PROBLEM SET 3

Choose half of the problems to submit.

Problem 1: Let (X,T) be a system and $Y \subset X$ a T-invariant compact subset. Recall Y is called locally maximal (or isolated) if there exists an open neighborhood V of Y such that

$$Y = \cap_{n\in\mathbb{Z}} T^n(V).$$

Show that a subshift (Z,σ) which is locally maximal in a full shift must be a shift of finite type, and conversely, any shift of finite type may be realized as a locally maximal invariant set in a full shift.

Problem 2: Show that if (X_A,σ_A) is an irreducible shift of finite type with positive entropy and $Y \subset X$ is a proper subshift, then the entropy of (Y,σ) is strictly less than the entropy of σ_A.

Problem 3: For each of the following pairs of matrices, determine whether they are shift equivalent over $\mathbb{Z}_+$:

(1) $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 4 \\ 1 & 1 \end{pmatrix}$.

(2) $A = \begin{pmatrix} 3 & 5 \\ 2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}$.

Problem 4: Compute the dimension groups for $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix}$.

Problem 5: Show that the dimension groups associated to $\begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}$ and $\begin{pmatrix} 2 & 1 \\ 0 & 5 \end{pmatrix}$ are not isomorphic.

Problem 6: Suppose A is a square $\mathbb{Z}_+$-matrix and $\det(I - tA) = 1 - nt$ where n is some positive integer. Prove that (X_A, σ_A^k) is topologically conjugate to a full shift for all sufficiently large k.

Problem 7: Let A be an irreducible square $\mathbb{Z}_+$-matrix and recall the construction of the dimension group (D_A, δ_A) in terms of rays and beams given in class. A continuous map $(X_A,\sigma_A) \xrightarrow{\pi} (X_B,\sigma_B)$ is called *u-bijective* if for any

$x \in X_A$, π restricted to $W^u(x) = \{y \in X_A \mid y_{(-\infty,n]} = x_{(-\infty,n]}$ for some $n\}$ maps bijectively onto $W^u(\pi(x))$ in X_B. Show that if $\pi\colon (X_A, \sigma_A) \to (X_B, \sigma_B)$ is u-bijective, then it induces a well-defined homomorphism between the dimension groups $\pi_*\colon (D_A, \delta_A) \to (D_B, \delta_B)$, and hence the dimension group is functorial with respect to u-bijective maps.

Problem 8: Let $(\Sigma_n, \hat{T}_n)$ be the n-adic solenoid constructed in class. Let $\mathbb{Z}_{(n)}$ denote the n-adic integers; that is, $\mathbb{Z}_{(n)}$ is the inverse limit of the system

$$\mathbb{Z}/n \xleftarrow{\text{mod } n} \mathbb{Z}/n^2 \xleftarrow{\text{mod } n^2} \mathbb{Z}/n^3 \xleftarrow{\text{mod } n^3} \cdots .$$

Recall the *suspension* of a system (X, T) is the topological space defined by $(X \times [0,1]) / \sim$ where $(x, 1) \sim (T(x), 0)$. Show that the suspension of the n-adic integers is homeomorphic to the n-adic solenoid.

Bonus problem: Show that $A = \begin{pmatrix} 19 & 5 \\ 4 & 1 \end{pmatrix}$ is not shift equivalent to its transpose, and conclude that (X_A, σ_A) is not topologically conjugate to its inverse (X_A, σ_A^{-1}).