epsilon.$$

In particular, pick such a point $x_n \in E$ for each $\delta = \frac{1}{n}$ with $n \in \mathbb{N}$. The resulting sequence (x_n) satisfies

$$0 < d_X(x_n,p) < \frac{1}{n} \text{ and } d_Y(f(x_n),q) \geq \epsilon.$$

The first inequality implies that $x_n \to p$, and the second that $f(x_n) \not\to q$ since the $f(x_n)$ get no closer than $\epsilon > 0$ away from q. Thus there exists a sequence converging to p, none of whose elements equal p, but for which the image sequence does not converge to q, which gives the contrapositive of the backwards direction.

Continuous functions. A function $f : X \to Y$ is said to be *continuous* at $p \in X$ if

$$\lim_{x\to p} f(x) = f(p).$$

We say that f is *continuous* on X if it is continuous at all points of X. The notion of a continuous function is perhaps the most important in all of analysis. The definition says that the limit of the function as you approach a point should be precisely the value of the function *at* that point. Thus for a continuous function, the value of the function at a point is *completely* determined by its values at points nearby: the value of $f(p)$ can be recovered from the values of $f(x)$ for x "close" to p via the limit $\lim_{x\to p} f(x)$.

If we write out the limit statement above in ϵ-δ form, we get the ϵ-δ definition of continuity: f is continuous at p if for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$\text{if } d_X(x,p) < \delta, \text{ then } d_Y(f(x),f(p)) < \epsilon.$$

(Note that we no longer have to require that $0 < d_X(x,p)$ here, since $d_X(p,p) < \delta$ is always true that $d_Y(f(p),f(p)) < \epsilon$ is always true as well.) Here then is the practical takeaway: for a continuous function, we can control how far away from $f(p)$ the point $f(x)$ is by controlling instead how far away from p the point x is, or informally, we can control distances between outputs by controlling distances between inputs. Having this type of "control" is what makes continuity such a useful concept.

Using the sequential approach to limits of functions, we then get the following equivalent sequential definition of continuity: f is continuous at p if and only if whenever $x_n \to p$ in X, then $f(x_n) \to f(p)$ in Y. This says that continuous functions are precisely the ones which "preserve" limits of convergent sequences. This version of the definition also matches some intuition we might expect: changing the input of a continuous function by a small amount should also change the value by only a (relatively) small amount, since as x_n gets close to p, $f(x_n)$ gets close to $f(p)$.

Example. The example we did last time of $\lim_{x\to a} x^3 = a^3$ shows that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ is continuous at all points. This is an important example to understand since it provides a good illustration of how to work with the ϵ-δ definition of continuity in general.

We can also justify this fact using sequences by piecing together things we've done previously. Namely, suppose $x_n \to a$ in $\mathbb{R}$. Then by what showed earlier about products of convergent sequences, we have that

$$x_n x_n x_n \to aaa,$$

or in other words $x_n^3 \to a^3$, which is what continuity of $f(x) = x^3$ requires.

Arithmetic and compositions. Using the same types of arithmetic sequences properties we saw before (i.e. taking sums, products, and reciprocals), we immediately get that sums and products of continuous functions are continuous. Also, reciprocals of nonzero continuous functions are continuous as well.

Compositions of continuous functions are also continuous. That is, if $f : X \to Y$ is continuous at $p \in X$ and $g : Y \to Z$ is continuous at $f(p) \in Y$, then $g \circ f : X \to Z$ is continuous at p. Indeed, let $\epsilon > 0$. Since g is continuous at $f(p)$, there exists $\delta' > 0$ such that

$$d_Y(y, f(p)) < \delta' \implies d_Z(g(y), g(f(p))) < \epsilon.$$

But then since f is continuous at p, there exists $\delta > 0$ such that

$$d_X(x, p) < \delta \implies d_Y(f(x), f(p)) < \delta'.$$

Thus if $d_X(x, p) < \delta$, we have $d_Y(f(x), f(p)) < \delta'$, which by the choice of δ' implies that

$$d_Z(g(f(x)), g(f(p))) < \epsilon,$$

which shows that $g \circ f$ is continuous at p as claimed.

Another example. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ by the function defined by

$$f(x, y) = (x^3 + y, xy, 2y - x).$$

We claim that this is continuous. Indeed, fix $(p, q) \in \mathbb{R}^2$ and suppose $(p_n, q_n) \to (p, q)$ in $\mathbb{R}^2$. We need to show that

$$f(p_n, q_n) = (p_n^3 + q_n, p_n q_n, 2q_n - p_n) \to f(p, q) = (p^3 + q, pq, 2q - p).$$

By what we know about convergent sequences in $\mathbb{R}^n$, to show this we need only show that each component sequence on the left converges to the corresponding component on the right. But this just again follows from some arithmetic properties: since $p_n \to p$ and $q_n \to q$ in $\mathbb{R}$, we have

$$p_n^3 \to p, \text{ so } p_n^3 + q_n \to p^3 + q,$$

and we have $p_nq_n \to pq$ and $2q_n - p_n \to 2q - p$. Hence by componentwise convergence, $f(p_n, q_n) \to f(p,q)$, so f is continuous.

In fact, there was nothing special about the components $x^3 + y, xy, 2y - x$ of this particular functions. More generally, these could have been *any* continuous expressions, and moreover we can look at $\mathbb{R}^n$ for other n as well. The general fact is that $f : \mathbb{R}^m \to \mathbb{R}^n$ is continuous if and only if each component function of f is continuous, where by component functions we mean the functions $f_i : \mathbb{R}^m \to \mathbb{R}$ that give each component of the values of f:

$$f(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_n(\mathbf{x})).$$

Discrete example. As a final example, we determine the possible continuous functions $f : X \to Y$ in the case where X has the discrete metric. In fact, we claim that *all* functions with discrete domains are actually continuous! Indeed, let $\epsilon > 0$, fix $p \in X$, and set $\delta = \frac{1}{2}$. The continuity condition we need is

$$d_X(x,p) < \frac{1}{2} \implies d_Y(f(x), f(p)) < \epsilon.$$

But for the discrete metric the only possible x satisfying $d_X(x,p) < \frac{1}{2}$ is $x = p$, so this is the *only* point we need to check in the condition above. For this point it is certainly true that $d_Y(f(p), f(p)) < \epsilon$ simply because this distance is zero, so we have continuity at p.

If we phrase this in terms of sequences instead, it comes down to a property we saw previously: the only convergent sequences in a discrete space are those which are eventually constant. If $p_n \to p$ in X, then $p_n = p$ for all n past some N, which case $f(p_n) = f(p)$ for all n past N as well. But this means that the image sequence $(f(p_n))$ is eventually constant, so it converges to the eventual constant value $f(p)$, which is what we need in continuity. The upshot is that continuity of a function depends *heavily* on the metric being considered!

Lecture 21: More on Continuity

Warm-Up. Fix $a \in \mathbb{R}$ and consider the function $T : C_b(\mathbb{R}) \to \mathbb{R}$ which sends a function $f \in C_b(\mathbb{R})$ to its value at a:

$$T(f) = f(a).$$

We claim that this is continuous when we equip $C_b(\mathbb{R})$ with the sup norm. Let $\epsilon > 0$ and fix $g \in C_b(\mathbb{R})$. To show that T is continuous at g means that there should exist $\delta > 0$ such that

$$d(f,g) < \delta \implies |T(f) - T(g)| < \epsilon,$$

where d denotes the sup metric. Given the definition of T as "evaluation at a", what this actually means is that

$$d(f,g) < \delta \implies |f(a) - g(a)| < \epsilon.$$

But note that $|f(a) - g(a)|$ is precisely one of the values of which $d(f,g)$ is the supremum:

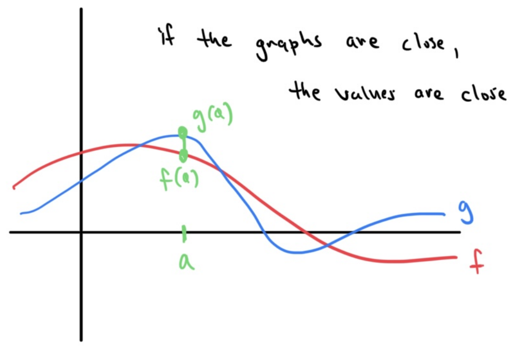
$$d(f,g) = \sup_{x \in \mathbb{R}} |f(x) - g(x)|.$$

This means that $|f(a) - g(a)| \leq d(f,g)$, so $\delta = \epsilon$ satisfies our requirement. That is, if $d(f,g) < \delta = \epsilon$, then

$$|T(f) - T(g)| = |f(a) - g(a)| \leq d(f,g) < \epsilon$$

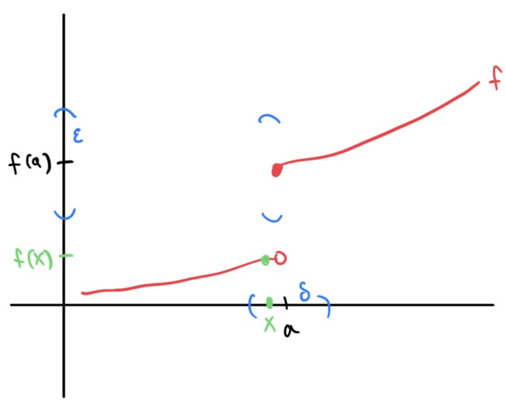
, so T is continuous at g. (The real tricky part of this problem is in understanding what the notation all means, since we are looking at a function that takes as inputs other functions! Thinking about "spaces of functions" in general takes some good effort to understand.)

The intuition behind this is as follows. Recall that, visually, $d(f, g)$ measures the "maximal" vertical distance between points on the graphs of f and g:



The value of $|T(f) - T(g)|$ is just one of these vertical distances, so continuity in this case just says that if the graphs of f and g are close to each other, then certainly the their values at any specific point are also close to each other.

Visualizing discontinuities. To get a basic sense of what it means for a function to *not* be continuous at a point, consider the following standard example of a function that fails to be continuous at a point:



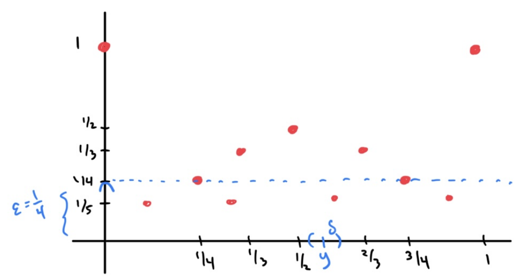
In calculus you would informally say that this function is discontinuous at $x = a$ due to the "jump" in the graph at $x = a$. To see that this in fact does not satisfy the formal ϵ-δ definition of continuity, consider the ϵ-interval around $f(a)$ drawn above. The claim is that no matter what $\delta > 0$ we take around a, it will never be true that all points within δ of a will be sent to points within ϵ of $f(a)$. Indeed, for any $\delta > 0$, picking a point x in the left half of the δ-interval around a gives a value $f(x)$ that lies further than ϵ away from $f(a)$, which is why continuity at a fails. (This is a good picture to have in mind, but is not completely illustrative of all ways in which a function can fail to be continuous, since there can also be "oscillatory discontinuous" in addition to "jump discontinuities". We'll some examples later.)

My favorite function. And now for my favorite example of all time, based on my favorite function of all time. (In every single analysis course I've ever taught, this is always a fundamental example since it illustrates, I think, incredibly well how to think about the ϵ-δ definition of continuity.) Note first that any rational number in $[0,1]$ can be written in a unique way as $\frac{p}{q}$ with p,q positive integers (except take $p=0$ and $q=1$ when the rational is zero) with no common factors apart from 1—simply "cancel" common factors in the numerator and denominator until you can't anymore. Define $f:[0,1]\to\mathbb{R}$ by

$$f(x)=\begin{cases}0 & \text{if } x\notin\mathbb{Q}\\ 1 & \text{if } x=0\\ \frac{1}{q} & \text{if } x=\frac{p}{q} \text{ where } p,q\in\mathbb{N} \text{ have no common factors}\end{cases}.$$

The claim is that this function is *discontinuous* at each rational in $[0,1]$, but actually *continuous* (!!!) at each irrational in $[0,1]$. (This is my favorite function since it exhibits seemingly strange continuity phenomena.)

You will prove this on the homework, but here is a quick word about how to think about the claim that f is continuous at each irrational, which is the harder part. Visually, this function has a graph that looks like



(Perhaps you can see why this function is often called the "popcorn" function.) Suppose we fix an irrational y and want to check the continuity condition for $\epsilon=\frac{1}{4}$. Then we need $\delta>0$ such that

$$|x-y|<\delta \implies |f(x)-f(y)|<\frac{1}{4}.$$

Since y is irrational, $f(y)=0$, so the second inequality is just $f(x)<\frac{1}{4}$. (The values of f are never negative, so we can just ignore the absolute value.) Certainly for irrational x in $(y-\delta,y+\delta)$, we have $f(x)=0<\frac{1}{4}$, so the point is that this continuity condition comes down to guaranteeing that *rationals* r in $(y-\delta,y+\delta)$ also satisfy $f(r)<\frac{1}{4}$.

But how many rationals r are there for which this is *not* true? The only possible values $f(r)$ can have are reciprocals like

$$\frac{1}{1},\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \frac{1}{5},\ \ldots,$$

and the first four here specifically are the only values of $f(r)$ for which $f(r)<\frac{1}{4}$ is not true. In order for $f(r)$ to be one of these values requires that the denominator of r be one of $1,2,3,4$, and thus the only values of r for which $f(r)<\frac{1}{4}$ is not true are

$$r=0,\ 1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{2}{3},\ \frac{1}{4},\ \frac{3}{4}.$$

Hence, by picking a small enough $\delta>0$ around y, we can make sure that none of these rationals are within δ from y, which implies that any rational satisfying $|r-y|<\delta$ for this δ will in turn satisfy $f(r) < \frac{1}{4}$, as continuity at y requires. That's the basic idea, and an argument along these lines works for any $\epsilon > 0$.

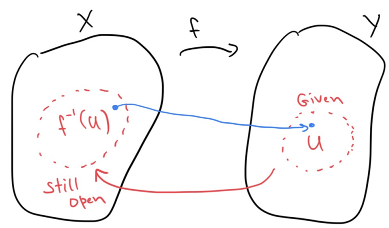
Continuity via topology. We have seen the definition of continuity phrased in terms of limits, ϵ-δ (which is just the same as the limit definition), and sequences, and now we give one more characterization, this time via open sets. The claim is that:

$f : X \to Y$ is continuous if and only if $f^{-1}(U)$ is open in X whenever U is open in Y.

Here, $f^{-1}(U)$ denotes the *preimage* of U under f, which is the set of all elements in the domain X which get sent to something in U:

$$f^{-1}(U) := \{x \in X \mid f(x) \in U\}.$$

Note the direction here: we start with an open subset of Y on the *right*, and "pull it back" to X via taking its preimage, and the result should still be open:



This might seem like a strange phrasing of continuity, and indeed the proof might be tricky to follow at first, but I claim that this is essentially the same as the ϵ-δ definition when phrased in terms of open balls. In particular, saying

$$q \in B_\delta(p) \implies f(q) \in B_\epsilon(f(p))$$

as the ϵ-δ definition requires means that q is in the preimage of $B_\epsilon(f(p))$, since $f(q)$ in the set $B_\epsilon(f(p))$ of which is the preimage is being taken. So $B_\delta(p)$ is an open ball around p which is fully contained in this preimage, which is indeed look like the definition of "open" applied to this preimage. The proof is tricky only because it requires jumping back and forth between various definitions, but the key point is that mentioned above.

Here is the proof. Suppose that f is continuous and that $U \subseteq Y$ is open. Let $p \in f^{-1}(U)$. Then $f(p) \in U$, so since U is open in Y there exists $\epsilon > 0$ such that $B_\epsilon(f(p)) \subseteq U$. Now, since f is continuous at p, there exists $\delta > 0$ such that

$$q \in B_\delta(p) \implies f(q) \in B_\epsilon(f(p)).$$

But since $B_\epsilon(f(p)) \subseteq U$, this says that anything in $B_\delta(p)$ is sent to something in U, so that all of $B_\delta(p)$ is contained in the preimage of U. Thus there exists an open ball $B_\delta(p) \subseteq f^{-1}(U)$, so $f^{-1}(U)$ is open in X.

Conversely suppose that the preimage of any open subset of Y is open in X. Let $p \in X$ and let $\epsilon > 0$. Since $B_\epsilon(f(p))$ is open in Y, its preimage $f^{-1}(B_\epsilon(f(p)))$ is open in X. Thus since $p \in f^{-1}(B_\epsilon(f(p)))$, there exists $\delta > 0$ such that $B_\delta(p) \subseteq f^{-1}(B_\epsilon(f(p)))$. This means that any $q \in B_\delta(p)$ is in the preimage of $B_\epsilon(f(p))$, so for any such q we have $f(q) \in B_\epsilon(f(p))$. Thus

$$q \in B_\delta(p) \implies f(q) \in B_\epsilon(f(p)),$$

showing that f is continuous at p.

After taking complements, we immediately also get a characterization of continuity in terms of *closed* sets, again in terms of taking preimages:

$$f : X \to Y \text{ is continuous if } f^{-1}(A) \text{ is closed in } X \text{ whenever } A \text{ is closed in } Y.$$

The proof uses the fact that A is closed if and only if A^c is open, and the fact that the operation of taking preimages behaves well with respect to complements in the sense that the preimage of a complement is the complement of the preimage: $f^{-1}(A^c) = (f^{-1}(A))^c$.

Example. Here is a typical application of this new (and final) characterization of continuity. Consider the set

$$S = \{x \in \mathbb{R} \mid x^3 + x > 2\}.$$

We claim that this is open in $\mathbb{R}$. Showing this directly is not impossible but takes some effort since it is not easy to describe x satisfying $x^3 + x > 2$ so concretely, and really any direct proof of this will end up essentially proving that $f(x) = x^3 + x$ is continuous anyway, in a roundabout way. The goal is to express S as being the preimage of an open set under a continuous function. Take $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3 + x$. The points in S are precisely those for which $f(x) > 2$, or in other words those for which $f(x) \in (2, \infty)$. Thus, S is the preimage of $(2, \infty)$ under this function:

$$S = f^{-1}((2, \infty)).$$

Since $(2, \infty)$ is open in $\mathbb{R}$ and f is continuous (each of x^3 and x are continuous, and sums of continuous functions are continuous), we get that S is also open in $\mathbb{R}$ as claimed. The practical point is that changing the value of an x satisfying $x^3 + x > 2$ by a small enough amount produces a number that still satisfies this same inequality. That is, given x such that $x^3 + x > 2$, there is an interval interval around x containing numbers y satisfying $y^3 + y > 2$.

In the same vein, the set of numbers satisfying $x^3 + x = 2$ is closed. (Of course, there are not many numbers that satisfy this—there's only a finite number!—but how many there actually are is not important here.) If we denote this set by A, then A is the preimage of $\{2\}$ under the same function as above:

$$A = f^{-1}(\{2\}).$$

Since $\{2\}$ is closed in $\mathbb{R}$, we thus get that A is closed in $\mathbb{R}$ as well.

Lecture 22: Continuity and Compactness

Warm-Up. Let $M_2(\mathbb{R})$ denote the set of 2×2 matrices with real entries. By thinking of a 2×2 matrix as a vector in $\mathbb{R}^4$ via

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightsquigarrow \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix},$$

we can equip $M_2(\mathbb{R})$ with the Euclidean metric by defining the distance between two matrices to be the Euclidean distance between the corresponding vectors in $\mathbb{R}^4$. We claim that then the set of *invertible* 2×2 matrices, commonly denoted by $GL_2(\mathbb{R})$:

$$GL_2(\mathbb{R}) := \{A \in M_2(\mathbb{R}) \mid A \text{ is invertible}\},$$

is open in $M_2(\mathbb{R})$. (A practical consequence of this result is that changing the entries of an invertible matrix by a small enough does not effect invertibility. The notation "GL" stands for "general linear", but why this name is used is beyond the scope of this course.) We prove this via the relation between continuous functions and open sets given last time.

The key point is that a matrix is invertible if and only if its determinant is nonzero, which in the 2×2 case means that

$$ad - bc \neq 0.$$

Note that the determinant here on the left is nothing but a polynomial expression in terms of the entires of the matrix, which implies that it is continuous with respect to the Euclidean metric on $M_2(\mathbb{R})$! That is, the function that sends a matrix to its determinant:

$$\det : M_2(\mathbb{R}) \to \mathbb{R} \text{ defined by } A \mapsto \det A$$

is continuous. (Said another way, the functions sending a vector in $\mathbb{R}^4$ to one of its coordinates are continuous, and $\det A$ is a difference of products of these continuous functions, so it is continuous too.) To say that $\det A \neq 0$ means that $\det A \in (-\infty, 0) \cup (0\infty)$, so we can express $GL_2(\mathbb{R})$ as the preimage of this union:

$$GL_2(\mathbb{R}) = \det{}^{-1}((-\infty, 0) \cup (0, \infty)).$$

Since det is continuous and $(-\infty, 0) \cup (0, \infty)$ is open in $\mathbb{R}$, $GL_2(\mathbb{R})$ is open in $M_2(\mathbb{R})$ as claimed.

Actually, the same holds for larger matrices as well. Viewing the entries of an $n \times n$ matrix as the entries of a vector in $\mathbb{R}^{n^2}$ gives a way to equip $M_n(\mathbb{R})$ (space of $n \times n$ matrices) with a metric. The determinant of an $n \times n$ matrix is still a polynomial expression (i.e. made up of sums and products) of the entries of the matrix, so the function which sends an $n \times n$ matrix to its determinant is still continuous, and the same reasoning as above shows that $GL_n(\mathbb{R})$ (space of invertible $n \times n$ matrices) is open in $M_n(\mathbb{R})$ as well.

Another important subset of $M_n(\mathbb{R})$ is $SL_n(\mathbb{R})$, which is the set of matrices of determinant 1:

$$SL_2(\mathbb{R}) := \{A \in M_n(\mathbb{R}) \mid \det A = 1\}.$$

("SL" stands for "special linear".) In this case, $SL_n(\mathbb{R})$ is *closed* in $M_n(\mathbb{R})$ since it is the preimage under det of the closed subset $\{1\}$ of $\mathbb{R}$. Practically, this means that if you have a convergent sequence of matrices of determinant 1, its limit also has determinant 1. Good stuff!

Watch the direction. As we noted last time, the characterization of continuity in terms of open and closed sets uses a "backwards" direction: given an open/closed subset of Y on the right, its preimage on the left (so, right to left) is open in X. The analogous claims in the "forward" direction are not true in that a continuous function does *not* automatically send open sets *to* open sets, nor closed sets to closed sets.

Here are some examples. First, take $f : \mathbb{R} \to \mathbb{R}$ to be a constant function, say $f(x) = 1$ for all x. This is continuous, but given any open set U in $\mathbb{R}$ (the domain), the image $f(U)$ consists of only 1, and $f(U) = \{1\}$ is not open in $\mathbb{R}$. So, the *image* (going forward) of an open set under a continuous function is not necessarily open. Second, take $g : (0, \infty) \to \mathbb{R}$ to be defined by $g(x) = \frac{1}{x}$. Then $[1, \infty)$ is closed in the domain, but its image is$f([1, \infty)) = (0, 1]$, which is not closed in $\mathbb{R}$. Hence again, the image of a closed set under a continuous function is not necessarily closed. The upshot is that the direction matters (right to left when taking preimages vs left to right when taking images) when asking about continuity in terms of open and closed sets!

Continuity and compactness. We now come to, truly, one of the most important results in all of analysis, which gives the effect of continuous functions on compact sets. The claim is that continuous functions send compact sets to compact sets:

If $f : X \to Y$ is continuous and $K \subseteq X$ is compact, then the image $f(K) \subseteq Y$ of K under f is also compact.

(Note there the direction here *does* go from left to right, as opposed to continuity in terms of open and closed sets.) We can see that this is true in some basic examples, say $f(x) = x^3$ on $\mathbb{R}$. The image of the compact interval $[0, 2]$ is $[0, 8]$, which is compact as well, and the image of $[-3, 1]$ is $[-27, 1]$, which is also compact. The main consequence of this result is what's called the *Extreme Value Theorem*, which we will get to after some proofs.

We give two proofs of this result, one using open covers and the other using sequences. Both approaches are important to understand. In both cases, the key is to take data on the codomain side on the right, and use it to get data on the domain side on the left, where we can then apply compactness of K. First, suppose that $\{U_\alpha\}$ is an open cover of $f(K)$, so that

$$f(K) \subseteq \bigcup_\alpha U_\alpha.$$

Since f is continuous, each preimage $f^{-1}(U_\alpha)$ is open in X, so the collection $\{f^{-1}(U_\alpha)\}$ forms an open cover of K:

$$K \subseteq \bigcup_\alpha f^{-1}(U_\alpha).$$

(To be clear, for any $p \in K$, $f(p) \in f(K)$ is in some U_α, so that p is then in some preimage $f^{-1}(U_\alpha)$.) Now, since K is compact, this open cover has a finite subcover, say:

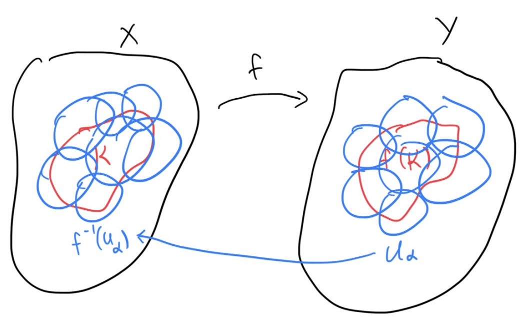
$$K \subseteq f^{-1}(U_1) \cup \cdots \cup f^{-1}(U_n),$$

which implies that

$$f(K) \subseteq U_1 \cup \cdots \cup U_n.$$

(Again, to be clear, anything in $f(K)$ is of the form $f(p)$ or some $p \in K$, and if this p is in $f^{-1}(U_i)$, $f(p)$ is in U_i.) Thus $\{U_1, \ldots, U_n\}$ is a finite subcover of the open cover $\{U_\alpha\}$ of $f(K)$, so $f(K)$ is compact as claimed.

Here is a picture illustrating the ideas above:



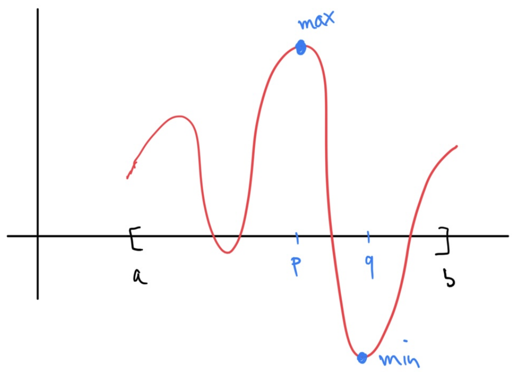
We begin with open sets on the right covering $f(K)$, take preimages to get open sets on the left covering K, apply compatness to reduce this to a finite number, which then also reduces the cover we had on the right. (The result goes left to right but the proof goes right to left!)

Now for sequences. Suppose (q_n) is a sequence in $f(K)$. Then each q_n can be written as $q_n = f(p_n)$ for some $p_n \in K$ since each q_n is in the image of K. Since K is compact, the sequence (p_n) in K has a convergent subsequence, say $p_{n_k} \to p \in K$. Since f is continuous, we then have $f(p_{n_k}) \to f(p) \in f(K)$, so $(f(p_{n_k}))$ is a convergent subsequence of $(q_n) = (f(p_n))$ in $f(K)$, so $f(K)$ is compact.

Extreme Value Theorem. In the special where $Y = \mathbb{R}$ above, we get as a consequence the *Extreme Value Theorem*:

> Any real-valued continuous function $f : K \to \mathbb{R}$ on a compact space K attains a maximum and a minimum. (So, f attains its "extreme" values.)

To be clear, to say that f has a *maximum* is to say that there exists $p \in K$ such that $f(p) \geq f(x)$ for all $x \in K$—so that the value at p is the largest value f can have—and to say that f has a *minimum* means there exists $q \in K$ such that $f(q) \leq f(x)$ for all $x \in K$, so that the value of f at q is the smallest value f can have. There might be more than one point at which each of these maximal or minimal values occur, but they exist. The most important case of this result will be where $K = [a, b]$, so that any continuous functions $f : [a, b] \to \mathbb{R}$ has a maximum and a minimum. This makes sense intuitively, at least, if we draw the graph of any such continuous function:



Here's the proof. Since f is continuous and K is compact, $f(K) \subseteq \mathbb{R}$ is compact. But then $f(K)$ is bounded (compact implies bounded), so it has a supremum and an infimum, and it is closed (compact implies closed), these supremums and infimums are actually in $f(K)$ itself. (Recall that the supremum and infimum of a set can both be obtained as limits of sequences within that set, and a closed set contains all such limits.) The point $p \in K$ for which $f(p)$ is $\sup f(K)$ is then the point at which the maximum is attained (a maximum is just a supremum that belongs to the set itself), and the point $q \in K$ for which $f(q) = \inf f(K)$ is where the minimum occurs, so maxima and minima both exist.

Example. Here is a first (informal) application of the Extreme Value Theorem. The surface of the Earth is a metric space (!), since we can easily measure distances in the usual way. (The surface of the Earth is *almost* like a sphere, or perhaps more precisely an ellipsoid.) Consider the function that sends a point on the surface of the Earth to its altitude, meaning its distance above (or below!) sea level:

$$\text{Earth} \to \mathbb{R}, \quad p \mapsto \text{altitude at } p.$$

This function is continuous since moving from a point p to a nearby point only changes the altitude by a (very) small amount, which is the type of thing continuity requires. The Earth is compact since it is closed and bounded in $\mathbb{R}^3$ (Heine-Borel), so the Extreme Value Theorem guarantees that there exist points of maximal and minimal altitude. (The maximal altitude is teh summit of Mount Everest, and the minimal altitude—below sea level—is known as the *Challenger deep* and is in the Mariana trench in the Pacific ocean.)

Another example. For an example more relevant to this course, take a compact subset K of X and a fixed point $a \in X$. Consider the function that sends a point of K to its distance to a:

$$K \to \mathbb{R},\ q \mapsto d(a, q).$$

A minimum for this function is a point $p \in K$ that minimizes all such $d(a, q)$, and this minimum value is precisely what we've previously called the *distance* from a to K and denoted by $d(a, K)$. Thus, saying that this function has a minimum is the claim that there exists $p \in K$ such that $d(a, p) = d(a, K)$.

But proving the existence of such a $p \in K$ was precisely a problem on a recent homework! Indeed, the real point of that problem was—even though not phrased in this language—to show that the function $q \mapsto d(a, q)$ above is in fact continuous. If so, then the Extreme Value Theorem immediately gives the minimum we desire. To say that this function is continuous is to say that for any convergent sequence $q_n \to q$ in K, it should be true that

$$d(a, q_n) \to d(a, q)$$

in $\mathbb{R}$. If you go back and check the homework solution for this problem, you will see that this sequence statement is *precisely* what it proven there. (The idea is to use the reverse triangle inequality to bound $|d(a, q_n) - d(a, q)|$ by $d(q_n, q)$.) Thus, that entire problem can be rephrased as a statement about continuity and the Extreme Value Theorem. This is indicative of a general phenomenon: many problems which state the existence of some maximal or minimal quantity can often be phrased as an application of the Extreme Value Theorem.

Lecture 23: Uniform Continuity

Warm-Up. We show that if K is nonempty and compact, then the diameter of K is attained as the distance between point elements of K: there exists $p, q \in K$ such that $d(p, q) = \operatorname{diam} K$. This is something you already did on the homework, but here we do it as a consequence of the Extreme Value Theorem. Consider the function

$$K \times K \to \mathbb{R},\ \text{defined by } (a, b) \mapsto d(a, b).$$

We claim that this function is continuous, and that $K \times K$ is compact, so that the Extreme Value Theorem guarantees the existence of a maximum, which is precisely what we want since $\operatorname{diam} K$ is then that maximal quantity.

The fact that the function above is continuous is precisely what was shown on the relevant homework problem, only without using the language of continuity. Indeed, the claim is that if $(a_n, b_n) \to (a, b)$ in $K \times K$, then

$$d(a_n, b_n) \to d(a, b)$$

in $\mathbb{R}$, and this is what was shown on the homework. The key is the following inequality

$$|d(a_n, b_n) - d(a, b)| \leq d(a_n, a) + d(b_n, b),$$

which comes from two applications of the triangle inequality. By making the terms on the right each smaller than $\frac{\epsilon}{2}$, we can make the term on the left smaller than ϵ, which is what $d(a_n, b_n) \to d(a, b)$ requires. Check the details in the homework solution.

Compactness of $K \times K$ is not something we have proven in full generality before, but we did give a proof as a Warm-Up one day in the case where $K = [a, b]$, and the general proof is the same.

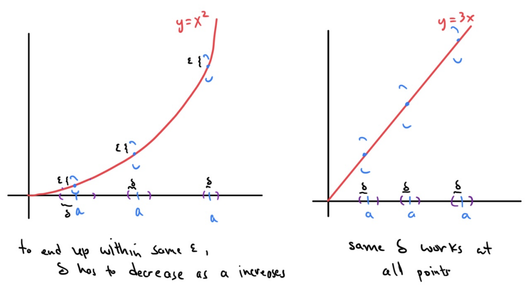
Namely, taken a sequence (p_n, q_n) in $K \times K$. Then (p_n) is a sequence in K, so it has a convergent subsequence (p_{n_k}) in K, converging to say $p \in K$. Then (q_{n_k}) is also a sequence in K, so it has its own convergent subsequence $(q_{n_{k_\ell}}$, converging to $q \in K$. Then $p_{n_{k_\ell}} \to p$, so $(p_{n_{k_\ell}}, q_{n_{k_\ell}})$ is a convergent subsequence of (p_n, q_n), converging to $(p, q) \in K \times K$, so $K \times K$ is compact. Hence the Extreme Value Theorem gives our claim.

Uniform continuity. Consider the functions $f(x) = x^2$ and $g(x) = 3x$, both defined on all of $\mathbb{R}$. Both of these are continuous at any $a \in \mathbb{R}$, and so for a fixed $\epsilon > 0$ there exists $\delta > 0$ satisfying the requirements in the ϵ-δ definition of continuous at a. In particular, if you work this out in each case, you'll find that for f the value

$$\delta = \frac{\epsilon}{2 + |a|}$$

works (for small enough ϵ) while for g the value $\delta = \frac{\epsilon}{3}$ works.

Here is the key observation: for f the value of δ we find depends on a—i.e. the point at which we're checking continuity—while for g it does not. As a gets larger and larger, the δ for f gets smaller and smaller, while the δ for g remains the same. In fact, because $\delta \to 0$ as $a \to \infty$ in the case of $f(x) = x^2$, it is not possible to find a "smallest" possible δ which will work for *all* $a \in \mathbb{R}$ at once for f since we want δ to be positive, while for g we do have one δ that works for all $a \in \mathbb{R}$:



This distinction is what tells us that f is not uniformly continuous on $\mathbb{R}$ but that g *is* uniformly continuous on $\mathbb{R}$.

Here is the definition, in the general metric space setting:

$f : X \to Y$ is *uniformly continuous* on X if for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\text{if } d_X(p, q) < \delta, \text{ then } d_Y(f(p), f(q)) < \epsilon.$$

This looks very similar to the usual definition of continuous, except that there we fixed a point $q \in X$ we were checking continuity at while here q is not fixed; there, only $p \in X$ was allowed to vary, while here both p and q are allowed to vary.

Practically, this means that in usual continuity δ can depend on ϵ and the point you're checking continuity at, while in uniform continuity δ can only depend on ϵ. Since the same δ works for all points in X, f is continuous in a "uniform" way across all of X. Geometrically, a continuous function fails to be uniformly continuous when it changes "too rapidly", such as when its graph gets steeper and steeper. This is what happens in the $f(x) = x^2$ on $\mathbb{R}$ case, but does not happen for $g(x) = 3x$ on $\mathbb{R}$. We will talk more about the relation between "uniformly continuous" and "steepness" when we talk about derivatives.

The domain matters. Consider the function $f(x) = x^2$ on the interval $[a, b]$. In this case, for $|x - y| < \delta$ we can bound $|f(x) - f(y)|$ as follows:

$$|x^2 - y^2| = |x - y||x + y| < \delta|x + y| \leq 2\max\{|a|, |b|\}\delta$$

since $|x+y| \leq |x|+|y| \leq \max\{|a|,|b|\}$ for $x, y \in [a, b]$. Thus for $\epsilon > 0$, $\delta = \frac{\epsilon}{2\max\{|a|,|b|\}}$ satisfies the ϵ-δ definition of continuous, so f is uniformly continuous on $[a, b]$.

The point is that when asking whether a function is uniformly continuous or not, the domain of the function matters: $f(x) = x^2$ is not uniformly continuous on all of $\mathbb{R}$, but it is uniformly continuous on $[a, b]$. Geometrically, when restricting the domain to be $[a, b]$ the graph of $f(x) = x^2$ does not get arbitrarily steep.

Example. The function $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0, 1)$. When going through a proof that f is continuous at $a \in (0, 1)$, for $\epsilon > 0$ you find that

$$\delta = \min\left\{\frac{a}{2}, \frac{a^2\epsilon}{2}\right\}$$

satisfies the required definition, since

$$\left|\frac{1}{x} - \frac{1}{a}\right| = \frac{|x-a|}{xa} \leq \frac{|x-a|}{a^2/2}$$

for $|x - a| < \frac{a}{2}$. However, note that $\delta \to 0$ as $a \to 0$, so there will not be a single positive δ of the form above which satisfies the required definition for all $a \in (0, 1)$ at once. This suggests that f is not uniformly continuous. (This isn't quite a definite proof yet since one could ask why there isn't some *other* δ not of the form above that *could* work. We'll come back to this in a bit.) Again, geometrically, note that the graph of f gets steeper and steeper as $a \to 0$.

Properties of uniformly continuous functions. Here are two basic properties of uniformly continuous functions, which hints at why uniform continuity is a nice property to have:

- If $f : X \to Y$ is uniformly continuous and (x_n) is a Cauchy sequence in X, then $(f(x_n))$ is Cauchy as well. Thus, uniformly continuous functions send Cauchy sequences to Cauchy sequences.

- If $f : (a, b) \to \mathbb{R}$ is uniformly continuous, then f can be "extended" to a continuous function $f : [a, b] \to \mathbb{R}$. Thus, uniformly continuous functions on open intervals can be defined at the endpoints so as to still remain continuous. (This generalizes to other metric spaces as well: if $f : E \subseteq X \to Y$ is uniformly continuous, then f can be extended to a continuous function on the closure $\overline{E}$ of E.)

Note that for the Cauchy sequence $\frac{1}{n+1}$ in $(0, 1)$, the function from the previous example has $f(\frac{1}{n+1}) = n + 1$, which is not Cauchy. Thus this function does not satisfy the first property above, which gives a proof that $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0, 1)$. This example also shows that the second property above fails for a non-uniformly continuous function, since $f(x) = \frac{1}{x}$ cannot be extended to be continuous at 0.

For the first property, suppose (x_n) is Cauchy in X and let $\epsilon > 0$. By uniform continuity there exists $\delta > 0$ such that

$$d_X(p, q) < \delta \implies d_Y(f(p), f(q)) < \epsilon.$$

Since (x_n) is Cauchy, there exists $N \in \mathbb{N}$ such that $d_X(x_n, x_m) < \delta$ for $m, n \geq N$. Hence for $m, n \in \mathbb{N}$ we then also have $d_Y(f(x_n), f(x_m)) < \epsilon$, which shows that $(f(x_n))$ is Cauchy in Y.

For the second property, take $p \in \overline{E}$ which is not in E; we want to show how to extend $f : E \to \mathbb{R}$ to be defined at p. Take a sequence (x_n) in E which converges to p. Then (x_n) is Cauchy in E, so

$(f(x_n))$ is Cauchy in $\mathbb{R}$ by the first property. Since $\mathbb{R}$ is complete, this sequence converges, and we then define $f(p)$ to be the limit of this sequence:

$$f(p) := \lim_{n\to\infty} f(x_n).$$

What remains to be shown is that this function is well-defined, in the sense that a difference choice of sequence (x_n) converging to p would give the same value for $f(p)$, and that the function so-defined is in fact continuous at p. We will omit these verifications here since they will not be crucial to how uniform continuity will be used going forward, but you should think about them for practice! Ultimately, for our purposes, uniform continuity will be important since it gives a way to control distances between outputs of a function by controlling distances between inputs, in a "uniform" way across the entire domain.

Continuous on a compact domain. The observation we made above—that $f(x) = x^2$, even though not uniformly continuous on $\mathbb{R}$, is uniformly continuous on $[a, b]$—is no coincidence, and reflects a general property of compactness. Indeed, the claim is that if $f : X \to Y$ is continuous and X is compact, then f is automatically uniformly continuous. So, for example, continuous functions on closed intervals are automatically uniformly continuous. You should view this as being one of the many properties (together with the Bolzano-Weierstrass Theorem and the Extreme Value Theorem) which make closed intervals special.

Here is a first proof of this fact, which is probably the most important but toughest to follow since relies on the open cover definition of compactness. Still, note that, in the end, compactness is used to turn an infinite set of radii into a finite set of radii, so that taking their minimum is possible. You will give another proof of this result on the homework using sequences, and we will sketch a possible third proof which highlights the intuition behind this result next time.

Fix $\epsilon > 0$. By ordinary continuity, we know that for any $a \in X$ there exists $\delta(a) > 0$ (delta might depend on a) such that

$$d_X(x, a) < \delta(a) \implies d_Y(f(x), f(a)) < \epsilon.$$

Here, a is fixed and x varies. Doing this for all $a \in X$ results in a corresponding $\delta(a)$ for each a, and then we consider the collection of open balls $\{B_{\delta(a)}(a)\}_{a\in X}$ in X given by these radii. This is an open cover of X since each $a \in X$ is in particular in the open ball $B_{\delta(a)}(a)$ centered at that point. Since X is compact, this open cover has a finite subcover

$$B_{\delta_1}(a_1), \dots, B_{\delta_N}(a_N)$$

where δ_j denotes $\delta(a_j)$. Each element of X is in at least one of these open balls.

Now, we want to come up with a single $\delta > 0$ such that

$$d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \epsilon.$$

The point of the finite cover obtained above is that now we have finitely many radii to deal with, so we can try to their minimum as the δ we need. However, we only know something about quantities of the form $d_Y(f(x), f(a_j))$ where a_j is one of the centers of the finitely many open balls derived above. We would like to use something like

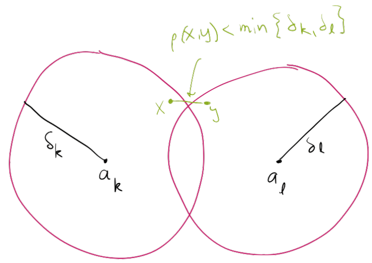
$$d_Y(f(x), f(y)) \le d_Y(f(x), f(a_j)) + d_Y(f(a_j), f(y)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

so as a first fix we go back and replace the ϵ we used in coming up with the radii $\delta(a)$ by $\frac{\epsilon}{2}$, as the book does in its proof.

But now the problem is that, although we know that both x and y will each be in *some* open ball among the finitely many we obtained above, and so we will get *some* inequalities of the form

$$d_Y(f(x), f(a_k)) < \frac{\epsilon}{2} \text{ and } d_Y(f(a_\ell), f(y)) < \frac{\epsilon}{2},$$

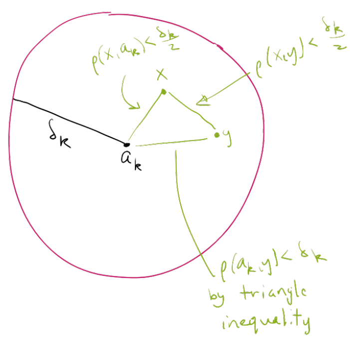
to get what we're doing to actually work we need the center points a_k and a_ℓ used here to be the *same*. Thus, we need to know that x and y are both in the *same* open ball among the finitely many obtained above. But when picking $\delta = \min\{\delta_1, \dots, \delta_N\}$, $d_X(x, y) < \delta$ does NOT guarantee that x and y will be in the same such ball, since we could have a picture like:



(ρ in this picture is d_X.) Thus this choice of δ is no good. But the fix is to go back and instead consider balls of radii $\frac{\delta(a)}{2}$, and after we get our finite subcover use $\delta = \min\{\frac{\delta_1}{2}, \dots, \frac{\delta_N}{2}\}$ instead as the book does. The point is that now having

$$d_X(x, y) \leq \min\left\{\frac{\delta_1}{2}, \dots, \frac{\delta_N}{2}\right\}$$

DOES guarantee that x and y will be in the same ball:



so that our approach will work out. (The book's proof should be easier to digest now, and although it is still quite challenging to grasp on the first read throughs, hopefully at least it's somewhat clearer now why the book uses $\frac{\epsilon}{2}$ and $\frac{\delta(a)}{2}$ instead of simply ϵ and $\delta(a)$ at the beginning.)

Lecture 24: Connected Sets

Warm-Up. We show that the function $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$ is uniformly continuous. The key inequality is the following:

$$|\sqrt{x} - \sqrt{y}| \leq \sqrt{|x - y|} \text{ for all } x, y \geq 0.$$

To see this, square both sides and consider $(\sqrt{x}-\sqrt{y})^2 = x - 2\sqrt{xy} + y$ vs $|x-y|$ instead. If $x \geq y$, then

$$x - 2\sqrt{xy} + y \leq x - 2\sqrt{yy} + y = x - y = |x-y|,$$

while if $x < y$, then

$$x - 2\sqrt{xy} + y < x - 2\sqrt{xx} + y = y - x = |x-y|,$$

so either way we have $(\sqrt{x}-\sqrt{y})^2 \leq |x-y|$, and taking square roots gives our desired inequality.

Now, let $\epsilon > 0$ and set $\delta = \epsilon^2$. Then if $x, y \in [0, \infty)$ satisfy $|x-y| < \delta = \epsilon^2$, we have

$$|\sqrt{x}-\sqrt{y}| \leq \sqrt{|x-y|} < \sqrt{\epsilon^2} = \epsilon,$$

so $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$ as claimed.

Back to previous theorem. As promised, here is a sketch of another "proof" that continuous functions on compact sets are uniformly continuous. I write "proof" in quotation marks here since what follows is not a precise proof because certain parts will be a bit hand-wavy, but I feel that this sketch better captures the underlying point of this result. There is a way to make what I'm going to outline precise and rigorous, but doing so fully probably isn't worth all the work involved. So, use this only to get some intuition for why this theorem is true: the point is that we want to find a single $\delta > 0$ which works for all points in our domain at once.

Suppose $f : X \to Y$ is continuous with X compact. Let $\epsilon > 0$. Then for any $y \in X$, f is continuous at y so there exists $\delta_y > 0$ such that

$$d_X(x,y) < \delta_y \text{ implies } d_Y(f(x), f(y)) < \epsilon.$$

(We're using δ_y to emphasize the δ depends on y, and different y's might require different δ's.) Now, view the assignment $y \mapsto \delta_y$ as defining a function $g : X \to \mathbb{R}$:

$$g(y) = \delta_y.$$

We claim (and this is the hand-wavy part) that g is continuous: intuitively, changing y by a small amount should only change δ_y by a small amount, and indeed in the examples we've seen where δ depends on a this has been the case. (There is another issue, in that δ_y isn't uniquely defined yet since there could be different δ's which satisfy the definition of continuity for the same y. This is easier to deal with: we can define δ_y to be $\frac{1}{2}$ the supremum among all δ's which work.) So, taking it for granted that there is a way to make $g(y) = \delta_y$ continuous, we push onward.

Since $g : X \to \mathbb{R}$ is continuous, it has a minimum value by the Extreme Value Theorem—call it δ. Note that $\delta > 0$ since it is the minimum of positive numbers. We claim that this one δ satisfies the definition of continuity at any $y \in X$. Indeed, suppose that $d_X(x,y) < \delta$. Since $\delta \leq \delta_y$, we then also have $d_X(x,y) < \delta_y$, so by the choice of δ_y we get

$$d_Y(f(x), f(y)) < \epsilon.$$

Thus $d_X(x,y) < \delta$ implies $d_Y(f(x), f(y)) < \epsilon$, so f is uniformly continuous as claimed.

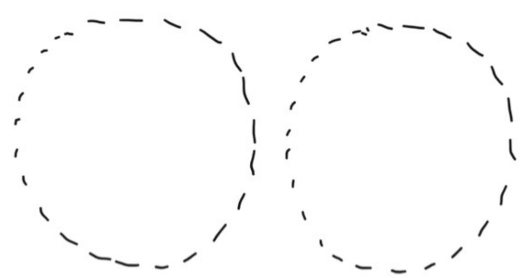
Connected sets. We now come back to one topic we glossed over previously, that of *connected sets*. We are doing this now in order to setup the discussion of properties of continuous functions on connected sets, with the Intermediate Value Theorem being the main goal. We briefly gave the definition of "connected" before, but let us start from scratch anyway.

A metric space X is said to be *disconnected* if there exist disjoint, nonempty open subsets U and V of X such that

$$X = U \cup V.$$

(Recall that saying U and V are disjoint means that $U \cap V = \emptyset$.) Intuitively, this means that we can "split" X into two pieces, U and V. A space is *connected* if it is not disconnected (beware the double negative!), meaning that such a "splitting" is not possible. Here is another way to phrase the definition of connected in a way which is more practically useful: X is connected if whenever we have $X = U \cup V$ with U and V disjoint and open in X, one of U or V must be empty.

Examples. The space consisting of the union of the following open disks is disconnected:



Indeed, each open disk in question is open in this space, is nonempty, and has nothing in common with the other one. Visually we can see the way in which disconnected spaces can be "split" up into multiples "pieces". Each individual open disk, however, is an example of a connected space.

The space $X = [0,1] \cup [2,3]$ is also disconnected, with respect to the standard absolute value metric. At first this might not seem to match the definition since the definition says we need to break our space up into *open* sets, but the point is that these sets are only required to be open in X itself. In particular, the intervals $[0,1]$ and $[2,3]$ are indeed open in X, so X is disconnected.

Finally, $\mathbb{Z}$ is disconnected, which again should be visually clear. To be precise, *every* subset of $\mathbb{Z}$ is open in $\mathbb{Z}$ so

$$\mathbb{Z} = \{\text{negative integers}\} \cup \{\text{nonnegative integers}\}$$

exhibits $\mathbb{Z}$ as the union of two nonempty, open and disjoint subsets, so $\mathbb{Z}$ is disconnected. In fact, the only connected subsets of $\mathbb{Z}$ are those which consist of a single point or are empty.

Clopen subsets. In the decomposition $X = U \cup V$ into disjoint open subsets, note that we can view each subset as the complement of the other. Since complements of open sets are closed, we see that U and V are both open *and* closed in X, so they are *clopen* subsets of X. Thus, saying that X is disconnected implies that it has a nonempty proper clopen subset, while conversely if X has a nonempty property clopen subset A,

$$X = A \cup A^c$$

exhibits X as the union of two nonempty, disjoint, open subsets, so X is disconnected.

Thus we have that X is disconnected if and only if it has a nonempty property clopen subset, or equivalently X is connected if and only if the only clopen subsets of X are $\emptyset$ and X itself. This gives a more succinct way of saying what disconnected/connected mean, although the definition we first gave is visually clearer. In the case of $X = [0,1] \cup [2,3]$, both $[0,1]$ and $[2,3]$ are clopen subsets of X, while in the case of the integers *every* subset is clopen.

Why we care. To motivate why we care about the notion of connected sets, consider the following question: if $f : U \to \mathbb{R}$ is a differentiable function on an open subset U of $\mathbb{R}$ with $f'(x) = 0$ for all $x \in U$, is it true that f must be constant? Your experience in calculus might lead you to believe that this is true, but in fact it is only true if U is connected! Indeed, take the function $f : U \to \mathbb{R}$ on $U = (-2, -1) \cup (1, 2)$ defined by

$$f(x) = \begin{cases} 1 & x \in (-2,-1) \\ -1 & x \in (1,2). \end{cases}$$

This is differentiable and has derivative equal to zero throughout U, but is clearly not constant, the issue being that U here is disconnected. In general, having derivative zero everywhere throughout a region only implies that your function is constant on each "connected piece" of that region, but the constant over different pieces can differ.

A similar thing will be true when we consider higher-dimensional derivatives, so the distinction between connected and disconnected spaces will pop-up next quarter as well, although only in the setting of $\mathbb{R}^n$ where things are easier to visualize.

Intervals are connected. Any interval I in $\mathbb{R}$ is connected, which should make intuitive sense visually. The one fact we need to prove this is that a compact subset of $\mathbb{R}$ always has a maximum element and a minimum element: indeed, a compact subset is bounded, so it has a supremum and an infimum, and a compact set is closed, so it will contain its supremum and its infimum. Note that by "interval" we mean any type: open, closed, half-open, half-closed, bounded or unbounded, so that in particular $\mathbb{R} = (-\infty, \infty)$ itself is connected.

The idea behind the proof is that given any two nonempty, open disjoint subsets of an interval, their union can never be the full interval. In the case of a closed interval, say we try to write it as

$$[x, y] = U \cup V$$

with U and V open, nonempty, and disjoint. Then we argue that both U and V are compact (since each is a closed subset of the compact set $[x, y]$), so they have a maximal and minimal element respectively. If we set this up correctly, then nothing between these two elements can be in $U \cup V$, which will lead to a contradiction. This makes sense visually: if you draw two disjoint open nonempty sets (for example open intervals) on a number line, there will always be points excluded by their union. The proof below is essentially reducing the case of an arbitrary interval to the case of a closed interval.

For a contradiction, suppose that $I = U \cup V$ where U and V are nonempty, disjoint, and open in I. Pick $x \in U$ and $y \in V$, and assume without loss of generality that $x < y$ and consider the smaller interval $[x, y] \subseteq I$. Then we have

$$[x, y] = ([x, y] \cap U) \cup ([x, y] \cap V).$$

Since $[x, y] \cap U$ is open in $[x, y]$, its complement $[x, y] \cap V$ is closed in $[x, y]$ and hence compact since a closed subset of a compact space is always compact. Thus $[x, y] \cap V$ has a minimum element, call it $b \in [x, y] \cap V$.

Consider now the interval $[x, b] \subseteq [x, y]$. We have

$$[x, b] = ([x, b] \cap U) \cup ([x, b] \cap V),$$

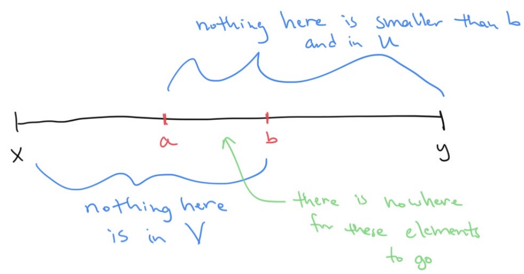
so the same argument as above shows that $[x, b] \cap U$ is compact and hence has a maximum element, call it $a \in [x, b] \cap U$. We have $a \leq b$, and thus $a < b$ since $a \neq b$ given that $a \in U$, $b \in V$, and U

and V are disjoint. Now, take any $a < c < b$. Since

$$x \leq a < c < b \leq y,$$

$c \in [x, y]$, and thus either $c \in [x, y] \cap U$ or $c \in [x, y] \cap V$. However, the first is not possible since then c is in $[x, b] \cap U$ and is greater than its largest element a, and the second is not possible since then c is in $[x, y] \cap V$ and is smaller than its largest element b. Thus we have a contradiction, so I must have been connected to begin with.

Here is a picture to illustrate where all the different elements considered above come from:



The point is that b is the smallest element of V in $[x, y]$ and a the largest element of U in $[x, b]$, so nothing between them is in $[x, y]$, which contradicts a basic property of intervals.

$\mathbb{R}^n$ **is connected.** (We didn't look at this specific claim in class, but let us include here for the sake of giving more examples of connected sets.) Suppose that A and B are connected subsets of $\mathbb{R}^2$ which are not disjoint. Then $A \cup B$ is also connected. This generalizes to other metric spaces as well, and gives a quick way of verifying that various sets are indeed connected.

To see this, suppose that $A \cup B = U \cup V$ where U and V are open in $A \cup B$ and disjoint. We must show that one of U or V is empty. Since A and B are not disjoint, there exists $p \in A \cap B$, and hence this element is in $U \cup V$ so either $p \in U$ or $p \in V$. Without loss of generality suppose that $p \in U$, in which case we must show that V is empty.

Now, we can write A as

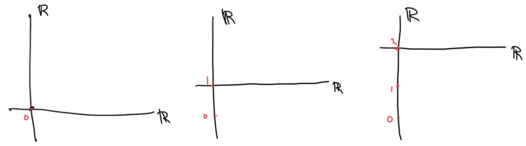
$$A = (A \cap U) \cup (A \cap V)$$

since any element of A must be in U or V. Since U and V are open in the larger space $A \cup B$, these intersections are each open in A. (In general, it is true that if $A \subseteq X$ and U is open in X, then $A \cap U$ is open in A, and in fact all open subsets of A arise in this way.) Since A is connected, one of these two open sets must be empty, and since $p \in A \cap U$ we must have $A \cap V = \emptyset$. Similarly, writing B as

$$B = (B \cap U) \cup (B \cap V)$$

and using the fact that $p \in B \cap U$, the fact that B is connected implies that $B \cap V = \emptyset$. But now we can conclude that $V = \emptyset$: if not, a point of V would be in A or B since $V \subseteq U \cup V = A \cup B$, in which case this point would be either in $A \cap V$ or $B \cap V$, neither of which are possible since both of these intersections are empty. Thus we conclude that $A \cup B$ is connected as claimed.

Now we consider $\mathbb{R}^2$. From last time, we know that $\mathbb{R}$ is connected. It does not matter if we draw $\mathbb{R}$ as a horizontal line or as some other kind of line, it will still be connected. Thus the x- and y-axes of $\mathbb{R}^2$ are connected, and by the fact above so is their union since they have a point in common. Similarly, the union of the y-axis with the line $y = 1$ is connected, and so is the union of the y-axis with *any* horizontal line:



But now, all of these connected "cross" shaped figures overlap with one another (they intersect on the y-axis), so again *their* union is connected. As we move the horizontal line up and down, we see that this union is all of $\mathbb{R}^2$, so $\mathbb{R}^2$ is connected. In a similar way, we can build up to show that $\mathbb{R}^3$ is connected, then $\mathbb{R}^4$, and so on, so $\mathbb{R}^n$ is connected in general.

Continuous sends connected to connected. The key relation between connected sets and continuous functions is the following claim: if $f : X \to Y$ is continuous and that $A \subseteq X$ is connected. Then $f(A) \subseteq Y$ is connected as well. Thus, a continuous function sends connected sets to connected sets. (This is similar to how continuous functions send compact sets to compact sets, although of course the proof is different.)

Suppose that $f(A) = U \cup V$ with U and V open in $f(A)$ and disjoint. We must show that one of U or V is empty. Since f is continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are open in A and we have

$$A = f^{-1}(U) \cup f^{-1}(V).$$

(Technically, here we are not really considering the original $f : X \to Y$ but rather its *restriction* to A, which is often denoted by $f|_A : A \to f(A)$. This is a minor point which we will not dwell on.) Since U and V are disjoint, their preimages are disjoint, so since A is connected we thus have that one of $f^{-1}(U)$ or $f^{-1}(V)$ is empty. Without loss of generality, say that $f^{-1}(U)$ is empty. Then U is empty as well, since if not there would exist $f(p) \in U \subseteq f(A)$ and this p would then be in $f^{-1}(U)$, which is empty. Thus we conclude that $f(A)$ is connected.

Intermediate Value Theorem. And finally we come to the main point, which is known as the *Intermediate Value Theorem*. Suppose that $f : A \to \mathbb{R}$ is continuous and that A is connected. Suppose further than $f(a) < f(b)$ in $\mathbb{R}$. Then the result is:

$$\text{for any } c \in \mathbb{R} \text{ such that } f(a) < c < f(b), \text{ there exists } p \in A \text{ such that } f(p) = c.$$

Hence, any "intermediate value" between $f(a)$ and $f(b)$ is attained as a value of f, and we say that f has the *intermediate value property*. In particular then, since intervals are connected we have that any continuous function $[a, b] \to \mathbb{R}$ has the intermediate value property. This makes sense intuitively if you draw a graph of a typical continuous function: given two points on the graph, there are points that give rise to all intermediate y-coordinates between them.

Here is the proof. Since f is continuous and A is connected, $f(A)$ is a connected subset of $\mathbb{R}$, so $f(A)$ must be an interval. But then if $f(a) < f(b)$ in this interval, any c between $f(a)$ and $f(b)$ remains in the interval, so $c \in f(A)$ and hence there exists $p \in A$ which is sent to c under f. (The claim that a connected subset of $\mathbb{R}$ is an interval is the converse to the claim that intervals are always connected. To see why this is true, note that if $J \subseteq \mathbb{R}$ is connected, then if J were not an interval there would exist $p < c < q$ in J such that $p, q \in J$ but $c \notin J$. This would give $J = [J \cap (-\infty, c)] \cup [J \cap (c, \infty)]$, which is a separation of J into nonempty disjoint open sets, contradicting connectedness. Hence J must be an interval.)

Lecture 25: Differentiable Functions

Warm-Up. We show that at any instant of time, there is a point p on the surface of the Earth such that the temperature at p is the same as the temperature at its antipodal point $-p$. (The *antipodal* point is the point directly on the opposite side of the Earth.) We are thinking of the "surface of the Earth" here really as being a sphere, and making use of the fact that spheres are connected. (We will explain why afterwards.)

Consider the function T : Earth $\to \mathbb{R}$ that sends a point on the Earth to the temperature measured at that point. This function is continuous, which we only informally justify by noting that moving from point to a nearby point only changes the temperature by a very very small amount. Now set f : Earth $\to \mathbb{R}$ to be the function

$$f(p) = T(p) - T(-p),$$

so that f measures the difference in temperatures at p vs its antipodal point $-p$. This function is also continuous since it is a sum of continuous functions, so it has the intermediate value property by the Intermediate Value Theorem. What we want then is a point such that $f(p) = 0$, which says precisely that $T(p) = T(-p)$ as we want.

Now, if f is always zero, then any point will work so there is nothing to show. If f is not always zero, there exists q such that $f(q) \neq 0$. Then one of $f(q), f(-q)$ is positive and the other is negative since

$$f(-q) = T(-q) - T(-(-q))) = -[T(q) - T(-q)] = -f(q),$$

so 0 is an intermediate value between $f(q)$ and $f(-q)$. Hence there exists p such that $f(p) = 0$ by the intermediate value property, as desired.

Path connectedness. The notion of a path-connected space is not mentioned in the book at this point yet, and indeed we did not mention it in class either. But let us briefly introduce it here to justify the fact about spheres made above (they are connected), and because it provides possibly simpler ways of showing that spaces are connected.

We say that a space X is *path-connected* if any points $p, q \in X$ of X can be joined by a continuous path: i.e. there exists a continuous function $\gamma : [a, b] \to X$ defined on some interval $[a, b]$ such that $\gamma(a) = p$ and $\gamma(b) = q$. The points $\gamma(t)$ in the image of γ trace out a path which starts at p and ends at q as t ranges from a to b. Visually, path-connectedness is a simple property to determine: just literally draw a continuous path from one point to the other. The point is that path-connectedness implies connectedness, and that path-connected is usually a much simpler property to check. Indeed, this gives an easier way of showing that $\mathbb{R}^n$ is connected, or that rectangles in $\mathbb{R}^2$ are connected, or even disks in $\mathbb{R}^2$. Essentially, any nice subset of $\mathbb{R}^2$ you can draw which "appears" to be path-connected will indeed be so, and will thus be connected as well. (Take note whoever that connected does not imply path-connected, with what's called the *topologist's sine curve* being the key counterexample. Look it up!)

Here is a proof that path-connected implies connected. Suppose X is disconnected. Then we claim that X is not path-connected. Since X is disconnected, we can write it as

$$X = U \cup V$$

for some disjoint, nonempty open subsets $U, V \subseteq X$. Pick $p \in U$ and $q \in V$. We claim that there is no continuous path then from p to q. Indeed, if there was such a continuous map $\gamma : [a, b] \to X$ with $\gamma(a) = p$ and $\gamma(b) = q$, then taking the preimages gives

$$[a, b] = \gamma^{-1}(U) \cup \gamma^{-1}(V).$$

Both sets on the right are open in $[a, b]$ since γ is continuous, and they are disjoint since U and V are disjoint. Moreover, they are both nonempty since $a \in \gamma^{-1}(U)$ and $b \in \gamma^{-1}(V)$, and this contradicts the fact that $[a, b]$ is connected. Hence no such γ can exist, so X is not path-connected as claimed. (Path-connectedness will show up at various points later, such as when discussing properties of higher-dimensional derivatives.)

Differentiable functions. Continuous functions are ones where we can control the behavior of outputs by controlling inputs, and the notion of differentiability will give us even more control over such things. For U an open subset of $\mathbb{R}$, we saw that a function $f : U \to \mathbb{R}$ is *differentiable* at $a \in U$ if

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists, in which case we call the value of this limit the *derivative of f at a* and denote it by $f'(a)$. This limit can equivalently be written as

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

after making the substitution $x = a + h$ and noting that then saying $x \to a$ and $h \to 0$ mean the same thing. We say that f is differentiable on U if it is differentiable at each $y \in U$. (The reason for asking that U be open is to guarantee that we can approach $a \in U$ from "both" sides; otherwise we potentially only get one-sided derivatives.)

This definition is no doubt one you've seen in a previous calculus course. The fraction we are taking the limit of is the slope of secant line passing through $(a, f(a))$ and $(x, f(x))$ on the graph of f in the first version and through $(a, f(a))$ and $(a + h, f(a + h))$ in the second; thus the limit, when it exists, indeed gives us the slope of the tangent line at a itself. We will take differentiability of many standard functions (x^n, e^x, trig functions, etc) for granted, and will focus on examples you likely haven't seen before and on what we can actually *do* with derivatives.

Example. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

We claim that this function is not differentiable at 0. Indeed, we have:

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x \sin \frac{1}{x}}{x} = \lim_{x \to 0} \sin \frac{1}{x},$$

and this limit does not exist due to the oscillatory behavior of $\sin \frac{1}{x}$ as $x \to 0$. (Note that when we are considering the limit as $x \to 0$, we are looking at values approaching 0 but never equal to zero itself, which is why we were able to substitute $f(x) = x \sin \frac{1}{x}$ for such x.) However, note that f is indeed continuous at 0 since $\lim_{x \to 0} f(x) = 0 = f(0)$.

Now consider the function $g : \mathbb{R} \to \mathbb{R}$ defined by the same formula as f only using $x^2 \sin \frac{1}{x}$ instead of $x \sin \frac{1}{x}$. In this case we end up with:

$$\lim_{x \to 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 \sin \frac{1}{x}}{x} = \lim_{x \to 0} x \sin \frac{1}{x} = 0,$$

so g is differentiable at 0 and $g'(0) = 0$. The function g is differentiable for $x \neq 0$ since near such values g is the same as the function

$$x^2 \sin \frac{1}{x}$$

which is differentiable at $x \neq 0$ as a consequence of some differentiantion rules, in particular the product and chain rules. (We will prove these next time.) Hence g is differentiable on all of $\mathbb{R}$. The value of $g'(x)$ for $x \neq 0$ is obtained by differentiating $x^2 \sin \frac{1}{x}$ for $x \neq 0$, and thus we find that the derivative of $g : \mathbb{R} \to \mathbb{R}$ is

$$g'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Note that this derivative g', however, is not continuous at 0 since $\lim_{x \to 0} g'(x)$ does not exist. (It should equal $g'(0)$ in order for g' to be continuous at 0.) This is due to the $\cos \frac{1}{x}$ term, which has no limit as $x \to 0$. We say that even though g is differentiable, it is not *continuously differentiable*. (In general, we say that a function is C^k, or *continuously k-times differentiable*, if it is k-times differentiable with a continuous k-th derivative.) As will show shortly, g' being discontinuous at 0 prevents it from being differentiable at 0, so g is not twice differentiable at 0.

The functions

$$f_k(x) = \begin{cases} x^k \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

in general then give examples of the various types of behaviors possible. (Above we considered f_1 and f_2.) The fact, as you will show on the homework, is that f_{2k} is an example of a function which is k-times differentiable but not continuously k-times differentiable, and f_{2k+1} is a function which is continuously k-times differentiable but not $(k+1)$-times differentiable.

Differentiable implies continuous. As a first basic fact, that claim that if $f : U \to \mathbb{R}$ is differentiable at $a \in \mathbb{R}$, then f is continuous at a. This should again be a well-known fact from calculus. The basic idea is that in order for the limit

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

to exist, the numerator had better approach 0 since the denominator does. (Otherwise with the denominator approaching 0 but not the numerator we would end up with a fraction whose limit did not exist.) Thus for this limit to exist we need $f(x) - f(a) \to 0$ as $x \to a$, so $\lim_{x \to a} f(x) = f(a)$ and thus f is continuous at a.

But we can be more precise (to avoid the vague "the numerator had better approach 0 since the denominator does") as follows. Consider the identity

$$f(x) = f(a) + \frac{f(x) - f(a)}{x - a}(x - a)$$

for $x \neq a$. If we take the limit of both sides as $x \to a$, the $\frac{f(x)-f(a)}{x-a}$ term approaches $f'(a)$ by differentiability, the $x - a$ term approaches 0, and the constant $f(a)$ term remains as is, so we get

$$\lim_{x \to a} f(x) = f(a) + f'(a) \cdot 0 = f(a),$$

which says that f is continuous at a.

Lecture 26: More on Differentiability

Warm-Up. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

We show that f is differentiable at 0 and only at 0. Indeed, to show that f is not differentiable at any $y \neq 0$, we show that it is not even continuous at any such y; this rules out f being differentiable at such y since differentiable implies continuous. If y is a nonzero rational number, take any sequence of irrationals (y_n) converging to y. (As we've seen before, such a sequence exists since the irrationals are dense in $\mathbb{R}$.) Then $f(y_n) = 0$ for all n so $f(y_n) \to 0$. Thus we have

$$y_n \to y \text{ but } f(y_n) \nrightarrow f(y) = y^2 > 0,$$

so f is not continuous at y. If y is an irrational number, take a sequence of rationals (y_n) converging to y. Then $f(y_n) = y_n^2$, which converges to $y^2 > 0$ according to some limit laws. Thus

$$y_n \to y \text{ but } f(y_n) \nrightarrow f(y) = 0,$$

so f is not continuous at y. Thus f is not continuous, nor then differentiable, at any $y \neq 0$.

Now, to determine differentiability at 0 we consider the limit

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{f(x)}{x}.$$

As $x \to 0$, $f(x)$ is either 0 (when x is irrational) or x^2 (when x is rational); in the first case we have $\frac{f(x)}{x} = 0$ and in the second $\frac{f(x)}{x} = x$. Thus for $\epsilon > 0$, $\delta = \epsilon$ satisfies

$$0 < |x - 0| < \delta \implies \left| \frac{f(x)}{x} - 0 \right| < \epsilon,$$

so the limit defining $f'(0)$ exists and equals zero.

Important Remark. At first glance, you might be tempted to say that since x^2 is differentiable at all $x \in \mathbb{Q}$ and since the constant 0 is differentiable as well, f is differentiable everywhere and

$$f'(x) = \begin{cases} 2x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

by computing the derivative of each term separately. However, this is total nonsense as our argument in the Warm-Up shows. The problem is that since the derivative is defined as a limit, it depends on values *near* the point we're approaching and not the value at that point itself. In other words, just knowing the value of f at a point x is not enough to determine the derivative at x; we need to know how f behaves "close" to x.

In this case, any interval around $x \in \mathbb{Q}$ will contain an irrational y, and at such points f will have the value 0 and *not* y^2. So f does not look like the function x^2 everywhere near $x \in \mathbb{Q}$, so we cannot just simplify use this expression itself to determine differentiability. Similarly, at an irrational f has the value 0 but it does not have the value 0 everywhere near an irrational since any interval around an irrational will contain a rational r, will f has the value r^2.

Comparing with a previous example where we had a function with the value $x^2 \sin \frac{1}{x}$ for $x \neq 0$ and 0 for $x = 0$, in that case at any $x \neq 0$ there is an interval consisting of only nonzero numbers y, and the value of f at those points is still given by $y^2 \sin \frac{1}{y}$. That is, in that case everywhere "near" some $x \neq 0$ the function in question was the same as the function $x^2 \sin \frac{1}{x}$ so we can use what we know about this function to say something about differentiability; that doesn't happen in the function in the first Warm-Up.

Product rule. Let us now justify the basic product rule from calculus. This is not difficult, and we do it really to highlight the use of continuity. The claim is that if f, g are both differentiable at a, then fg (the product function) is differentiable at a as well and

$$(fg)'(a) = f'(a)g(a) + f(a)g'(a).$$

This requires looking at the limit

$$\lim_{x\to a} \frac{f(x)g(x) - f(a)g(a)}{x-a}.$$

If we add and subtract $f(a)g(x)$ in the numerator, we get

$$\begin{aligned}\frac{f(x)g(x) - f(a)g(a)}{x-a} &= \frac{f(x)g(x) - f(a)g(x) + f(a)g(x) - f(a)g(a)}{x-a} \\ &= \frac{f(x)-f(a)}{x-a}g(x) + f(a)\frac{g(x)-g(a)}{x-a}.\end{aligned}$$

When we take the limit $x \to a$, we have

$$\frac{f(x)-f(a)}{x-a} \to f'(a), \quad \frac{g(x)-g(a)}{x-a} \to g'(a), \text{ and } g(x) \to g(a),$$

where the last one is where continuity of g at a (which follows from differentiability) is needed. This thus gives

$$\lim_{x\to a} \frac{f(x)g(x) - f(a)g(a)}{x-a} = f'(a)g(a) + f(a)g'(a)$$

as claimed.

Chain rule. The chain rule takes more effort to justify, but is important to understand for the sake of setting up the idea of using *errors*—which we will consider more carefully when discussing Taylor's theorem—and because this same idea is the one that will be needed to prove the multivariable chain rule next quarter, which is more important.

The claim is that if f is differentiable at a and g is differentiable at $f(a)$, then the composition $g \circ f$ is differentiable at a and

$$(g \circ f)'(a) = g'(f(a))f'(a).$$

We first note that if we introduce the "error"

$$\epsilon(x) = \frac{f(x)-f(a)}{x-a} - f'(a),$$

which measures how far off from $\frac{f(x)-f(a)}{x-a}$ the number $f'(a)$ really is, then the definition of differentiability of f at a can be rephrased as saying $\epsilon(x) \to 0$ as $x \to a$. Similarly, the error

$$\delta(y) = \frac{g(y) - g(f(a))}{y - f(a)} - g'(f(a))$$

approaches 0 as $y \to f(a)$, by differentiability of g at $f(a)$. With these errors we can write

$$f(x) - f(a) = (x-a)(f'(a) + \epsilon(x)) \text{ and } g(y) - g(f(a)) = (y - f(a))(g'(f(a)) + \delta(y)).$$

Taking $y = f(x)$ in the second gives

$$g(f(x)) - g(f(a)) = (f(x) - f(a))(g'(f(a)) + \delta(f(x))),$$

and then substituting in for $f(x) - f(a)$ gives

$$g(f(x)) - g(f(a)) = (x-a)(f'(a) + \epsilon(x))(g'(f(a)) + \delta(f(x))).$$

Thus

$$\frac{g(f(x)) - g(f(a))}{x-a} = (f'(a) + \epsilon(x))(g'(f(a)) + \delta(f(x)))$$

for $x \neq a$. As $x \to a$, we have $\epsilon(x) \to 0$ and $f(x) \to f(a)$ (continuity), so $\delta(f(x)) \to 0$ as well because $\delta(y) \to 0$ as $y \to f(a)$. Hence taking the limit gives

$$\lim_{x \to a} \frac{g(f(x)) - g(f(a))}{x-a} = (f'(a) + 0)(g'(f(a)) + 0) = g'(f(a))f'(a),$$

as the chain rule requires.

Derivative at extreme point. Now we justify one of the most basic properties of derivatives when it comes to optimization, namely the fact that if $f : U \to \mathbb{R}$ is differentiable and has a local maximum or minimum at a, then $f'(a) = 0$. To be clear: to say that f has a *local maximum* at a means that there exists an interval $(a-\delta, a+\delta)$ around a on which $f(a) \geq f(x)$ is true, and similarly for a local minimum with the inequality reversed. This is a very well-known fact from calculus, which is more-or-less clear intuitively when considering the graph of f, since the tangent line at a local max or min is horizontal. But of course, not every possible differentiable function will have an easy-to-draw graph, so that geometric intuition isn't enough to constitute a full justification.

So, suppose f has a local maximum at a. (The proof for the local minimum case is very similar, with inequalities below reversed.) Then there exists $\delta > 0$ such that $f(x) \geq f(a)$ for all $x \in (a-\delta, a+\delta)$. We know that

$$\lim_{x \to a} \frac{f(x) - f(a)}{x-a} = f'(a)$$

exists. In particular, this means that the limit as x approaches a from the left and right both exist and equal $f'(a)$. For $x \in (a-\delta, a)$, we have that $x - a < 0$ and $f(x) - f(a) \leq 0$, so the fraction in the above limit is positive for x to the left of a. Thus

$$\lim_{x \to a^-} \frac{f(x) - f(a)}{x-a} = f'(a) \geq 0$$

where $x \to a^-$ means the limit as we approach a from the left. For $x \in (a, a+\delta)$, $x - a > 0$ and $f(x) - f(a) \leq 0$ so the fraction in the above limit is negative. Thus

$$\lim_{x \to a^+} \frac{f(x) - f(a)}{x-a} = f'(a) \leq 0.$$

Since $f'(a) \geq 0$ and $f'(a) \leq 0$, we must therefore have $f'(a) = 0$ as claimed.

Lecture 27: Mean Value Theorem

Warm-Up. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function. We claim that f' has the following intermediate value property: whenever $f'(a) < y < f'(b)$, there exists c between a and b such that $f'(c) = y$. In other words, derivatives always attain any "intermediate" values. If f' is continuous this is just a consequence of the Intermediate Value Theorem, but the amazing fact here is that this is true even when f' is *not* continuous! (The result is often called *Darboux's theorem*, and a function with the intermediate value property in general is often called a *Darboux* function.) This places a big restriction on which types of functions can arise as derivatives of others, as we'll see.

Suppose without loss of generality that $a < b$ and let F be the function defined by $F(x) = f(x) - y$. Then F is differentiable since it is the difference of differentiable functions, and hence it is continuous on $[a, b]$. Thus by the Extreme Value Theorem F has a minimum at some point in $[a, b]$. Now, we claim that this minimum does not occur at a nor at b. Indeed, since

$$F'(a) = f'(a) - y < 0,$$

we have that

$$\frac{F(x) - F(a)}{x - a} < 0 \text{ for } x \text{ in some interval } (a - \delta, a + \delta) \text{ around } a.$$

(This is a fact we've seen before: the limit of the fraction above defines $F'(a)$, and if this limit is negative then the expression we take the limit of must be negative near the point a we're approaching.) In particular, for $x \in (a, a + \delta)$ we have that $x - a > 0$ so $F(x) - F(a) < 0$. Hence $F(x) < F(a)$ so the minimum of F does not occur at a. Similarly, since

$$F'(b) = f'(b) - y > 0,$$

we have

$$\frac{F(x) - F(b)}{x - b} > 0 \text{ for } x \text{ in some interval } (b - \delta, b + d\delta) \text{ around } b.$$

In particular for $x \in (b - \delta, b)$, $x - b < 0$ so $F(x) - F(b) < 0$ and again $F(x) < F(b)$ meaning that minimum of F does not occur at b. Thus the minimum of F must occur at some $c \in (a, b)$. At a minimum the derivative must be zero, so $F'(c) = f'(c) - y = 0$, meaning that $f'(c) = y$ and c is the desired point.

Derivatives do not have jump discontinuities. The fact that derivatives have the intermediate value property says that certain functions can never arise as the derivatives of other functions; in particular, any function with a "jump" discontinuity is not the derivative of anything else. For instance, consider the function

$$g(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0. \end{cases}$$

If there was a differentiable function G such that $G' = g$, we would have

$$G'(-1) = g(-1) = -1 < 0 < G'(1) = g(1) = 1,$$

so the Warm-Up would say that there should exist $c \in (-1, 1)$ such that $G'(c) = g(c) = 0$, which is nonsense. Thus there can be no such G.

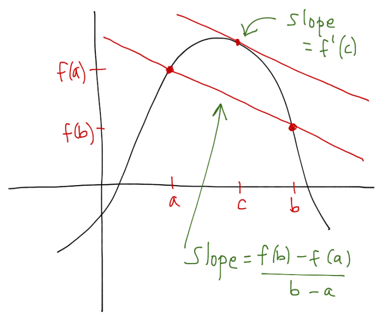
The problem is that g has a jump discontinuity at 0. Derivatives of course can have discontinuities, but the Warm-Up places restrictions on just what types of discontinuities those can be: the only way in which a derivative might fail to be continuous is because of some "oscillatory" behavior, such as what happens with the derivative of the function which is $x^2 \sin \frac{1}{x}$ for $x \neq 0$ and 0 at $x = 0$. Said another way, a function with a jump discontinuity such as g above does not have an *antiderivative*. This is surprising for the following reason: such a function, such as this explicit g, could very well have a well-defined *integral*, even without having an antiderivative. This seems to run counter to what you know from calculus, where "integral" and "antiderivative" are usually thought of as being synonymous with one another. The point is that "integration" and "anti-differentiation" really are two different things, and the fact that they are sometimes related as clarified by the *Fundamental Theorem of Calculus* truly is quite amazing. You'll see this next quarter when studying integration in detail.

Mean Value Theorem. We now come to the *Mean Value Theorem*, which is perhaps the entire point of differentiability in the first place, at least from the perspective of an *analysis* (not calculus) course. Ultimately, if continuity says that we can control how large $f(b) - f(a)$ is, the Mean Value Theorem (and hence differentiability) goes one step further by giving more *explicit* control over this expression.

Here is the claim. If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$. The "explicit control" mentioned above comes from this explicit expression for $f(b) - f(a)$, where by controlling the derivative we gain better control over this difference. The Mean Value Theorem is completely obvious if you draw a picture of what it says. Take the graph of a differentiable function f and draw the points $(a, f(a))$ and $(b, f(b))$ on the graph. The line passing through these two points has slope

$$\frac{f(b) - f(a)}{b - a},$$

and from the picture:



it looks as though there should be some point $c \in (a, b)$ at which the slope of the tangent line has same slope as this line above:

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

This is precisely what the Mean Value Theorem says, after rewriting this equation. The proof essentially amounts to "straightening out" the picture above and applying the Extreme Value Theorem in a way similar to the Warm-Up.

Here are the details. Set

$$g(x) = f(x) - \left[f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \right],$$

is still differentiable and continuous. The expression in brackets above is precisely the secant line connecting $(a, f(a))$ to $(b, f(b))$ in the picture above, so g measures the difference between this line and f. Note that in the picture it seems we should thus have $g(a) = 0 = g(b)$ since the secant line and f agree at these points, and this is indeed evident from the definition of g as well:

$$g(a) = f(a) - \left[f(a) + \frac{f(b) - f(a)}{b - a}(a - a) \right] = f(a) - f(a) = 0$$

and

$$g(b) = f(b) - \left[f(a) + \frac{f(b) - f(a)}{b - a}(b - a)\right] = f(b) - [f(a) + f(b) - f(a)] = 0.$$

(Geometrically, as alluded to above, g has the effect of "straightening out" the graph of f by putting the points of intersection between the graph and the line down on the x-axis.)

Now, since g is continuous on $[a, b]$, it has a maximum and a minimum on $[a, b]$. If both occurred at the endpoints, then $g(a) = 0 = g(b)$ would be both the maximum and minimum value of g, so that g would be constant. In this case, f is exactly equal to the secant line, so the derivative of f is equal to the slope of the secant line at *all* points, which is better than what Mean Value Theorem asks for. If g is not constant, at least one of the max or min (in the "straightened out" version of the picture above it would be the max) occurs in (a, b). If this extreme point is attainted at $c \in (a, b)$, then we have $g'(c) = 0$ by what we already know about the value of derivatives at local extrema, and this translates to

$$0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}.$$

This is what the Mean Value Theorem asks for, after rearranging terms.

Basic mean value consequences. As stated before, the Mean Value Theorem gives a way to directly relate values of the function to one another using properties of the derivative. The first few basic consequences are simple ones you expect from calculus, but whose justification depends on this result.

For example, if $f' > 0$ at all points, then f is an increasing function. Indeed, for any a, b the Mean Value Theorem gives c such that

$$f(b) - f(a) = f'(c)(b - a).$$

But $f'(c) > 0$, so if $a \leq b$ then $f(a) \leq f(b)$ as well since both sides above should be nonnegative. This says that f is increasing. If instead $f' < 0$, the equation above gives that if $a \leq b$ then $f(a) \geq f(b)$, so f is decreasing in this case.

Finally, if $f' = 0$ at all points, then we have $f(b) - f(a) = 0(b - a) = 0$ as a consequence of the mean value identity above, so $f(a) = f(b)$ for all a, b, which says that f is constant. Again these are all basic facts from calculus, but they cannot be justified formally without some version of the Mean Value Theorem. For example, knowing that f has zero derivative on an interval means that for any a in that interval

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = 0.$$

But, this doesn't say much about the fraction we're taking the limit of since you can definitely have a nonzero expression which gives a limit of zero. So, we can't directly conclude that the numerator must be zero; we need some way of comparing $f(x) - f(a)$ to $x - a$, which is precisely what the Mean Value Theorem gives us.

Lecture 28: More on Mean Value

Warm-Up. Suppose that f is continuous on $\mathbb{R}$, differentiable at all $x \neq a$, and that $\lim_{x \to a} f'(x) = L$ exists. Then f is differentiable at a as well and $f'(a) = L$. Before looking at the proof, note that this too places a restriction on how badly derivatives can actually behave, similar to the "no jump discontinituies" fact we saw before. In particular, if f' exists everywhere near a point a and the limit of the derivative exists as you approach a, this fact says that f' is actually continuous at a! A function who graph thus has a "hole" at a point then cannot be the derivative of another function, and so has no antiderivative.

To prove the claim, consider the limit defining $f'(a)$:

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

For any fixed x, f is differentiable on the open interval between x and a so the Mean Value Theorem says that there exists some c_x in this interval such that

$$f(x) - f(a) = f'(c_x)(x - a).$$

Substituting this above gives

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{f'(c_x)(x - a)}{x - a} = \lim_{x \to a} f'(c_x).$$

Since c_x is sandwiched between x and a, as $x \to a$ we also have $c_x \to a$. Thus the above limit is the same as

$$\lim_{c_x \to a} f'(c_x),$$

which exists and equals L by our assumption on f. Hence $f'(a)$ exists and $f'(a) = L$.

Bounded derivative implies uniformly continuous. Now we expand on something we saw when introducing uniform continuity, namely that functions which do not "change" too rapidly should be uniformly continuous. The precise version of this statement is that a differentiable function $f : \mathbb{R} \to \mathbb{R}$ which has bounded derivative everywhere is in fact uniformly continuous. Say that M is a bound for f', so $|f'(x)| \leq M$ for all $x \in \mathbb{R}$. For any x and y, by the Mean Value Theorem says there exists $c \in \mathbb{R}$ such that

$$|f(x) - f(y)| = |f'(c)||x - y| \leq M|x - y|.$$

(A function satisfying this type of inequality in a general metric space is said to be *Lipschitz* with *Lipschitz constant* M. For example, the contractions you saw on a recent homework are Lipschitz with Lipschitz constants smaller than 1.) Thus, if $\epsilon > 0$, $\delta = \frac{\epsilon}{M} > 0$ has the property that

$$|x - y| < \delta \implies |f(x) - f(y)| \leq M|x - y| < M\delta = \epsilon,$$

so that f is uniformly continuous. (Lipschitz functions are uniformly continuous in general.)

Even though this result is good for visualizing what the graph of a uniformly continuous function might look like, it does not describe all possible uniformly continuous functions. For one thing, not every uniformly continuous function is differentiable, and this fact only applies to differentiable functions. More importantly, the converse of this fact is not true: if f is differentiable and uniformly continuous, it is not necessarily true that f' must be bounded. For instance, the function $f(x) = \sqrt{x}$ is uniformly continuous on $(0, \infty)$ but its derivative is unbounded there. (Actually, it is only for differentiable Lipschitz functions that we can guarantee the derivative has to be bounded.)

Second derivative test. Here is another basic fact from calculus whose proof uses the Mean Value Theorem. Suppose that f is differentiable, that $f'(a) = 0$, and that $f''(a) > 0$. We claim that f then has a *local minimum* at a, meaning there exists an interval $(a - \delta, a + \delta)$ around a such that $f(a) \leq f(x)$ for all $x \in (a-\delta, a+\delta)$. (This is often referred to as the *second derivative test* for classifying local extrema via concavity. The case where $f''(a) < 0$ implies that f has a local maximum at a is on the final homework.)

To prove this, start with

$$\lim_{x \to a} \frac{f'(x) - f'(a)}{x - a} = f''(a) > 0.$$

Since this limit is positive, there exists $\delta > 0$ such that

$$\frac{f'(x) - f'(a)}{x - a} > 0 \text{ for } x \in (a-\delta, a+\delta).$$

Recall that $f'(a) = 0$, so this inequality means that

$$f'(x) \text{ and } x - a \text{ have the same sign for } x \in (a-\delta, a+\delta).$$

Now, take any $x \in (a-\delta, a+\delta)$. By the Mean Value Theorem there exists c between x and a such that

$$f(x) - f(a) = f'(c)(x-a).$$

Now, either $x < c < a$ or $a < c < x$, and either way $x - a$ and $c - a$ have the same sign, and thus $f'(c)$ and $x - a$ have the same sign. This means that

$$f(x) - f(a) = f'(c)(x-a) \geq 0, \text{ so } f(x) \geq f(a) \text{ for } x \in (a-\delta, a+\delta)$$

and hence f has a local minimum at a as claimed.

Error estimates. When viewed in the right way, the Mean Value Theorem, and its higher-order generalization *Taylor's theorem* (which we'll look at next time), is really a statement about *errors*. To setup the context let us take a step back to ordinary continuity. For $x \neq a$, we can write

$$f(x) = f(a) + \epsilon_0(x)$$

for the "zeroth order error" $\epsilon_0(x) := f(x) - f(a)$. This zeroth order error describes how far off we are when approximating $f(x)$ using the *constant* function $f(a)$. The property of being continuous at a is then the claim that

$$\epsilon_0(x) \to 0 \text{ as } x \to a,$$

so that this "constant" approximation gets better the closer we are to a. The upshot is that the Mean Value Theorem then gives an exact form for this error. Indeed, if f is differentiable, we have

$$f(x) = f(a) + \underbrace{f'(c)(x-a)}_{\epsilon_0(x)}$$

for some c between x and a, so that the explicit form $\epsilon_0(x) = f'(c)(x-a)$ for the zeroth order error is really the point of the Mean Value Theorem from this perspective.

Now, if f is differentiable, we get an even better approximation than the constant one. Indeed, write

$$f(x) = f(a) + f'(a)(x-a) + \epsilon_1(x)$$

where $\epsilon_1(x)$ is now the *first-order* error ($\epsilon_1(x)$ is literally just $f(x)$ minus $f(a) + f'(a)(x-a)$). The first-order error thus tells us how far off the *linear* approximation $f(a) + f'(a)(x-a)$ is from the actual value of $f(x)$. The definition of differentiable then says precisely that

$$\frac{\epsilon_1(x)}{x-a} \to 0 \text{ as } x \to a,$$

so that not only does $\epsilon_1(x)$ approach 0, it does so "faster" than $x - a$. The upshot is that the first-order error approaches 0 more rapidly than the zeroth-order error, so the linear (or "tangent line") approximation should be better than the constant approximation.

If f is twice-differentiable (which implies in particular that f' is continuous), Taylor's theorem will give an explicit form for *this* second-order error as

$$f(x) = f(a) + f'(a)(x-a) + \underbrace{\frac{f''(c)}{2}(x-a)^2}_{\epsilon_1(x)}$$

for some c between x and a. And so on, Taylor's theorem in general is all about saying what happens with the higher-order errors. At the next step, being twice-differentiable gives

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \epsilon_2(x)$$

for some second-order error $\epsilon_2(x)$ that controls the *quadratic* approximation $f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$ to f near a. This second-order error satisfies

$$\frac{\epsilon_2(x)}{(x-a)^2} \to 0 \text{ as } x \to a,$$

so that $\epsilon_2(x)$ goes to zero even more rapidly than $(x-a)^2$, which is less than $x - a$ for x close enough to a. If f is three-times differentiable (let's just say C^3), Taylor's theorem then gives

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \underbrace{\frac{f'''(c)}{3!}(x-a^3)}_{\epsilon_2(x)}$$

for some c between x and a, and so on. The overarching idea is that each time we introduce a new order of differentiability, we gain more control over $f(x) - f(a)$, or indeed over $f(x)$ minus higher-order approximations. We'll state the full form of Taylor's theorem, prove it, and look at some applications next time. Good stuff!

Lecture 29: Taylor's Theorem

Warm-Up. Suppose f and g are differentiable and fix $x \neq a$. We show there exists c between x and a such that

$$g'(c)(f(x) - f(a)) = f'(c)(g(x) - g(a)).$$

Now, before doing so, it is natural to wonder why this type of result might be useful since it seems like quite a random thing to consider. Note first that when g is the function $g(x) = x$, the equation above is precisely the one given in the Mean Value Theorem: we have $g'(c) = 1$, so the left side above is $f(x) - f(a)$ and the right side is $f'(c)(x - a)$. So, this result can be viewed as a generalization of the Mean Value Theorem, and indeed it is often called the *generalized Mean Value Theorem*. If we assume that $g'(c) \neq 0$ and $g(x) - g(a) \neq 0$, we can rewrite the given equation as

$$\frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(c)}{g'(c)},$$

so that if we think of the left side as some kind of "slope/mean" of f "with respect to g" (whatever that means), the claim is that this "mean" value is attained as an actual value of the derivative, again in some sense "with respect to g". (You will consider such values "with respect to g" next quarter when discussing a general form of integration.) Note that if we simply apply the Mean Value Theorem to f and g separately:

$$f(x) - f(a) = f'(c)(x-a) \text{ and } g(x) - g(a) = g'(d)(x-a)$$

for some c and d, we get

$$\frac{f(x)-f(a)}{g(x)-g(a)} = \frac{f'(c)}{g'(d)},$$

but with possibly *different* inputs on the right. Thus the real point of the result in the Warm-Up is that we guarantee this equation holds with the *same* input on the right.

Set F to be the function (of y) defined by

$$F(y) = g(y)[f(x) - f(a)] - f(y)[g(x) - g(a)].$$

Since f and g are differentiable, F is differentiable as well. We have

$$F(a) = g(a)f(x) - f(a)g(x) = F(x).$$

By the Mean Value Theorem there exists c between x and a such that

$$F(x) - F(a) = F'(c)(x-a),$$

which, since the left side is zero and $x \neq a$, gives $F'(c) = 0$. Explicitly we have

$$F'(c) = g'(c)[f(x) - f(a)] - f'(c)[g(x) - g(a)],$$

so $F'(c) = 0$ gives $g'(c)[f(x) - f(a)] = f'(c)[g(x) - g(a)]$ as desired.

One quick application of this result is *L'Hopital's rule*, which you might remember from calculus. The statement is that if f, g are differentiable and $f(a) = 0 = g(a)$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Indeed, since $f(a) = 0 = g(a)$, we have

$$\frac{f(x)}{g(x)} = \frac{f(x)-f(a)}{g(x)-g(a)},$$

which by our result is

$$\frac{f(x)-f(a)}{g(x)-g(a)} = \frac{f'(c)}{g'(c)}$$

for some c between x and a. Since c is between x and a, as $x \to a$ we also have $c \to a$, so

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{c \to a} \frac{f'(c)}{g'(c)}$$

as desired. The point is that the Warm-Up gives a way to control $f(x) - f(a)$ and $g(x) - g(a)$ in a *simultaeneous* way.

Taylor's theorem. As explained at the end of last time, the Mean Value Theorem can be viewed as a giving an explicit expression for the "zeroth-order errors" obtained when approximating functions by constant functions. The general version of this result for higher-order errors is *Taylor's theorem.* Here is the statement:

Suppose f is a C^{n+1} function. (Actually, Taylor's theorem applies in the more general situation where is C^n and $(n+1)$-times differentiable, without assuming that the $(n+1)$-st derivative is continuous, but saying all this is more of a mouthful than just saying "C^{n+1}", which is the only reason why I'm using C^{n+1} as my assumption.) Then for any $x \neq a$, there exists c between x and a such that

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

The summation in the first term on the right is a polynomial:

$$\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

and is called the *n-th order Taylor polynomial of f centered at a*. Taylor's theorem thus gives an expression for the error $\epsilon_n(x)$ (also called the *n-th order Taylor remainder*) obtained when approximating f via this Taylor polynomial:

$$\epsilon_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \text{ for some } c \text{ between } x \text{ and } a,$$

where $\epsilon_n(x)$ is the difference between $f(x)$ and its n-th order Taylor polynomial approximation. Note that the error in the n-th order Taylor approximation is described in terms of the derivative for the *next larger* order $n+1$. The Mean Value Theorem is precisely the $n=0$ case of Taylor's theorem, and, as we'll see, also plays a key role in its proof.

Taylor polynomials. Before proving Taylor's theorem, let us comment on why Taylor polynomials specifically are the ones that appear. That is, if we are trying to approximate a function f near a point a using a polynomial

$$c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n,$$

why should it be the case that c_n is given by $\frac{f^{(n)}(a)}{n!}$? The point is that the polynomial with these specific coefficients is the unique one which agrees with f "up to order n" at a. This means that if we denote the polynomial with these coefficients by $P_n(x)$, then $P_n(x)$ is the only polynomial of degree n whose derivatives at a all the way up to the n-th derivative agree with those of f:

$$f^{(k)}(a) = P_n^{(k)}(a) \text{ for } k = 0, \ldots, n.$$

Indeed, set $P_n(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n$. Then $P_n(a) = c_0$, so if this is meant to agree with $f(a)$, then we must have $c_0 = f(a)$. Next, $P_n'(a) = c_1$, so if this agrees with $f'(a)$ we have $c_1 = f'(a)$. Then $P_n''(a) = 2c_2$, so we must have $c_2 = \frac{f''(a)}{2}$ in order for $P_n''(a) = f''(a)$ to hold. And so on: in general $P_n^{(k)}(a) = k!c_k$, so

$$f^{(n)}(a) = P_n^{(k)}(a) \iff c_k = \frac{f^{(k)}(a)}{k!}.$$

Thus $P_n(x)$ is the n-th order Taylor polynomial of f at a as claimed.

Proof of Taylor's theorem. Let us first prove Taylor's theorem in the $n = 1$ case to get a feel for the general argument. In this case, the claim is that if f is twice differentiable, then for any x and a there is some c between x and a satisfying

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(c)}{2!}(x-a)^2.$$

Consider the function h of a variable y defined by

$$h(y) = \epsilon_1(y) - \frac{\epsilon_1(x)}{(x-a)^2}(y-a)^2,$$

where $\epsilon_1(x) = f(x) - [f(a) + f'(a)(x-a)]$ is the first-order Taylor error/remainder of f at a. The function h is twice differentiable with respect to y and its derivatives with respect to y are:

$$\begin{aligned} h'(y) &= \epsilon_1'(y) - 2\frac{\epsilon_1(x)}{(x-a)^2}(y-a), \text{ and} \\ h''(y) &= \epsilon_2''(y) - 2\frac{\epsilon_1(x)}{(x-a)^2}. \end{aligned}$$

Since f and its first-order Taylor approximation agree up to first-order at a, we have that $\epsilon_1(a)$ and $\epsilon_1'(a)$ are both zero, so the function h satisfies:

$$\begin{aligned} h(a) &= \epsilon_1(a) - \frac{\epsilon_1(x)}{(x-a)^2}(a-a)^2 = 0 - 0 = 0, \\ h'(a) &= \epsilon_1'(a) - 2\frac{\epsilon_1(x)}{(x-a)^2}(a-a) = 0 - 0 = 0, \text{ and} \\ h(x) &= \epsilon_1(x) - \frac{\epsilon_1(x)}{(x-a)^2}(x-a)^2 = \epsilon_1(x) - \epsilon_1(x) = 0. \end{aligned}$$

By the Mean Value Theorem applied to h there exists c_1 between x and a such that

$$0 - 0 = h(x) - h(a) = h'(c_1)(x-a), \text{ so } h'(c_1) = 0.$$

Now applying the Mean Value Theorem to h' says that there exists c between c_1 and a such that

$$0 - 0 = h'(c_1) - h'(a) = h''(c)(c_1 - a), \text{ so } h''(c) = 0.$$

But $\epsilon_1(y) = f(y) - [f(a) + f'(a)(y-a)]$, so $\epsilon_1''(y) = f''(y)$ and thus the equation $h''(c) = 0$ is the same as

$$f''(c) - 2\frac{\epsilon_1(x)}{(x-a)^2} = 0, \text{ or } \epsilon_1(x) = \frac{f''(c)}{2}(x-a)^2,$$

which is the desired claim. Since c is between c_1 and c, and x_1 is between x and a, c is indeed between x and a, so Taylor's theorem holds in the $n = 1$ case.

The proof in the general case is similar, where we introduce an auxiliary function h, and then "bootstrap" by applying the Mean Value Theorem first to h, and then h', then h'', and so on all the way up to $h^{(n)}$. Namely, set

$$h(y) = \epsilon_n(y) - \frac{\epsilon_n(x)}{(x-a)^n}(y-a)^n$$

where ϵ_n is the n-th order Taylor error:

$$\epsilon_n(y) = f(y) - \sum_{k=0}^{n} \frac{f^{(n)}(a)}{n!}(y-a)^n.$$

This function satisfies

$$h(x) = 0,\ h(a) = 0,\ h'(a) = 0,\ h''(a) = 0, \ldots, h^{(n)}(a) = 0$$

since f and the n-th order Taylor polynomial agree at a to order n, and applying the Mean Value Theorem to h, then h', then h'', etc in the end produces c between x and a such that $h^{(n+1)}(c) = 0$, in a manner exactly analogous to the $n = 1$ case above. The equation $h^{(n+1)}(c) = 0$ then precisely works out to be (by direct computation)

$$\epsilon_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1},$$

which is what Taylor's theorem states.