

FINITE GROUP EXTENSIONS OF SHIFTS OF FINITE TYPE: K-THEORY, PARRY AND LIVŠIĆ

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ABSTRACT. This paper extends and applies algebraic invariants and constructions for mixing finite group extensions of shifts of finite type. For a finite abelian group G , Parry showed how to define a G -extension S_A from a square matrix over \mathbb{Z}_+G , and classified the extensions up to topological conjugacy by the strong shift equivalence class of A over \mathbb{Z}_+G . Parry asked in this case if the dynamical zeta function $\det(I - tA)^{-1}$ (which captures the “periodic data” of the extension) would classify up to finitely many topological conjugacy classes the extensions by G of a fixed mixing shift of finite type. When the algebraic K-theory group $NK_1(\mathbb{Z}G)$ is nontrivial (e.g., for $G = \mathbb{Z}/n$ with n not squarefree) and the mixing shift of finite type is not just a fixed point, we show the dynamical zeta function for any such extension is consistent with infinitely many topological conjugacy classes. Independent of $NK_1(\mathbb{Z}G)$, for every nontrivial abelian G we show there exists a shift of finite type with an infinite family of mixing nonconjugate G extensions with the same dynamical zeta function. We define computable complete invariants for the periodic data of the extension for G not necessarily abelian, and extend all the above results to the nonabelian case. There is other work on basic invariants. The constructions require the “positive K-theory” setting for positive equivalence of matrices over $\mathbb{Z}G[t]$.

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1. INTRODUCTION

One part of the celebrated paper [26] of Livšic shows that for certain hyperbolic dynamical systems $T : X \rightarrow X$, if the restrictions of Hölder functions f and g to the periodic points are cohomologous as point set maps (i.e. ignoring topology), then they are Hölder cohomologous in (X, T) — i.e., $f = g + r \circ T - r$, with the transfer function r being Hölder continuous. (For an excellent introduction to the Livšic theory and to cocycles in dynamical systems, see [19].) The proof of Livšic works for functions into a metrizable abelian group. This result was generalized to nonabelian groups for shifts of finite type by Parry (see Remark 4.7) and Schmidt [32, 39], and to more sophisticated systems by various authors (e.g. [32, 39, 18, 38]).

Parry posed a bold related question in the case G is finite abelian. For (X, T) a mixing SFT and $f : X \rightarrow G$, a suitable dynamical zeta function ζ_f encodes for all n, g the number of periodic orbits of size n and weight g . Then $\zeta_f = \zeta_g$ if and only if there is a bijection $\beta : \text{Per}(X) \rightarrow \text{Per}(X)$ such that $f \circ \beta$ and g are cohomologous as point set maps. Parry asked, for $f : X \rightarrow G$ continuous and G a finite abelian group: does the set of continuous $g : X \rightarrow G$ with $\zeta_g = \zeta_f$ contain only finitely many continuous cohomology classes? Parry's question probed not only a possible direction for extending the Livšic result, but also the strength of conjugacy invariants for mixing SFTs and their group extensions. (The classification of cohomology classes of functions from X into a group is a version of the classification of group extensions of a system (X, T) .)

We will show that for many groups G (the finite groups G with $NK_1(\mathbb{Z}G) \neq 0$), the answer to Parry's question is negative for *every* nontrivial dynamical zeta function. The ingredients for this are the following.

- (1) Generalizing the Williams' theory for SFTs, Parry showed that any G -extension of an SFT (X, S) can be presented by a square matrix A over \mathbb{Z}_+G , and two such group extensions are isomorphic if and only if their presenting matrices are strong shift equivalent (SSE) over the positive semiring \mathbb{Z}_+G of the integral group ring $\mathbb{Z}G$. The dynamical zeta function, with coefficient ring $\mathbb{Z}G$, is then $\zeta(z) = (\det(I - zA))^{-1}$. Parry's theory, which he never published, is presented in [11] (in Appendix A, we correct an error in the presentation in [11]).¹

¹The algebraic invariants here over \mathbb{Z} (shift and strong shift equivalence, $\det(I - tA)$), are paralleled in the study of shifts of finite type with Markov measure, where a finitely generated abelian group

- (2) By Theorem 2.2, taken from [10], for any ring \mathcal{R} and shift equivalence (SE) class \mathcal{C} of matrices over \mathcal{R} , the collection of SSE classes over \mathcal{R} of matrices in \mathcal{C} is in bijective correspondence with the group $\mathrm{NK}_1(R)$ of algebraic K-theory. If $\mathrm{NK}_1(R)$ is not trivial, then it is not finitely generated as a group [13, 47]. We give more background on $\mathrm{NK}_1(R)$ in Appendix C, and give some concrete examples in Appendix C.
- (3) In this paper, given $\mathrm{NK}_1(\mathbb{Z}G)$ nontrivial, we construct, for any nontrivial mixing SFT (X, S) , infinitely many G -extensions of (X, S) defined by matrices which pairwise are SE over \mathbb{Z}_+G but are not SSE over $\mathbb{Z}G$ (and hence are not SSE over \mathbb{Z}_+G). Consequently, these extensions pairwise are eventually conjugate; are not conjugate; and have the same isomorphism class of conjugacy classes (in the abelian case, this means they have the same dynamical zeta function). The construction arguments, carried out in Section 5, use constructive tools available in the polynomial matrix setting.

In Section 4, we discuss Parry’s question in more detail, and we use the structure of shift equivalence of matrices over $\mathbb{Z}G$ to address and clarify some other cases of Parry’s question (Sec. 4). We show that for every nontrivial finite group G , there is an infinite collection of matrices which are not SE- $\mathbb{Z}G$ and which can be realized in mixing extensions of SFTs with the same periodic data. Consequently, for every nontrivial finite abelian group G , there is a dynamical zeta function compatible with infinitely many SE- $\mathbb{Z}G$ classes which can be realized in mixing extensions of SFTs. On the other hand, we give a class of mixing examples for which the dynamical zeta function determines the SE- $\mathbb{Z}G$ class (regardless of $\mathrm{NK}_1(\mathbb{Z}G)$). For such a class, known invariants do not provide a negative answer to Parry’s question. In no nontrivial case do known constructions provide a positive answer to Parry’s question.

One purpose of this paper is to summarize and extend our understanding of the algebraic invariants for and approaches to mixing finite group extensions of shifts of finite type (which we need anyway for Parry’s question). (In particular, for not necessarily abelian finite groups G , we give complete and computable invariants for the periodic data of the G extension of a shift of finite type.) There are two parallel formulations for this. One involves SSE of matrices over $\mathbb{Z}G$ (Section 2). The other formulation is in terms of the “positive K-theory” of polynomial matrix presentations (Section 3). In Appendix B, we work out results involving primitivity (some of which we need for proofs) and shift equivalence to extend the theory parallel to the theory over \mathbb{Z} . In Appendix A we review the basic connection of matrices over \mathbb{Z}_+G to G -extensions, and correct a mistake in [11]. (The mistake is only that the defining matrix should be

appears in place of the finite group G [28, 33], and positivity issues around $\det(I - tA)$ and shift equivalence become more analytic and formidable [16].

associated to a left action of G , not a right action.) Some open problems are listed in Section 7.

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2. FINITE GROUP EXTENSIONS OF SFTS VIA MATRICES OVER $\mathbb{Z}G$

In this section we give basic definitions for finite group extensions; describe the presentation of group extensions of SFT by matrices over \mathbb{Z}_+G ; and describe algebraic invariants of defining matrices which correspond to invariants of the group extensions. Cocycles and the group extension construction are an important tool much more generally in dynamics (topological, measurable and smooth), but for simplicity, we restrict definitions to our special case. We recommend [19] for an introduction to cocycles in dynamics; [11] is a reference with proofs adapted to some of the items below, as indicated by references.

Standing assumption. Unless indicated otherwise, from here G denotes a finite group. All G actions are assumed to be continuous and free unless indicated.

Basic definitions [11]. Let a pair (X, S) represent a homeomorphism $S : X \rightarrow X$. We will be interested in only two cases: either (X, S) is a shift of finite type, or it is a countable union of finite orbits, with the discrete topology (i.e., we neglect topology). A *group extension* of (X, S) by G is a pair (Y, T) together with a continuous map $\pi : (Y, T) \rightarrow (X, S)$ such that $S\pi = \pi T$; two points have the same image under π if and only if they are in the same G -orbit; and π is a covering map (for each point x of X , there is a neighborhood V such that there are $|G|$ disjoint neighborhoods in Y such that the restriction of π to each is a homeomorphism onto V). If (X, S) is SFT, then a G extension of (X, S) is a free G -SFT, i.e. an SFT (X, S) together with a continuous free action of G which commutes with the shift. We will always take G acting *from the left*, for a correct matrix correspondence in the case G is nonabelian – see Appendix A for an explanation, which corrects the choice “from the right” in [11].

Two G extensions $(Y_1, T_1), (Y_2, T_2)$ are *conjugate*, or *isomorphic*, if there is a homeomorphism $\phi : Y_1 \rightarrow Y_2$ such that $\phi T_1 = T_2 \phi$ and $\phi(gy) = g\phi(y)$ for all $y \in Y_1$. Equivalently, they are isomorphic as G -SFTs. A G extension of (X, S) may be constructed from a continuous function $\tau : X \rightarrow G$ (a *skewing function*) as follows. Let $Y = X \times G$ and define $T : Y \rightarrow Y$ by the rule $(x, g) \mapsto (S(x), g\tau(x))$, with $\pi : X \times G \rightarrow X$ the obvious map $(x, g) \mapsto x$. Every G -extension of an SFT is isomorphic to one constructed in this way, and for brevity we may refer to such a group extension as (X, S, τ) .

We say G -extensions (X_1, S_1, τ_1) and (X_2, S_2, τ_2) are *eventually conjugate* if for all but finitely many $n > 0$ the G -extensions $(X_1, (S_1)^n, \tau_1)$ and $(X_2, (S_2)^n, \tau_2)$ are conjugate.

In a system (X, S) , continuous functions τ_1 and τ_2 from X to G are *cohomologous* if there is a continuous function $\gamma : X \rightarrow G$ such that for all x , $\tau_1(x) = (\gamma(x))^{-1}(\tau_2(x))\gamma(Sx)$ in the group G . For G -extensions (X_1, S_1, τ_1) and (X_2, S_2, τ_2) , the following are equivalent:

- (1) The two G -extensions are isomorphic.
- (2) There is a homeomorphism $\phi : X_1 \rightarrow X_2$ such that $\phi S_1 = S_2 \phi$ (i.e. ϕ is a topological conjugacy) and the functions $\tau_2 \circ \phi$ and τ_1 are cohomologous in (X_1, S_1) .

A *mixing* G -extension of (X, S) is a G -extension (Y, T) of (X, S) such that (Y, T) is topologically mixing. This is distinctly a stronger assumption than the assumption that (X, S) is mixing. The mixing G -extensions are the fundamental, central case. (The papers [2, 1] of Adler, Kitchens and Marcus describe invariants with which the classification of some G extensions of SFTs can be reduced to this central case.)

Presentation by matrices over \mathbb{Z}_+G [11]. Suppose A is a square matrix with entries in \mathbb{Z}_+G . Then A may be viewed as the adjacency matrix of a labeled directed graph, with adjacency matrix \overline{A} defining an edge SFT (X, S) , by setting

$$(2.1) \quad \tau(x) = \text{the label of the edge } x_0 .$$

Then (X, S, τ) is a group extension of the SFT (X, S) . Every group extension of an SFT is isomorphic to one of this type.

Mixing. For an element $x = \sum_g n_g g$ of $\mathbb{Z}G$, we write $x \gg 0$ if $n_g > 0$ for every g , and say x is G -positive. For a matrix A over $\mathbb{Z}G$, $A \gg 0$ means every entry is $\gg 0$. We define a G -primitive matrix to be a square matrix over \mathbb{Z}_+G such that $A^n \gg 0$ for some $n > 0$.

A nonzero square matrix A contains a maximum principal submatrix with no zero row and no zero column; this is the *nondegenerate core* of A . For a property P , a matrix A is *essentially P* if its nondegenerate core is P . A matrix A over \mathbb{Z}_+G defines a mixing G -extension if and only if it is essentially G -primitive (Proposition B.8).

NOTE: The \mathbb{Z}_+ matrix \overline{A} being primitive does not guarantee that A is primitive. (E.g., $A = (e + e)$ over $\mathbb{Z}G$ with $G = \mathbb{Z}/2\mathbb{Z}$.)

Conjugacy and eventual conjugacy. G -extensions of SFTs presented by matrices A, B over \mathbb{Z}_+G are conjugate if and only if the matrices A, B are strong shift equivalent (SSE) over \mathbb{Z}_+G . This theory, due to Parry and never published by him, is presented in [11]. By Proposition B.11, these G -extensions are eventually conjugate if and only if A, B are shift equivalent (SE) over \mathbb{Z}_+G . By Proposition B.12, two G -primitive matrices are SE over \mathbb{Z}_+G if and only if they are SE over $\mathbb{Z}G$.

Refinement of SE-ZG by SSE-ZG. For any ring \mathcal{R} , the refinement of SE- \mathcal{R} by SSE- \mathcal{R} is captured by the group $\mathrm{NK}_1(\mathcal{R})$ of algebraic K-theory, as follows.

Theorem 2.2. [10] *Suppose A is a square matrix over a ring \mathcal{R} .*

- (1) *If B is SE over \mathcal{R} to A , then there is a nilpotent matrix N over \mathcal{R} such that B is SSE over \mathcal{R} to the matrix $A \oplus N = \begin{pmatrix} A & 0 \\ 0 & N \end{pmatrix}$.*
- (2) *The map $[I - tN] \rightarrow [A \oplus N]_{\mathrm{SSE}}$ induces a bijection from $\mathrm{NK}_1(\mathcal{R})$ to the set of SSE classes of matrices over \mathcal{R} which are in the SE- \mathcal{R} class of A .*

For more on NK_1 , see Appendix C.

Periodic data and trace series. We consider G -extensions (X, S, τ) such that (X, S) has only finitely many orbits of size n , and formulate “periodic data” which give a complete invariant of isomorphism for the group extension obtained by restriction of S and τ to the periodic points of S , with the discrete topology. (Caveat: in the context of a Livšic type theorem, “periodic data” may refer to the cohomology class of the restriction of τ to the periodic points, with discrete topology [38]. Our series definition (2.4) is equivalent for the case we consider, being a complete invariant for that class.)

Definition 2.3. For $g \in G$, let $\kappa(g)$ denote the conjugacy class of g in G ($= \{g\}$ if G is abelian). Let $\mathbb{Z}\mathrm{Conj}G$ denote the free abelian group with generators the conjugacy classes of G . We also let κ denote the induced group homomorphism $\mathbb{Z}G \rightarrow \mathbb{Z}\mathrm{Conj}G$ given by $\sum n_g g \mapsto \sum n_g \kappa(g)$. We use κ similarly for other induced maps.

If (X_1, S_1, τ_1) is a G extension and $x \in \mathrm{Fix}(S^n)$, set $w(x) = \tau(x)\tau(Sx) \dots \tau(S^{n-1}x)$ and $\kappa_n(x) = \kappa(w(x))$. If a topological conjugacy $\phi : X_1 \rightarrow X_2$ sends τ_1 to a function cohomologous to τ_2 , and $x \in \mathrm{Fix}(S^n)$, then $\kappa_n(x) = \kappa_n(\phi(x))$. Given a G -extension of (X, S) defined by τ and a conjugacy class c from G , define the *periodic data* to be the formal power series with coefficients in $\mathbb{Z}\mathrm{Conj}G$,

$$(2.4) \quad P_\tau = \sum_{n=1}^{\infty} \left(\sum_{x \in \mathrm{Fix}(S^n)} \kappa(\tau(x)\tau(Sx) \dots \tau(S^{n-1}x)) \right) t^n .$$

Then for G extensions (X_1, S_1, τ_1) and (X_2, S_2, τ_2) , a necessary and sufficient condition for isomorphism of the G extensions obtained by restriction to their periodic points (neglecting topology) is that $P_{\tau_1} = P_{\tau_2}$.

Definition 2.5. Let A be a square matrix over a ring. The *trace series* of A is

$$(2.6) \quad \mathcal{T}_A = \sum_{n=1}^{\infty} \mathrm{tr}(A^n) t^n .$$

For A a matrix over $\mathbb{Z}G$, the *conjugacy class trace series* of A is

$$(2.7) \quad \kappa\mathcal{T}_A = \sum_{n=1}^{\infty} \kappa(\operatorname{tr}(A^n))t^n .$$

The trace series of A and B are *conjugate* if $\kappa\mathcal{T}_A = \kappa\mathcal{T}_B$.

We relate $\kappa\mathcal{T}_A$ to existing K -theory invariants [43] in Proposition 3.13. If the extension (X, S, τ) is defined by a matrix A over \mathbb{Z}_+G , then

$$(2.8) \quad P_\tau = \mathcal{T}_A .$$

Periodic data for G abelian. If G is abelian, we identify $\kappa(g)$ with $g \in \mathbb{Z}G$. Then the periodic data P_τ for the extension (X, S, τ) is encoded by the usual dynamical zeta function, taken with coefficients in $\mathbb{Z}G$,

$$\zeta_\tau(z) = \exp\left(\sum_{n=1}^{\infty} \sum_{x: S^n x=x} \tau(x)\tau(Sx)\cdots\tau(S^{n-1}x)\frac{z^n}{n}\right) .$$

When $\tau : X \rightarrow G$ is constructed from a matrix A over \mathbb{Z}_+G as above,

$$(2.9) \quad \zeta_\tau(t) = \exp \sum_{n=1}^{\infty} \frac{1}{n} \operatorname{tr}(A^n)t^n = (\det(I - tA))^{-1}$$

and $\det(I - tA)$ is a complete invariant for the periodic data. (Here, ζ_τ is an example of a dynamical zeta function. There is a huge literature using variants of such functions; one survey for nonexperts is [35].)

Periodic data for general G . Suppose A has entries in \mathbb{Z}_+G where G need not be abelian. The usual polynomial $\det(I - tA)$ need not be well defined. Nevertheless, by Proposition B.3, the finite sequence $(\kappa(\operatorname{tr}(A^k)))_{1 \leq k \leq mn}$ determines all of $\kappa\mathcal{T}_A$, and the sequence $(\kappa(\operatorname{tr}(A^k)))_{1 \leq k < \infty}$ satisfies a readily computed recursion relation with coefficients in \mathbb{Z} . A connection of $\kappa\mathcal{T}_A$ and K -theory is described in Proposition 3.13.

Periodic data, SE and SSE. If A, B are SSE over $\mathbb{Z}G$, then $\kappa\mathcal{T}_A = \kappa\mathcal{T}_B$ (Proposition B.3). If G is a finite abelian group, then $\det(I - tA)$ is an invariant of SE over $\mathbb{Z}G$, as follows. With B SE over $\mathbb{Z}G$ to A , by Theorem 2.2 there exists a nilpotent matrix N such that $A \oplus N$ is SSE over $\mathbb{Z}G$ to B , and then

$$\det(I - tB) = \det(I - tA) \det(I - tN) = \det(I - tA)$$

with the second equality holding by Proposition C.1.

For G not abelian, $\mathbb{Z}G$ might contain nonzero nilpotent elements (for example $\mathbb{Z}[D_4]$, where D_4 is the dihedral group of order 4, contains nilpotent elements), and in this case the periodic data will not be invariant under SE over $\mathbb{Z}G$. In any case, if A and B are

SE over $\mathbb{Z}G$ with lag ℓ , then $\kappa(\text{tr}(A^k)) = \kappa(\text{tr}(B^k))$ for all $k \geq \ell$, and then $\kappa\mathcal{T}_A = \kappa\mathcal{T}_B$ if and only if $\kappa(\text{tr}(A^k)) = \kappa(\text{tr}(B^k))$ for all $k < \ell$.

Flow equivalence. Complete invariants of G -equivariant flow equivalence for G -SFTs are known in terms of algebraic invariants associated to a presenting \mathbb{Z}_+G matrix A (see [11] for the case \bar{A} primitive and [4] for the general case).

3. FINITE GROUP EXTENSIONS OF SFTs VIA MATRICES OVER $\mathbb{Z}G[t]$

Invariants of group extensions of SFTs can be developed via matrices over \mathbb{Z}_+G with the SSE/SE approach, or via matrices with entries from the polynomial ring $\mathbb{Z}_+G[t]$ with the “positive K-theory” approach of [5, 6]). In this section we recall and develop what we need of the positive K-theory for constructions, and summarize algebraic invariants in this setting.

In this paper, we formulate positive equivalence in terms of finite matrices. The equivalent infinite matrix formulation of positive equivalence described later in this section is used in [5, 6]. Other formulations vary a bit among [5], [6] and the present paper, but they are equivalent where they overlap. The paper [5] is written for matrices over \mathbb{Z} and \mathbb{Z}_+ , outside of Section 7, which address matrices over integral group rings.

Positive equivalence.

Let R be a ring (always assumed to contain 1). A *basic elementary matrix* over R is a square matrix over R equal to the identity except perhaps in a single offdiagonal entry.

Below, 0_n is the $n \times n$ zero matrix, I_n is the $n \times n$ identity matrix, and $0, I$ represent zero, identity matrices of appropriate sizes.

Let \mathcal{M} be a set of square matrices $I - A$ over R such that

$$I - A \in \mathcal{M} \implies I - (A \oplus 0_n) \in \mathcal{M} \quad , \quad \text{for all } n > 0 \quad .$$

Let \mathcal{S} be a subset of R containing zero and one. A *basic elementary equivalence over \mathcal{S} in \mathcal{M}* is an equivalence of the form $I - A \mapsto U(I - A) = I - B$ or $I - A \mapsto (I - A)U = I - B$ such that U is a basic elementary matrix, and both $I - A$ and $I - B$ are in \mathcal{M} . An equivalence $I - A \mapsto U(I - A)V = I - B$ is an *elementary equivalence over \mathcal{S} in \mathcal{M}* if for some k , $(U \oplus I_k, V \oplus I_k) : I - (A \oplus I_k) \rightarrow I - (B \oplus I_k)$ is a composition of basic elementary equivalences over \mathcal{S} in \mathcal{M} . We say square matrices $I - A, I - B$ are *elementary equivalent over \mathcal{S} in \mathcal{M}* if there exist j, k such that there is an elementary equivalence over \mathcal{S} in \mathcal{M} from $I - (A \oplus I_j)$ to $I - (B \oplus I_k)$.

Definition 3.1. Suppose R is an ordered ring with R_+ containing 0 and 1. A square matrix A over $R_+[t]$ has the NZC property if for all $n \geq 0$, every diagonal entry of A^n has constant term zero. $\text{NZC}(R_+[t])$ is the set of square matrices A over $R_+[t]$ having the NZC property.

For example, the matrix $\begin{pmatrix} t & 3+t^3 \\ 2t^5 & t \end{pmatrix}$ is in $\text{NZC}(\mathbb{Z}_+[t])$; the matrix $\begin{pmatrix} t & 3+t^3 \\ 1+2t & t \end{pmatrix}$ is not. The square matrices over $tR_+[t]$ are contained in $\text{NZC}(R_+[t])$.

Definition 3.2. Suppose R is an ordered ring with R_+ containing 0 and 1. With respect to this ordered ring, two matrices are *positive equivalent* if they are elementary equivalent over R_+ in \mathcal{M} , where \mathcal{M} is the set of square matrices of the form $I - A$ with A in $\text{NZC}(R_+)$.

In this paper, positive equivalent without modifiers means positive equivalent with respect to $R = \mathbb{Z}G[t]$ and $R_+ = \mathbb{Z}_+G[t]$.

Positive equivalence and SSE. The next result is a trivial corollary of [5, Theorem 7.2], but it takes a little space to explain why this is so.

Theorem 3.3. [5, Theorem 7.2] *Let G be a group and $\mathbb{Z}G$ its integral group ring. Let A, B be matrices in $\text{NZC}(\mathbb{Z}_+G[t])$ and let A^\diamond, B^\diamond be square matrices over \mathbb{Z}_+G such that $I - A$ and $I - B$ are (respectively) positive equivalent to $I - tA^\diamond$ and $I - tB^\diamond$. Then the following are equivalent.*

- (1) A^\diamond and B^\diamond are SSE over \mathbb{Z}_+G .
- (2) $I - A$ and $I - B$ are positive equivalent.

Moreover, for every matrix A in $\text{NZC}(\mathbb{Z}_+G[t])$, there is a matrix A^\diamond over \mathbb{Z}_+G such that $I - A$ is positive equivalent to $I - tA^\diamond$.

Proof. The construction in [5, Sec. 7.2] produces from A in $\text{NZC}(\mathbb{Z}_+G[t])$ a matrix A^\sharp over \mathbb{Z}_+G such that there is a positive equivalence from $I - A$ to $I - tA^\sharp$. Then [5, Theorem 7.2] states (with different terminology) that $I - A$ and $I - B$ are positive equivalent if and only if A^\sharp and B^\sharp are SSE- \mathbb{Z}_+G . Now assume the Claim: for any square matrix M over \mathbb{Z}_+G , $(tM)^\sharp$ is SSE- \mathbb{Z}_+G to M . Then we have

$$\begin{aligned} & A^\diamond \text{ and } B^\diamond \text{ are SSE over } \mathbb{Z}_+G \\ \iff & (tA^\diamond)^\sharp \text{ and } (tB^\diamond)^\sharp \text{ are SSE over } \mathbb{Z}_+G \\ \iff & I - tA^\diamond \text{ and } I - tB^\diamond \text{ are positive equivalent} \\ \iff & I - A \text{ and } I - B \text{ are positive equivalent.} \end{aligned}$$

It suffices then to prove the Claim.

Suppose M is square over \mathbb{Z}_+G . Let \mathcal{G} be the G -labeled graph with adjacency matrix M . Let \mathcal{H} be the G -labeled graph with adjacency matrix C such that the vertices of \mathcal{H} are the edges of \mathcal{G} , and C is zero except that $C(a, b)$ is the label $g = g_a$ of edge a in \mathcal{G} if the terminal vertex of a equals the initial vertex of b . By definition in [5, Sec.7] (note the ‘‘Special Case’’ remark above [5, (2.6)]), $(tM)^\sharp$ will be the adjacency matrix C of \mathcal{H} . (The chosen ordering of indices to define an actual matrix won’t affect the SSE- \mathbb{Z}_+G class.) Explicitly, define matrices R, S , which are zero except for: $R(i, a) = 1$

if i is the initial vertex of a ; $S(a, j) = g_a$ if j is the terminal vertex of the edge a . Then $M = RS$ and $C = SR$. \square

Notational Convention 3.4. For a matrix A in $\text{NZC}(\mathbb{Z}_+G[t])$, we will use A^\diamond to denote a matrix over \mathbb{Z}_+G such that $I - tA^\diamond$ is positive equivalent to $I - A$.

The connection to shifts of finite type explained in [5] is less straightforward for $\text{NZC}(\mathbb{Z}_+G[t])$ than for matrices over $t\mathbb{Z}_+G[t]$. However, $\text{NZC}(R_+[t])$ is good for constructions (e.g., it is necessary for Proposition 3.9). Most importantly: if in the definition 3.2 of positive equivalence we replace $\text{NZC}(R_+)$ with the set of square matrices over $t\mathbb{Z}_+G[t]$, then the implication (1) \implies (2) of Theorem 3.3 would fail (see [5, Remark 6.4]).

The setting of positive equivalence has been useful for constructing conjugacies between SFTs and between G -SFTs [21, 22, 23, 27]. Positive equivalence constructions with matrices over \mathbb{Z}_+G (not over $\mathbb{Z}_+G[t]$) are fundamental for the classification of G -SFTs up to equivariant flow equivalence in [11, 4].

Recall that a matrix is *nondegenerate* if it has no zero row and no zero column. If row i or column i of a matrix is zero, then we say that the index i is removable. For a square matrix A , let $A = A_0$. Given A_k , define $A_{k+1} = (0)$ if every index of A_k is removable; otherwise, define A_{k+1} to be the principal submatrix of A_k on the nonremovable indices. For some k , $A_k = A_{k+1}$, and we call this matrix the *core* of A . A square matrix over \mathbb{Z}_+G is always SSE over \mathbb{Z}_+G to its core.

By Theorem 3.3, all matrices A^\diamond over \mathbb{Z}_+G with $I - tA^\diamond$ positive equivalent to a given $I - A$ lie in the same SSE- \mathbb{Z}_+G class. So, given A , whether the core of A^\diamond is G -primitive does not depend on the choice of A^\diamond . Similarly, given A , the following are equivalent: The choices are for A^\diamond , not for the core once A^\diamond has been chosen.

- (1) Some choice of A^\diamond has core zero.
- (2) Every choice of A^\diamond has core zero.
- (3) Every A^\diamond is SSE over \mathbb{Z}_+G to (0) .
- (4) $I - A$ is positive equivalent to I .

Some technical results. The main purpose of this subsection is to prove its Propositions, which we need later in proofs.

Suppose A is a square matrix over $t\mathbb{Z}_+G[t]$, say $A = \sum_{i=1}^k A_i t^i$, with the A_i matrices over \mathbb{Z}_+G . As in [9], define the matrix

$$(3.5) \quad A^\square = \begin{pmatrix} A_1 & A_2 & A_3 & \dots & A_{k-2} & A_{k-1} & A_k \\ I & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & I & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & I & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & I & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & I & 0 \end{pmatrix}.$$

Remark 3.6. If B is a matrix with all entries in $\mathbb{Z}_+G[t]$, \overline{B} is the matrix defined by applying the augmentation $\mathbb{Z}G \rightarrow \mathbb{Z}$ entrywise (Definition B.1). Then for A over $t\mathbb{Z}_+G[t]$, we have $\overline{(A^\square)} = (\overline{A})^\square$, and the notation $\overline{A^\square}$ is unambiguous.

Lemma 3.7. *Suppose A is a square matrix over $t\mathbb{Z}_+G[t]$. Then the matrices $I - A$ and $I - tA^\square$ are positive equivalent.*

Proof. The proof is clear from the case $k = 3$, as follows. The given multiplications by elementary matrices can be factored as a composition of basic positive equivalences.

$$\begin{aligned} \begin{pmatrix} I - tA_1 & -tA_2 & -tA_3 \\ -tI & I & 0 \\ 0 & -tI & I \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & tI & I \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ tI & I & 0 \\ 0 & 0 & I \end{pmatrix} &= \begin{pmatrix} I - A & -tA_2 - t^2A_3 & -tA_3 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \\ \begin{pmatrix} I & A_2 & 0 \\ 0 & I & A_3 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} I - A & -tA_2 & -tA_3 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} &= \begin{pmatrix} I - A & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \end{aligned}$$

□

The next proposition is used in the proof of Lemma 5.3.

Proposition 3.8. *Suppose A is an $n \times n$ matrix in $\text{NZC}(\mathbb{Z}_+G[t])$ and d is the maximum degree of an entry of A . Then there is a matrix A^\diamond over \mathbb{Z}_+G such that the following hold.*

- (1) $I - tA^\diamond$ is positive equivalent to $I - A$.
- (2) A^\diamond is $m \times m$ with $m \leq nd$.

If $I - A$ is not positive equivalent to I , then in addition A^\diamond can be chosen to be nondegenerate.

Proof. First suppose $A \in \text{NZC}(\mathbb{Z}_+G[t])$. We claim $I - A$ is positive equivalent to a matrix over $t\mathbb{Z}_+G[t]$. This is stated for $\mathbb{Z}G = \mathbb{Z}$ in [5, Prop. 4.3], but the argument is for our purposes quite indirect, so we will sketch a proof. Suppose for a row i , the

indices $j = j_1, \dots, j_t$ are those such that $A(i, j)$ has nonzero constant term, $c_{i,j} \neq 0$. For $1 \leq s \leq t$, let E_s be the $n \times n$ basic elementary matrix with $E(i, j_s) = c_{i,j_s}$. Then there is a positive equivalence from $I - A$ to $E_1 E_2 \cdots E_t (I - A) := I - B_1$. A and B_1 are equal outside row i . Now, if $M_i(A)$ denotes the maximum integer k such that an entry of row i of A^k has nonzero constant term, then $M_i(B_1) \leq M_i(A) - 1$. Thus by iterating this process, we can produce an $n \times n$ matrix B over $t\mathbb{Z}_+[t]$ such that $I - B$ is positive equivalent to $I - A$. Let d_B be the maximum degree of an entry of B ; then $d_B \leq d$.

Now by Lemma 3.7, the matrix $I - tB^\square$ is positive equivalent to $I - B$ and hence to $I - A$, with size $nd_B \leq nd$. Set $A^\diamond = B^\square$. For the nondegeneracy condition, let A^\diamond be the core of B^\square . \square

The next proposition is used in the proof of Theorem 5.6.

Proposition 3.9. *Suppose $I - A, I - B$ are matrices in $\text{NZC}(\mathbb{Z}_+G[t])$ such that A and B are SSE over $\mathbb{Z}_+G[t]$. Suppose A', B' are matrices over \mathbb{Z}_+G such that $I - tA'$ and $I - tB'$ are positive equivalent respectively to $I - A$ and $I - B$. Then A' and B' are SSE over \mathbb{Z}_+G .*

Proof. It suffices to prove the proposition in the case that there are matrices R, S over $\mathbb{Z}_+G[t]$ such that $A = RS$ and $B = SR$. By Theorem 3.3, it suffices to show that $I - A$ is positive equivalent to $I - B$. To see this, using the ‘‘polynomial strong shift equivalence equations’’ of [5, Sec.4], we multiply by matrices below in the order given by subscripts. Each multiplication gives a positive equivalence.

$$\begin{aligned} \begin{pmatrix} I & 0 \\ S & I \end{pmatrix}_4 \begin{pmatrix} I & -R \\ 0 & I \end{pmatrix}_2 \begin{pmatrix} I - RS & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -S & I \end{pmatrix}_1 \begin{pmatrix} I & R \\ 0 & I \end{pmatrix}_3 &= \begin{pmatrix} I & 0 \\ 0 & I - SR \end{pmatrix} \\ \begin{pmatrix} I & B \\ 0 & I \end{pmatrix}_4 \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix}_2 \begin{pmatrix} I & 0 \\ 0 & I - B \end{pmatrix} \begin{pmatrix} I & -B \\ 0 & I \end{pmatrix}_1 \begin{pmatrix} I & I \\ 0 & I \end{pmatrix}_3 &= \begin{pmatrix} I - B & 0 \\ 0 & I \end{pmatrix} \end{aligned}$$

\square

Infinite matrices.

Let R be a ring. $\text{El}_n(R)$ is the group of $n \times n$ matrices which are products of basic elementary matrices over R . $\text{GL}_n(R)$ is the group of $n \times n$ matrices invertible over R . For R commutative, $\text{SL}_n(R)$ is the subgroup of matrices in $\text{GL}_n(R)$ with determinant 1. The group $\text{GL}(R)$ is the direct limit group defined by the maps $\text{GL}_n(R) \rightarrow \text{GL}_{n+1}(R)$, $U \mapsto U \oplus 1$. $\text{El}(R)$ and (for R commutative) $\text{SL}(R)$ are the subgroups of $\text{GL}(R)$ defined as direct limits of the groups $\text{El}_n(R)$ and $\text{SL}_n(R)$. We define finite square matrices $I - A, I - B$ to be $\text{El}(R)$ equivalent if there exist j, k, n and matrices U, V in $\text{El}_n(R)$ such that $U(I - (A \oplus 0_j))V = I - (B \oplus 0_k)$. $\text{GL}(R)$ equivalence and $\text{SL}(R)$ equivalence are defined in the same way.

For a finite square matrix M , let M_∞ denote the infinite matrix which has upper left corner M and agrees with I in all other entries. The elements of $\mathrm{GL}(R)$ are naturally identified with the matrices U_∞ such that U is invertible. Similarly for $\mathrm{SL}(R)$ and $\mathrm{El}(R)$.

An equivalence $U(I-A)V$ with U and V in $\mathrm{GL}_n(R)$ produces an equivalence $U_\infty(I-A)_\infty V_\infty$ by matrices U, V in $\mathrm{GL}(R)$. Likewise for $\mathrm{El}(R)$ and $\mathrm{SL}(R)$. Basic elementary equivalence, ZNC and positive equivalence can be defined for these infinite matrices in the obvious way, such that finite square matrices $I-A$ and $I-B$ are positive equivalent if and only if $(I-A)_\infty$ and $(I-B)_\infty$ are positive equivalent.

Algebraic invariants via polynomial matrices.

In this subsection we look at the earlier algebraic invariants in terms of the polynomial matrix presentations.

Definition 3.10. For a ring R we say square matrices M, N are $\mathrm{El}(R)$ equivalent if there are positive integers j, k, n and matrices U, V in $\mathrm{El}_n(R)$ such that $U(M \oplus I_j)V = N \oplus I_k$. $\mathrm{GL}(R)$ equivalence and $\mathrm{SL}(R)$ equivalence are defined in the same way.

Theorem 3.11 is an easy corollary of the main result of [10].

Theorem 3.11. [10, Corollary 6.6] *Suppose R is a ring. Suppose $I-A$ and $I-B$ are matrices over $R[t]$; A', B' are square matrices over R ; and $I-A$ and $I-B$ are respectively $\mathrm{El}(R[t])$ equivalent to $I-tA'$ and $I-tB'$. Then the following are equivalent.*

- (1) A' and B' are SSE over R .
- (2) $I-tA$ and $I-tB$ are $\mathrm{El}(R[t])$ equivalent.

If A is $n \times n$ over the group ring $\mathbb{Z}G[t]$, then matrix multiplication defines $(I-A) : (\mathbb{Z}G[t])^n \rightarrow (\mathbb{Z}G[t])^n$ and thereby the $\mathbb{Z}G[t]$ module $\mathrm{cok}(I-A)$. (The isomorphism class of the module depends in general on whether one chooses multiplication of row vectors or column vectors.)

Proposition 3.12. [10, Theorem 5.1]² *Suppose A and B are square matrices over a ring R . Then the following are equivalent.*

- (1) A and B are SE over R .
- (2) The $R[t]$ modules $\mathrm{cokernel} \mathrm{cok}(I-tA)$ and $\mathrm{cok}(I-tB)$ are isomorphic.
- (3) $I-tA$ and $I-tB$ are $\mathrm{GL}(R[t])$ equivalent.

Lastly, we consider the algebraic invariants for the periodic data. Proposition 3.12 (via condition (3)) shows that $\det(I-tA)$ is invariant under SE- R for any commutative ring R (e.g. $\mathbb{Z}G$ for G abelian). For any ring R , $R[[t]]$ denotes the ring of formal power series with coefficients in R , and the *generalized characteristic polynomial* $\mathrm{ch}(A)$

²See [10] for attributions; especially, (2) \iff (3) is due to Fitting [14].

[30, 31, 43, 44]) of a square matrix A over R is the element of $K_1(R[[t]])$ containing $I - tA$. Motivation for and a characterization of $\text{ch}(A)$ are in [43, 44]. If R is commutative, then $\det(I - tA)$ is a complete invariant for $\text{ch}(A)$.

Recall the definitions (2.6) and (2.7) for \mathcal{T}_A and $\kappa\mathcal{T}_A$. Given a ring R , let C denote the additive subgroup (not the ideal) of R generated by the set $\{ab - ba : a \in R, b \in R\}$. Let $\gamma : R \rightarrow R/C$ denote the corresponding epimorphism of additive groups. Let \mathcal{T}_A/C denote $\sum_{n=1}^{\infty} \gamma(\text{tr}A^n)t^n$. Following Sheiham [43, p.19], for a square matrix A over R define $\chi : A \mapsto \mathcal{T}_A/C$.

Proposition 3.13. *Suppose G is a group and A, B are square matrices over $\mathbb{Z}G$. Then*

$$(3.14) \quad \mathcal{T}_A/C = \mathcal{T}_B/C \iff \kappa\mathcal{T}_A = \kappa\mathcal{T}_B .$$

If $I - tA$ and $I - tB$ are $\text{El}(\mathbb{Z}G[t])$ equivalent, or even just $\text{El}(\mathbb{Z}G[[t]])$ equivalent, then $\kappa\mathcal{T}_A = \kappa\mathcal{T}_B$.

Proof. The proof of the first claim is straightforward. For the second claim, note that for any ring R , χ factors through $\text{ch}(A)$, as pointed out by Sheiham [43, Remark 2.9]. If $I - tA$ and $I - tB$ are $\text{El}(R[t])$ equivalent, then they are $\text{El}(R[[t]])$ equivalent, so $\text{ch}(A) = \text{ch}(B)$. In the case $R = \mathbb{Z}G$, this means $\mathcal{T}_A/C = \mathcal{T}_B/C$. \square

With Theorem 3.11, Proposition 3.13 gives an alternate proof that $\mathcal{T}_A = \mathcal{T}_B$ when A and B are SSE over R . For G a nonabelian group, we do not know if $\kappa\mathcal{T}_A$ determines $\text{ch}(A)$.

4. PARRY'S QUESTION AND SE- $\mathbb{Z}G$

Parry's Question 4.1. Suppose G is a finite abelian group, (X, S) is a mixing SFT and ζ is a fixed dynamical zeta function. Must there be only finitely many topological conjugacy classes of G extensions of (X, S) , with ζ_τ constructed from a skewing function τ as in (2.1), such that $\zeta_\tau = \zeta$?

Slightly different versions of Parry's question were recorded in [6, Sec. 5.3], [7, Question 31.1] and [34, Sec. 4.4, p.331]. The version above is matched to our notation. The other versions are equivalent, except that the SFT (X, S) might be assumed mixing or only irreducible. Because (X, S) is fixed, a map $X \rightarrow X$ implementing an isomorphism of (X, S, τ_1) and (X, S, τ_2) would have to be an automorphism of (X, S) , as in the language of [34, Sec. 4.4]. (For work on a related problem, in which the skewing function f is Hölder into the real numbers, see [36].)

We will address the following version of Question 4.1.

Question 4.2. Suppose G is a nontrivial finite group and A is a G -primitive matrix over \mathbb{Z}_+G . Let $\mathfrak{M}(A)$ be the collection of G -primitive matrices B over \mathbb{Z}_+G such that

- (1) the matrices \overline{A} and \overline{B} are SSE over \mathbb{Z}_+ , and
- (2) the matrices B and A have the same periodic data, $P_B = P_A$ as in (2.8)
(if G is abelian, this means $\det(I - tB) = \det(I - tA)$).

Must $\mathfrak{M}(A)$ contain only finitely many SSE- \mathbb{Z}_+G classes?

In Question 4.2, the condition that A be G -primitive adds the requirement that the extension be a mixing extension – the central case. A negative answer to (4.2) gives a negative answer to (4.1). The condition that \overline{A} and \overline{B} are SSE over \mathbb{Z}_+ captures up to isomorphism the extensions of Question 4.1 (we can recode them to this form) and also includes every (X', S', τ') such that (X', S') is topologically conjugate to (X, S) and τ' gives the correct periodic data. This does not change the set of isomorphism classes of extensions, because isomorphism classes of G -extensions of (X', S') pull back bijectively under topological conjugacy to isomorphism classes of G -extensions of (X, S) . Also, we have broadened Parry's question to include nonabelian groups. We add the condition that G be nontrivial for linguistic simplicity. If G is trivial, then the answer to (4.1) is trivially 'yes', so we no longer need to exclude this case when giving a negative answer. If G is nontrivial and A is G -primitive, then the extension must have positive entropy, and there is nothing more to say about excluding a case of finitely many orbits.

Parry³ with an unpublished example showed that nonisomorphic skew products over a mixing SFT could share the same zeta function ζ_τ . His question followed the study of dozens of examples, and grew out a study of cocycles describing how Markov measures change under a flow equivalence of SFTs as in [3].

A natural way to attack Question 4.2 is to consider how the algebraic relations SE- $\mathbb{Z}G$ and SSE- $\mathbb{Z}G$ can refine a prescribed $\det(I - tA)$. If the refinement is infinite, then there is an issue of constructing G -primitive matrices realizing an infinite class on which the algebraic invariants differ. In Section 5, we'll carry out this program at the level of SSE- $\mathbb{Z}G$, when $NK_1(\mathbb{Z}G)$ is not trivial. In this case, for *every* A the answer is negative.

If $NK_1(\mathbb{Z}G)$ is trivial, then SE- $\mathbb{Z}G$ and SSE- $\mathbb{Z}G$ are equivalent, by Theorem 2.2. By appeal to SE- $\mathbb{Z}G$ invariants, Theorem 4.3 below gives a negative answer to Parry's question for every G , regardless of whether $NK_1(\mathbb{Z}G)$ is trivial. However, in contrast to the SSE- $\mathbb{Z}G$ invariants, the SE- $\mathbb{Z}G$ invariants do not provide an infinite refinement of the periodic data of A for *every* A . We will give examples for which the data $\det(I - tA)$ determines the SE- $\mathbb{Z}G$ class of A .

$\mathfrak{M}(A)$ in the statement of Theorem 4.3 was defined in Question 4.2.

³Descriptions of Parry's work and motivation are based on a review of email correspondence 2002-2006 between Boyle and Parry.

Theorem 4.3. *Suppose G is a nontrivial finite group. There is a G -primitive matrix A over $\mathbb{Z}G$ such that $\mathfrak{M}(A)$ contains infinitely many SE- $\mathbb{Z}G$ equivalence classes.*

Proof. We will define some matrices over $\mathbb{Z}G[t]$. Let $u = \sum_{g \in G} g \in \mathbb{Z}G$. Fix g an element of G distinct from the identity e . Set $s = ut \in \mathbb{Z}G[t]$ and $w = et$. Below, p_k in $\mathbb{Z}G[t]$ will depend on $k \in \mathbb{Z}_+$, with $p_0 = 0$. Given r , $E_{ij}(r)$ denotes the basic elementary matrix of appropriate size which equals r in the i, j entry and otherwise equals I . Define 5×5 matrices equal to I except that $U(3, 4) = U(3, 5) = 1 = V(5, 1) = V(4, 1)$. U will act by adding column 3 to columns 4 and 5. V will act by adding row 1 to rows 4 and 5. Define

$$C_k = \begin{pmatrix} 4s & s & s & 0 & 0 \\ 4s & s & s & 0 & 0 \\ 4s & 2s & 2s & 0 & 0 \\ 0 & 0 & 0 & w & p_k \\ 0 & 0 & 0 & 0 & w \end{pmatrix}$$

$$D_k = U^{-1}C_kU = \begin{pmatrix} 4s & s & s & s & s \\ 4s & s & s & s & s \\ 4s & 2s & 2s & 2s - w & 2s - w - p_k \\ 0 & 0 & 0 & w & p_k \\ 0 & 0 & 0 & 0 & w \end{pmatrix}$$

$$F_k = VD_kV^{-1} = \begin{pmatrix} 2s & s & s & s & s \\ 2s & s & s & s & s \\ 2w + p_k & 2s & 2s & 2s - w & 2s - w - p_k \\ 2s - w - p_k & s & s & s + w & s + p_k \\ 2s - w & s & s & s & s + w \end{pmatrix}.$$

We will choose p_k to be a sum of k monomials, $p_k = (e - g)(t^{n_1} + \cdots + t^{n_k})$. Define $A = F_0$. Then $A = tA^\square$ and A^\square is G -primitive. For each k , we have $\overline{p_k} = 0$, and therefore $\overline{F_k} = A$. We will arrange the following.

- (1) The $\mathbb{Z}G[t]$ modules $\text{cok}(I - C_k)$ are pairwise not isomorphic.
- (2) For each k , there is a matrix B_k over $t\mathbb{Z}_+G[t]$ and a finite string of matrices $F_k = B_{(0)}, B_{(1)}, \dots, B_{(m)} = B_k$ such that the following hold.
 - (a) B_k^\square is G -primitive.
 - (b) For $1 \leq i \leq m$, $I - B_{(i)}$ equals $E_i(I - B_{(i-1)})$ or $(I - B_{(i-1)})E_i$, for some basic elementary matrix E_i with offdiagonal entry in $t\mathbb{Z}G[t]$.
 - (c) For $0 \leq i \leq m$, $\overline{B_{(i)}}$ has all entries in $t\mathbb{Z}_+[t]$.

Suppose we have these conditions. For each k , the $\mathbb{Z}G[t]$ modules $\text{cok}(I - C_k)$, $\text{cok}(I - F_k)$ and, by 2(b), $\text{cok}(I - B_k)$ are isomorphic. Therefore the $\mathbb{Z}G[t]$ modules $\text{cok}(I -$

B_k), are, by (1), pairwise not isomorphic. Therefore the G -primitive matrices B_k^\square are pairwise not shift equivalent over $\mathbb{Z}G$. However, the elementary equivalences of 2(b) over $\mathbb{Z}G[t]$ push down to elementary equivalences over $\mathbb{Z}[t]$, and by 2(c) these are positive equivalences over $\mathbb{Z}[t]$. Therefore each $\overline{B_k}$ is SSE over $\mathbb{Z}_+[t]$ to A , and the first condition in the Question 4.2 definition of $\mathfrak{M}(A)$ is satisfied. For the second condition, note by 2(b) and Proposition 3.13 that for each k the matrices B_k^\square and F_k^\square have the same periodic data. F_k^\square and C_k^\square also have the same periodic data. By the block structure of C_k , the entry p_k has no effect on the traces of powers of C_k . Thus every C_k has the periodic data of C_0 , which is that of A . This shows the second condition in the Question 4.2 definition of $\mathfrak{M}(A)$ is satisfied. So, it remains to arrange the conditions (1) and (2) above.

For condition (2), consider the multiplication of $I - F_k$ from the right by matrices $E_{25}(s), E_{25}(s^2), \dots, E_{25}(s^k)$, producing say a matrix $I - G_k$. These push down to a positive equivalence from $I - \overline{F_k}$ to $I - \overline{G_k}$. We have

$$\begin{aligned} G_k(3, 5) &= (2s - w - p_k) + 2s^2 + 2s^3 + \dots + 2s^{k+1} \\ G_k(4, 5) &= (s + p_k) + s^2 + s^3 + \dots + s^{k+1} . \end{aligned}$$

Thus for suitable p_k of the specified form, these two entries of G_k will lie in $\mathbb{Z}_+G[t]$. Apply the same procedure with E_{21} in place of E_{25} to likewise address the sign issue for the 1,3 and 1,4 entries. The resulting matrix is our B_k .

Finally, we address condition (1). For h in G , let \tilde{h} be the $|G| \times |G|$ permutation matrix which is the image of G under the left regular representation. This induces a map $M \mapsto \widetilde{M}$ sending 5×5 matrices over $\mathbb{Z}G[t]$ to $5|G| \times 5|G|$ matrices over $\mathbb{Z}[t]$. Suppose there is an isomorphism of $\mathbb{Z}G[t]$ modules $\text{cok}(I - C_k) \xrightarrow{\sim} \text{cok}(I - C_j)$. Let the homomorphism $\mathbb{Z}[t] \rightarrow \mathbb{Z}$ induced by $t \mapsto 1$ send a matrix $I - \widetilde{C}$ to $I - C'$. Then there is an induced isomorphism of \mathbb{Z} modules (abelian groups), $\text{cok}(I - C'_j) \rightarrow \text{cok}(I - C'_k)$. The lower right $2|G| \times 2|G|$ block of $I - C'_k$ has the block form $\begin{pmatrix} 0 & k(I-P) \\ 0 & 0 \end{pmatrix}$, where P is \tilde{g} . From the block diagonal form of C_j and C_k we conclude that $\text{cok}(k(I - P))$ and $\text{cok}(j(I - P))$ are isomorphic groups.

But, let m be the order of g in G and let $c = |G|/m$. P is conjugate by a permutation matrix to the direct sum of c copies of a matrix C , where C is an $m \times m$ cyclic permutation matrix. $I_m - C$ is $\text{SL}_m\mathbb{Z}$ -equivalent to $I_{m-1} \oplus 0_1$. Therefore $\text{cok}(k(I - P))$ is isomorphic to $(\mathbb{Z}/k\mathbb{Z})^{(m-1)c} \oplus \mathbb{Z}^c$, and for positive integers $j \neq k$, $\text{cok}(j(I - P))$ and $\text{cok}(k(I - P))$ cannot be isomorphic. This contradiction finishes the proof. \square

Lemma 4.4. *Suppose G is a finite group, and let $u = \sum_g g$. Suppose A and B are matrices over $\mathbb{Z}G$ with some powers A^p, B^q all of whose entries lie in $u\mathbb{Z}$. Suppose that \overline{A} and \overline{B} are SE over \mathbb{Z} . Then A and B are SE over $\mathbb{Z}G$.*

Proof. For any matrix M over $\mathbb{Z}G$, we have $uM = u\overline{M}$. So, $A^p = u(1/|G|)\overline{A^p} = u(1/|G|)\overline{A}^p$, with $(1/|G|)\overline{A}^p$ having integer entries. For $k > 0$, $A^{p+k} = u(1/|G|)\overline{A}^{p+k}$. Without loss of generality, we suppose $p = q$. Suppose R, S gives an SE over \mathbb{Z} of \overline{A}^ℓ and \overline{B}^ℓ :

$$\overline{A}^\ell = RS, \quad \overline{B}^\ell = SR, \quad \overline{A}R = R\overline{B}, \quad S\overline{A} = \overline{B}S.$$

Define $\tilde{R} = A^p R = u(1/|G|)\overline{A}^p R$ and $\tilde{S} = B^p S = u(1/|G|)\overline{B}^p S$. Then

$$\begin{aligned} \tilde{R}\tilde{S} &= \left(u\left(\frac{1}{|G|}\overline{A}^p R\right)\right)\left(u\left(\frac{1}{|G|}\overline{B}^p S\right)\right) = u\left(\frac{1}{|G|}\overline{A}^p R\overline{B}^p S\right) \\ &= u\left(\frac{1}{|G|}\overline{A}^p R S \overline{A}^p\right) = u\left(\frac{1}{|G|}\overline{A}^{2p+\ell}\right) = A^{2p+\ell}, \quad \text{and} \\ A\tilde{R} &= A\left(u\left(\frac{1}{|G|}\overline{A}^p R\right)\right) = u\frac{1}{|G|}\overline{A}^{p+1} R \\ &= u\frac{1}{|G|}\overline{A}^p R\overline{B} = u\frac{1}{|G|}\overline{A}^p R B = \tilde{R}B \end{aligned}$$

(for the last line, note that u lies in the center of $\mathbb{Z}G$). Likewise, $\tilde{S}\tilde{R} = B^{2p+\ell}$ and $B\tilde{S} = \tilde{S}A$. \square

It is easy to construct matrices A over \mathbb{Z}_+G such that some power A^p has all entries in $u\mathbb{Z}G$. For example, take A over $u\mathbb{Z}G$; or let $A = B + N$ where B is over $u\mathbb{Z}_+G$ and N over $\mathbb{Z}G$ is nilpotent with $uN = 0$. If B here is also G -primitive and $B - N$ has all entries over \mathbb{Z}_+G , then A will be G -primitive.

Lemma 4.5. *Suppose A is $n \times n$ over $\mathbb{Z}G$, with $m = |G|$. Let τ_k denote $\text{tr}(A^k)$, with $\tau_{k,g}$ the integers such that $\tau_k = \sum_{g \in G} \tau_{k,g}g$. Then the following are equivalent.*

- (1) *There is p in \mathbb{N} such that A^p has all entries in $u\mathbb{Z}G$.*
- (2) *$m\tau_{k,e} = \overline{\tau_k}$, for $1 \leq k \leq mn$,*

Now suppose a positive power of A has all entries in $u\mathbb{Z}G$ and B is a matrix over $\mathbb{Z}G$ such that (i) B and A have the same periodic data or (ii) B is SE over $\mathbb{Z}G$ to A . Then some positive power of B has all entries in $u\mathbb{Z}G$. Consequently, for $\mathcal{R} = \mathbb{Z}$ or $\mathcal{R} = \mathbb{Z}_+$: if \overline{A} and \overline{B} are SE- \mathcal{R} , then A and B are SE- $\mathcal{R}G$.

Proof. We use $\tilde{A}: \mathbb{Z}^{mn} \rightarrow \mathbb{Z}^{mn}$ constructed as in Appendix B. Let W be the subspace of \mathbb{Z}^{mn} corresponding to $(u\mathbb{Z}G)^n$. A has a positive power with all entries in $u\mathbb{Z}G$ if and only if \tilde{A} has a power which maps \mathbb{Z}^{mn} into W if and only if \tilde{A} restricted to the complementary invariant subspace is nilpotent. This holds if and only if the sequences $(\text{tr}(\tilde{A}^k))_{1 \leq k \leq mn}$ and $(\text{tr}((\tilde{A}|_W)^k))_{1 \leq k \leq mn}$ are equal. We have $\text{tr}(\tilde{A}^k) = m\tau_{k,e}$

and (because A acts on $(u\mathbb{Z}G^n)$ exactly as \bar{A} acts on \mathbb{Z}^n) $\text{tr}((\tilde{A}|_W)^k) = \bar{\tau}_k$. This proves the equivalence of (1) and (2).

Then (i) holds because (1) \iff (2) shows (1) depends only on the periodic data. Although the periodic data need not be an invariant of SE- $\mathbb{Z}G$ when G is nonabelian, if matrices A, B are SE- $\mathbb{Z}G$ then for every large enough $\ell \in \mathbb{N}$ there are R, S over $\mathbb{Z}G$ such that $A^\ell = RS$ and $B^\ell = SR$, and then $A^{2\ell} = (A^\ell R)S$ and $B^{2\ell} = S(A^\ell R)$. Clearly if A^ℓ is over $u\mathbb{Z}G$, then so is $B^{2\ell}$. The final claim follows now from Lemma 4.4. \square

Proposition 4.6. *Suppose G is a finite abelian group. Set $u = \sum_g g$. Let \mathcal{Z}_u denote the set of polynomials of the form $1 + \sum_{i=1}^k c_i u^i$, with each c_i in \mathbb{Z} . Suppose A and B are square matrices over $\mathbb{Z}G$ such that $\det(I - tA)$ and $\det(I - tB)$ lie in \mathcal{Z}_u , and \bar{A} and \bar{B} are SE over \mathbb{Z} . Then A and B are SE over $\mathbb{Z}G$.*

Proof. By the Cayley-Hamilton Theorem, for all large n the matrices A^n and B^n have entries in $u\mathbb{Z}$. The theorem then follows from Lemma 4.4. \square

Proposition 4.6 applies to any A all of whose entries are integer multiples of u ; for example, $A = (e + g)$ with $G = \{e, g\} = \mathbb{Z}/2\mathbb{Z}$.

In the case of A satisfying the assumptions of Proposition 4.6, with $\text{NK}_1(\mathbb{Z}G)$ trivial, to answer Parry's question we are left with the open problem: for A G -primitive, can the refinement the SSE- $\mathbb{Z}G$ class of A by SSE- $\mathbb{Z}G_+$ be infinite? In the case $G = \{e\}$ ($\mathbb{Z}G = \mathbb{Z}$), that question remains open more than 40 years after Williams' original paper [48].

Remark 4.7. In [32, Theorem 7.1], Parry proved that for G compact and X an irreducible SFT if $f, g : X \rightarrow G$ are Hölder with equal weights on all periodic points, then f and g are Hölder cohomologous. If one assumes only that the f and g weights are conjugate, Parry shows then the existence of an isometric automorphism ϕ of G such that ϕf and g are cohomologous [32, Theorem 6.5]. Parry also gives an example with G finite of two cocycles having conjugate weights for which the isomorphism is necessary, although the example is not mixing [32, Section 10].

We note now that in general ϕ cannot be chosen to be the identity even if the extension is mixing (i.e., presented by a G -primitive matrix A over \mathbb{Z}_+G). For example, let G be a finite group having an outer automorphism φ for which φ preserves all conjugacy classes of G . Such groups exist (see [12]); for example, the group $LP(1, \mathbb{Z}/8)$ consisting of all linear permutations $x \mapsto \sigma x + \tau$ on $\mathbb{Z}/8$, with σ, τ in $\mathbb{Z}/8$, is such a group. Let A be primitive over \mathbb{Z}_+G , and τ denote the corresponding edge labeling on the graph of \bar{A} coming from A . Then $\phi\tau$ is another edge labeling, and $\phi\tau$ and τ have conjugate weights on all periodic points. However, $\phi\tau$ and τ are not cohomologous. If they were, then because they are defined by edge labelings of an irreducible graph, by

[32, Lemma 9.1] there would be a function $\gamma : X_A \rightarrow G$ such that $\gamma(x)$ depends only on the initial vertex of x and $\phi\tau = \gamma^{-1}\tau\gamma$. Let ν be a vertex and let g in G be such that $g = \gamma(x)$ when x_0 has initial vertex ν . Now for every word $x_0 \dots x_k$ beginning and ending at ν : if $h = \tau(x_0) \cdots \tau(x_k)$, then $\phi(h) = g^{-1}hg$. Because A is G -primitive, every element of G occurs as such an h , and therefore ϕ is an inner automorphism. This contradiction shows $\phi\tau$ and τ are not cohomologous.

5. PARRY'S QUESTION AND SSE- $\mathbb{Z}G$

In this section, we prove the following result, which gives a strong negative answer to Parry's question (4.1) whenever $\text{NK}_1(\mathbb{Z}G) \neq 0$. (See Appendix C for a description of the finite G with nontrivial $\text{NK}_1(\mathbb{Z}G)$.)

Theorem 5.1. *Let G be a finite group such that $\text{NK}_1(\mathbb{Z}G) \neq 0$. Let (X, σ) be a mixing shift of finite type and let $\tau : X \rightarrow G$ be a continuous function defining a mixing G -extension (X_τ, σ_τ) of (X, T) .*

Then there is an infinite family of G -extensions of (X, T) which are eventually conjugate as G -extensions to (X_τ, σ_τ) and which are pairwise not isomorphic G -extensions. If G is abelian, then they all have the same dynamical zeta function.

Theorem 5.1 will be proved as a corollary to the following result.

Theorem 5.2. *Suppose G is a finite group and A is a G -primitive matrix with spectral radius $\lambda > 1$ and $\text{NK}_1(\mathbb{Z}G) \neq 0$. Let A be a G -primitive matrix.*

Then there is an infinite family $\{A_i : i \in \mathbb{N}\}$ of G -primitive matrices which are pairwise not SSE over $\mathbb{Z}G$ but such that for all i the following hold:

- (1) A_i is SE over \mathbb{Z}_+G to A .
- (2) \overline{A}_i is SSE over \mathbb{Z}_+ to \overline{A} .
- (3) If G is abelian, then $\det(I - tA_i) = \det(I - tA)$.

To prove Theorem 5.2, we first will work to establish a rather technical result, Proposition 5.6. Below, we will use the notations of (B.2) and the definitions (3.5), (B.4) and (B.5) of a matrix A^\square , a G -primitive matrix and the spectral radius λ_A of a square matrix over $\mathbb{Z}G$ or $\mathbb{Z}G[t]$. For a polynomial p over $\mathbb{Z}G$, λ_p is the spectral radius of the 1×1 matrix (p) . For a polynomial matrix $M = M(t)$, we let $M(1)$ denote its evaluation at $t = 1$.

Lemma 5.3. *Suppose $n > 1$ and A is an $n \times n$ matrix over $t\mathbb{Z}_+G[t]$ with spectral radius $\lambda > 1$ and with A^\square G -primitive. Given $\epsilon > 0$, there exists a positive integer m_0 such that for any $d \geq m_0$ there is an $n \times n$ matrix C over $t\mathbb{Z}_+G[t]$ such that $I - C$ is positive equivalent to $I - tA$ and*

$$c_{11kg} > (\lambda - \epsilon)^k, \quad \text{for } m_0 \leq k \leq d, \quad \text{for all } g \in G.$$

Proof. We will produce C in three stages.

STAGE 1. Because $A(1)$ is G -primitive, by [11, Lemma 6.6] there is a positive equivalence with respect to the ordered ring $(\mathbb{Z}G, \mathbb{Z}_+G)$ from $I - A(1)$ to a matrix $I - H$ such that H is a matrix over \mathbb{Z}_+G with no zero entry. Lift this positive equivalence with respect to $(\mathbb{Z}G, \mathbb{Z}_+G)$ to a positive equivalence with respect to $(\mathbb{Z}G[t], \mathbb{Z}_+G[t])$ from $I - tA$ to a matrix $I - L$, with L a matrix over $t\mathbb{Z}_+G[t]$ with every entry nonzero.

STAGE 2. In this stage, given $\epsilon > 0$ we produce an $n \times n$ matrix B over $t\mathbb{Z}_+G[t]$ with no zero entry such that $I - tA$ is $\mathbb{Z}G[t]$ positive equivalent to $I - B$ and the $B(n, n)$ entry has spectral radius greater than $\lambda - \epsilon/2$.

For this, we define $n \times n$ matrices B_1, B_2, \dots recursively. We set B_1 to be the matrix L produced in Stage 1. In block form, let $B_1 = \begin{pmatrix} M & u \\ v & f \end{pmatrix}$, in which f is 1×1 . A matrix B_k will have a block form

$$(5.4) \quad B_k = \begin{pmatrix} M & u \\ v^{(k)} & f^{(k)} \end{pmatrix} .$$

Given B_k , define B_{k+1} by the equivalence

$$I - B_{k+1} = \begin{pmatrix} I & 0 \\ v^{(k)} & I \end{pmatrix} \begin{pmatrix} I - M & -u \\ -v^{(k)} & 1 - f^{(k)} \end{pmatrix} = \begin{pmatrix} I - M & -u \\ -v^{(k)}M & 1 - f^{(k)} - v^{(k)}u \end{pmatrix} .$$

This defines a positive equivalence from $I - B_k$ to $I - B_{k+1}$. By induction, for all k , B_k is in \mathcal{M} and has no zero entry; B_k^\square is G -primitive; and B_{k+1} is a matrix over $t\mathbb{Z}_+G[t]$ with block form

$$B_{k+1} = \begin{pmatrix} M & u \\ vM^k & f + v(I + M + \dots + M^{k-1})u \end{pmatrix} .$$

Because A is G -primitive, by the condition (3) in Theorem B.6 we have a positive real number c such that $\text{tr}(A^j) > c\lambda^j(g_1 + \dots + g_m)$ for all large j . Because M^\square is a proper principal submatrix of the G -primitive matrix $(B_1)^\square$, which has spectral radius λ , we have $\lambda_M < \lambda$. Choose $\delta > 0$ such that $\delta < \epsilon/2$ and $\lambda_M < \lambda - \delta$. For all large j ,

$$\text{tr}(M^j) < (\lambda - \delta)^j(g_1 + \dots + g_m) .$$

Because M^k has entries in $t^k\mathbb{Z}G[t]$ and u has entries in $t\mathbb{Z}G[t]$, if $j \leq k$ then

$$\text{tr}((A^\square)^j) = \text{tr}((M^\square)^j) + \text{tr}((f^{(k)})^\square)^j) .$$

It follows that for all large k , for $j \in \{k-1, k\}$,

$$(5.5) \quad \text{tr}(((f^{(k)})^\square)^j) \geq (\lambda - \delta)^j(g_1 + \dots + g_m) .$$

Consequently, $f^{(k)}$ is G -primitive for all large k .

Let $\lambda^{(k)}$ be the spectral radius of $(f^{(k)})$. Let d be the maximum degree of an entry of B_1 . From the block form (5.4) we see that $f^{(k)}$ has degree at most dk . Then by

Proposition 3.8, we can use a version of p^\square which is a $dk \times dk$ matrix Q over \mathbb{Z}_+G . Then for $q = dk/m$, the matrix \overline{Q} is $q \times q$ over \mathbb{Z}_+ with spectral radius $\lambda_Q = \lambda^{(k)}$. Using (5.5), we have

$$\lambda^{(k)} = \lambda_Q \geq \left(\frac{1}{q} \text{tr}(Q^k) \right)^{1/k} \geq \left(\frac{m}{dk} (\lambda - \delta)^k m \right)^{1/k}.$$

Because $0 < \delta < \epsilon/2$, It follows that $\lambda^{(k)} > \lambda - \epsilon/2$ for all large k .

STAGE 3. We define $n \times n$ matrices P_1, P_2, \dots over $t\mathbb{Z}_+[t]$ recursively. The recursive step is the same as in Stage 2, but with row 1 in Stage 3 playing the role of row n in Stage 2. In block form, we write $P_1 = \begin{pmatrix} s & w \\ x & Q \end{pmatrix}$, with s being 1×1 . We take $P_1 = B$ from Stage 2 and set $q = P_1(n, n)$. The 1×1 matrix (q) has q^\square G -primitive with spectral radius λ_q such that $0 < \lambda - \lambda_q < \epsilon/2$.

A matrix P_k will have a block form

$$P_k = \begin{pmatrix} s^{(k)} & w^{(k)} \\ x & Q \end{pmatrix}$$

and given P_k we define P_{k+1} by

$$I - P_{k+1} = \begin{pmatrix} 1 & w^{(k)} \\ 0 & I \end{pmatrix} \begin{pmatrix} 1 - s^{(k)} & -w^{(k)} \\ -x & I - Q \end{pmatrix} = \begin{pmatrix} 1 - s^{(k)} - w^{(k)}x & -w^{(k)}Q \\ -x & I - Q \end{pmatrix}.$$

By induction,

$$P_{k+1} = \begin{pmatrix} s + w(I + Q + \dots + Q^{k-1})x & wQ^k \\ x & Q \end{pmatrix}$$

and $q = P_{k+1}(n, n)$. As in Proposition B.9, let (τ_j) be the sequence from \mathbb{Z}_+G such that

$$\sum_{k=1}^{\infty} q^k = \sum_{j=1}^{\infty} \tau_j t^j.$$

Appealing to Proposition B.9, choose positive c', d' such that $\tau_j > c'(\lambda_q)^j(g_1 + \dots + g_m)$ for all $j \geq d'$. Pick g, h in G and positive integers n_1, n_2 satisfying $gt^{n_1} \leq P_1(1, n)$ and

$ht^{n_2} \leq P_1(n, 1)$. Then for $k > d'$,

$$\begin{aligned}
P_{k+1}(1, 1) &\geq P_1(1, n) (Q(n, n))^{k-1} P_1(n, 1) \\
&\geq gt^{n_1} \left(\sum_{j=d'}^{k-1} c'(\lambda_q)^j (g_1 + \cdots + g_m)t^j \right) ht^{n_2} \\
&\geq \sum_{j=d'+n_1+n_2}^{k+n_1+n_2-1} \left(\frac{c'}{(\lambda_q)^{n_1+n_2}} \right) (\lambda_q)^j (g_1 + \cdots + g_m)t^j \\
&> \sum_{j=d'+n_1+n_2}^{k+n_1+n_2-1} \left(\frac{c'}{(\lambda_q)^{n_1+n_2}} \right) \left(\lambda - \frac{\epsilon}{2} \right)^j (g_1 + \cdots + g_m)t^j .
\end{aligned}$$

Let m_0 be the smallest j such that $j \geq d' + n_1 + n_2$ and

$$\left(\frac{c'}{(\lambda_q)^{n_1+n_2}} \right) \left(\lambda - \frac{\epsilon}{2} \right)^j > (\lambda - \epsilon)^j .$$

Then given $d \geq m_0$, for $k = d$ we have $P_k(1, 1) > \sum_{j=m_0}^d (\lambda - \epsilon)^j (g_1 + \cdots + g_m)t^j$. This finishes the proof of the lemma. \square

Proposition 5.6. *Suppose A is an $n \times n$ G -primitive matrix over \mathbb{Z}_+G , $n > 1$ and $1 < \beta < \lambda_A$. Then there is a positive integer r_0 such that the following holds. If $r \geq r_0$ and $I - Q$ is a matrix in $\text{GL}(k, \mathbb{Z}[t])$ such that*

- (i) $|q_{ijsg}| \leq \beta^s$ for all i, j, s, g , and
- (ii) $Q \in \mathcal{M}(t^r \mathbb{Z}[t])$

then the matrix $\begin{pmatrix} I - Q & 0 \\ 0 & I - tA \end{pmatrix}$ is $\text{El}(\mathbb{Z}G[t])$ equivalent to an $(m+k) \times (m+k)$ matrix $I - B$ over $\mathbb{Z}G[t]$ such that

- (1) B has entries in $t\mathbb{Z}_+G[t]$
- (2) B^\square is G -primitive
- (3) if $\overline{Q} = 0$, then \overline{B}^\square is SSE over \mathbb{Z}_+ to \overline{A} .

Proof. We use \sim to denote $\text{El}(\mathbb{Z}G[t])$ equivalence. First, note that if $I - F$ is a matrix over $\mathbb{Z}G[t]$ with block form $I - F = \begin{pmatrix} I - Q & -X \\ 0 & I - C \end{pmatrix}$ such that $I - C \sim I - tA$, then

the invertibility of $I - Q$ implies $I - F \sim \begin{pmatrix} I - Q & 0 \\ 0 & I - tA \end{pmatrix} = I - (Q \oplus tA)$, since

$$\begin{aligned} \begin{pmatrix} I - Q & -X & 0 \\ 0 & I - C & 0 \\ 0 & 0 & I \end{pmatrix} & \begin{pmatrix} (I - Q)^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & (I - Q) \end{pmatrix} \begin{pmatrix} I & X & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} I - Q & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & (I - Q)^{-1} \end{pmatrix} \\ & = \begin{pmatrix} I - Q & 0 & 0 \\ 0 & I - C & 0 \\ 0 & 0 & I \end{pmatrix}. \end{aligned}$$

Next, given β , let $\epsilon = (\lambda_A - \beta)/2$ and let m_0 be the integer of the conclusion of Lemma 5.3 given A and ϵ . Suppose $I - Q \in \text{GL}(k, \mathbb{Z}[t])$ and Q satisfies (i) and (ii). Pick $r \geq m_0$ such that for all $s \geq r$, $(\lambda_A - \epsilon)^s > 2k\lceil\beta^s\rceil + 1$. Let d be an integer such that $d > r$ and $d \geq \text{degree}(Q)$. Now take $I - C$ from Lemma 5.3, positive equivalent to $I - tA$, such that

$$c_{11gm} \geq 2k\lceil\beta^m\rceil + 1, \text{ for } r \leq m \leq d \text{ and for all } g.$$

Let $u = \sum_g g$. Let $\alpha = \sum_{m=r}^d \lceil\beta^m\rceil ut^m$. Consider a matrix in block form,

$$H = \begin{pmatrix} Q & X \\ 0 & C \end{pmatrix} = \left(\begin{array}{cccc|cccc} q_{11} & q_{12} & \cdots & q_{1k} & \alpha & 0 & \cdots & 0 \\ q_{21} & q_{22} & \cdots & q_{2k} & \alpha & 0 & \cdots & 0 \\ \vdots & & & \vdots & \vdots & & & \vdots \\ q_{k1} & q_{k2} & \cdots & q_{kk} & \alpha & 0 & \cdots & 0 \\ \hline 0 & 0 & \cdots & 0 & c_{11} & c_{12} & \cdots & c_{1n} \\ 0 & 0 & \cdots & 0 & c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & & & \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 & c_{n1} & c_{n2} & \cdots & c_{nn} \end{array} \right).$$

Define a matrix V with matching block structure, $V = \begin{pmatrix} I_k & 0 \\ Y & I_n \end{pmatrix}$, in which the top row of Y has every entry 1 and the other entries of Y are zero, and in which I_j as usual

denotes a $j \times j$ identity matrix. Define $B = V^{-1}HV$. We have

$$(5.7) \quad B = \left(\begin{array}{cccc|cccc} q_{11} + \alpha & q_{12} + \alpha & \cdots & q_{1k} + \alpha & \alpha & 0 & \cdots & 0 \\ q_{21} + \alpha & q_{22} + \alpha & \cdots & q_{2k} + \alpha & \alpha & 0 & \cdots & 0 \\ \vdots & & & \vdots & \vdots & & & \vdots \\ q_{k1} + \alpha & q_{k2} + \alpha & \cdots & q_{kk} + \alpha & \alpha & 0 & \cdots & 0 \\ \hline x - \eta_1 & x - \eta_2 & \cdots & x - \eta_k & x & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{21} & \cdots & c_{21} & c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & & & \vdots & \vdots & & & \vdots \\ c_{n1} & c_{n1} & \cdots & c_{n1} & c_{n1} & c_{n2} & \cdots & c_{nn} \end{array} \right)$$

in which $x = c_{11} - k\alpha$ and $\eta_j = q_{1j} + q_{2j} + \cdots + q_{kj}$. Then $x \geq (k+1)u(t^d + \cdots + t^r)$, $x - \eta_j \geq u(t^d + \cdots + t^r)$ and $q_{ij} + \alpha \geq 0$. Because x is G -primitive and C is G -primitive, it follows easily that B is G -primitive. Also, since $I - B = V^{-1}(I - H)V$, the matrix $I - B$ is $\text{El}(\mathbb{Z}G[t])$ equivalent to $I - H$, and therefore to $I - t(Q \oplus C)$.

Finally, suppose $\overline{Q} = 0$. We must show \overline{B}^\square is SSE over \mathbb{Z}_+ to A . Clearly A and \overline{C}^\square are SSE over \mathbb{Z}_+ . The matrices B and C have all entries in $t\mathbb{Z}_+[t]$. Thus by Remark 3.6, $\overline{B}^\square = \overline{B}^\square$ and $\overline{C}^\square = \overline{C}^\square$. Therefore it suffices to show that \overline{B}^\square and \overline{C}^\square are SSE over \mathbb{Z}_+ . By Proposition 3.9, this will follow if we show \overline{B} is SSE over $\mathbb{Z}_+[t]$ to \overline{C} .

Because $\overline{Q} = 0$, we have $\overline{B} = \overline{H}'$, where H' is the matrix obtained from H by replacing the entries q_{ij} and η_j in the display (5.7) with zero. Let D be the lower right hand block of the 2×2 block matrix B . H' is SSE over $\mathbb{Z}_+G[t]$ to C , since

$$C = \begin{pmatrix} Y & I_n \end{pmatrix} \begin{pmatrix} X \\ D \end{pmatrix} \quad \text{and} \quad H' = \begin{pmatrix} X \\ D \end{pmatrix} \begin{pmatrix} Y & I_n \end{pmatrix}.$$

Therefore $\overline{B} = \overline{H}'$ is SSE over $\mathbb{Z}_+[t]$ to \overline{C} . This finishes the proof. \square

Lemma 5.8. *Suppose G is a finite group, N is nilpotent $n \times n$ over $\mathbb{Z}G$ and $r \in \mathbb{N}$. Then there is a matrix M_r over $t^r\mathbb{Z}G[t]$ such that $\overline{M}_r = 0$ and $I - M_r$ is $\text{El}(n, \mathbb{Z}G[t])$ -equivalent to $I - t^rN$. Given N , the matrices M_r can be chosen such that the coefficients of all entries are bounded above independent of r .*

Proof. Suppose N is $n \times n$. Because \overline{N} is nilpotent over \mathbb{Z} , we can take U in $\text{SL}_n(\mathbb{Z}) = \text{El}_n(\mathbb{Z})$ such that the matrix $N_1 = U^{-1}\overline{N}U$ is upper triangular with zero diagonal. Given r , for $1 \leq i < n$, let W be $n \times n$ with $W(i, j) = -t^rN_1(i, j)$ if $i < j$ and $W = I$ otherwise. Set $W = W_1W_2 \cdots W_{n-1}$; then $W \in \text{El}(n, \mathbb{Z}G[t])$ and $\overline{W(I - t^rN_1)} = I$. Let M_r be the matrix over $t\mathbb{Z}G[t]$ such that $I - M_r = WU^{-1}(I - t^rN)U = W(I - t^rN_1)$. Then $I - \overline{M}_r = \overline{I - M_r} = \overline{W(I - t^rN_1)} = I$, so $\overline{M}_r = 0$. The boundedness claim is clear from the construction. \square

Lemma 5.9. *Suppose G is a finite group and A is a G -primitive matrix with spectral radius $\lambda > 1$ and N is nilpotent over $\mathbb{Z}G$. Then for all sufficiently large r in \mathbb{N} , the matrix $\begin{pmatrix} I - tA & 0 \\ 0 & I - t^r N \end{pmatrix}$ is $\text{El}(\mathbb{Z}G[t])$ -equivalent to a matrix $I - B$ such that B has entries in $t\mathbb{Z}G_+[t]$ and B^\square is G -primitive and $\overline{B^\square}$ is SSE over \mathbb{Z}_+ to \overline{A} .*

Proof. Pick β such that $1 < \beta < \lambda$. Let r_0 be the integer of Proposition 5.6, which depends on A and β . Let $\{M_r\}$ be the uniformly bounded family given for $\{t^r N\}$ by Lemma 5.8. Then for all large $r \in \mathbb{N}$, $r \geq r_0$ and the matrix $Q = M_r$ satisfies $|q_{ijs}| \leq \beta^s$ for all i, j, g, s . Because $t^r N$ is nilpotent, the matrix $I - t^r N$ is invertible over $\mathbb{Z}G[t]$. Now Lemma 5.9 follows from Proposition 5.6. \square

Given $r \in \mathbb{N}$, define $V_r : NK_1(\mathbb{Z}G) \rightarrow NK_1(\mathbb{Z}G)$ by $V_r : [I - tN] \mapsto [I - t^r N]$, and $F_r : NK_1(\mathbb{Z}G) \rightarrow NK_1(\mathbb{Z}G)$ by $F_r : [I - tN] \mapsto [I - tN^r]$. The map V_r is often called the Verschiebung operator, and F_r the Frobenius operator.

Lemma 5.10. *Let G be a finite group and $r \in \mathbb{N}$ be such that r and $|G|$ are relatively prime. Then the map $V_r : NK_1(\mathbb{Z}G) \rightarrow NK_1(\mathbb{Z}G)$ is injective.*

Proof. One may check directly that $F_r V_r(x) = rx$ for all $x \in NK_1(\mathbb{Z}G)$. By a result of Weibel [45, 6.5, p. 490], the order of every element in $NK_1(\mathbb{Z}G)$ must be a power of $|G|$. Thus the map $F_r V_r$ is injective for r relatively prime to $|G|$, and V_r is as well. \square

Proof of Theorem 5.2. Because $NK_1(\mathbb{Z}G)$ is nontrivial, it is infinite [13]. Given $j \in \mathbb{N}$, let N_1, \dots, N_j be nilpotent over $\mathbb{Z}G$ with the matrices $I - tN_j$ representing distinct classes of $NK_1(R)$. For a sufficiently large such r , Lemma 5.9 applies to each $t^r N_i$, giving B_i satisfying the conclusions of the lemma. We take r which in addition is relatively prime to $|G|$; then the matrices $I - t^r N_i$ will represent distinct classes of $NK_1(\mathbb{Z}G)$, by Lemma 5.10. Let $A_i = B_i^\square$. Condition (2) holds as part of Lemma 5.9. Condition (1) holds because (i) adding a nilpotent direct summand to a matrix does not affect its SE class and (ii) for G -primitive matrices, SE over $\mathbb{Z}G$ is equivalent to SE over \mathbb{Z}_+G (Prop. B.12).

By Theorem 2.2, the matrices A_i are pairwise not SSE over $\mathbb{Z}G$. Condition (3) holds because $\det(I - tA_i) = \det(I - tA) \det(I - tN_i)$ and $\det(I - tN_i)$ here must be 1 by Prop. C.1. \square

Proof of Theorem 5.1. Let A be a G -primitive matrix defining a G extension which is isomorphic to that defined by τ and let A_i be the G -primitive matrices provided by Theorem 5.2. By condition (1) of Theorem 5.2 and Proposition B.11, these G extensions of (X, T) are all eventually conjugate to (X_τ, σ_τ) . By condition (2), the A_i define G extensions which are conjugate to G -extensions defined from (X, T) . Because the A_i are not SSE over $\mathbb{Z}G$, they cannot be SSE over \mathbb{Z}_+G , so their extensions (and

hence their conjugate extensions from (X, T) are pairwise not isomorphic. Lastly, they satisfy condition (3), which for abelian G is a well defined invariant of SSE over $\mathbb{Z}G$ (and even SE over $\mathbb{Z}G$) and therefore is carried over to the isomorphic versions defined over (X, T) . □

6. OPEN PROBLEMS

Realization Problems 6.1. This set of problems for the algebraic analysis of mixing finite group extensions of SFTs involves understanding the range of the algebraic invariants.

- (1) Suppose G is finite group, A is G -primitive and N is a nilpotent matrix over $\mathbb{Z}G$. Must $A \oplus N$ be SSE over $\mathbb{Z}G$ to a G -primitive matrix?
(The methods for Section 5 and [8, Radius Theorem] might be useful. The answer to the corresponding problem for matrices over subrings of \mathbb{R} is positive [9].)
- (2) Given a finite abelian group G , characterize the polynomials $\det(I - tA)$ arising from G -primitive matrices A over $\mathbb{Z}G$.
(For $\mathbb{Z}G = \mathbb{Z}\{e\} = \mathbb{Z}$, this is solved [24].)
- (3) Given a finite group G , characterize the trace series \mathcal{T}_A and conjugate trace series $\kappa\mathcal{T}_A$ arising from G -primitive matrices A over $\mathbb{Z}G$.
- (4) Let G be a finite abelian group. Suppose A is a G -primitive matrix over \mathbb{Z}_+G , and B is a matrix over $\mathbb{Z}G$ such that $\det(I - tA) = \det(I - tB)$. Must B be shift equivalent over $\mathbb{Z}G$ to a G -primitive matrix?
(For analogues involving \mathbb{R} and \mathbb{Z} , see [9].)
- (5) Let G be a finite group. Suppose A is a G -primitive matrix over \mathbb{Z}_+G , and B is a matrix over $\mathbb{Z}G$ with the same conjugate trace series (2.7), $\kappa\mathcal{T}_A = \kappa\mathcal{T}_B$. Must B be shift equivalent over $\mathbb{Z}G$ to a G -primitive matrix?

Algebraic Study 6.2. For square matrices A over $\mathbb{Z}G$, G a finite group, make a satisfactory algebraic study of the $\mathbb{Z}G[t]$ -modules $\text{cok}(I - tA)$ and the associated $\mathbb{Z}G$ -modules $\text{cok}(I - A)$. (The latter arise as invariants of G -equivariant flow equivalence [11].)

Sufficiency of invariants 6.3. The following questions are open even for $G = \{e\}$.

- (1) For G -primitive matrices, what invariants must be added to SSE- $\mathbb{Z}G$ to imply SSE- \mathbb{Z}_+G ?
- (2) Prove or disprove: for G nontrivial, every SSE- $\mathbb{Z}G$ class of G -primitive matrices contains infinitely many SSE- \mathbb{Z}_+G classes.

APPENDIX A. G -SFTs DEFINED FROM MATRICES: LEFT VS. RIGHT ACTION

In this section we describe how G extensions of SFTs are defined from matrices over \mathbb{Z}_+G , and the corresponding classifying role of strong shift equivalence of the matrices over \mathbb{Z}_+G (SSE- \mathbb{Z}_+G). In the process, we correct (see the Erratum A.2 below) an error in the corresponding definition in [11]. Given $X \times G$, the map $g : (x, h) \mapsto (x, hg)$ defines a right action of G on $X \times G$, and the map $g : (x, h) \mapsto (x, gh)$ defines a left action of G on $X \times G$.

There are corresponding notations for presenting a G extension. Suppose $T : X \rightarrow X$ is a homeomorphism and $\tau : X \rightarrow G$ is continuous. For the left action on $X \times G$ we define the group extension $T_{\ell, \tau} : X \times G \rightarrow X \times G$ by $T_{\ell} : (x, h) \mapsto (T(x), h\tau(x))$. For the right action we define $T_{r, \tau} : X \times G \rightarrow X \times G$ by $T_r : (x, h) \mapsto (T(x), \tau(x)h)$. Each commutes with its associated G action.

In the case of the left G action, continuous functions τ, τ' from $X \times G$ to G are *cohomologous* if there is a continuous $\gamma : X \rightarrow G$ such that for all x , $\tau'(x) = \gamma^{-1}(x)\tau(x)\gamma(Tx)$. In the case of the right action, the cohomology equation is $\tau'(x) = \gamma(Tx)\tau(x)\gamma^{-1}(x)$.

Now suppose A is square over \mathbb{Z}_+G . The matrix \bar{A} over \mathbb{Z}_+ is defined from A by applying the augmentation map $\sum_g n_g g \mapsto \sum_g n_g$ entrywise. We view \bar{A} as the adjacency matrix of a directed graph. If the set of edges from vertex i to vertex j is nonempty, label them by elements of G to match $A(i, j) = \sum_g n_g g$: for each g , exactly n_g edges are labeled g . Let $\tau_A : X_{\bar{A}} \rightarrow G$ be the continuous function which sends $x = \dots x_{-1}x_0x_1\dots$ to the label of the edge x_0 , denoted $\ell(x_0)$. We use $T_{\ell, A}$ and $T_{r, A}$ to denote $T_{\ell, \tau}$ and $T_{r, \tau}$ with $\tau = \tau_A$.

In the case of the left G action, with T the shift on X_A , for the corresponding G extension $T_{\ell, A}$ defined on $X_{\bar{A}} \times G$, for $n > 0$ we have

$$\begin{aligned} T_{\ell}^n : (x, h) &\mapsto (T^n x, h\tau_A(x) \cdots \tau_A(T^{n-1}x)) \\ &= (T^n x, h\ell(x_0) \cdots \ell(x_{n-1})) . \end{aligned}$$

Here a weight $w = \ell(x_0)\ell(x_1)\cdots\ell(x_{n-1})$ is the product of the labels along the edge-path $x_0x_1\cdots x_{n-1}$. If $A^n(i, j) = \sum_g n_g g$, then the number of edge paths with initial vertex i , terminal vertex j and weight g is equal to n_g . This is the connection of matrix and group extension behind the following result of Parry (see [11, Prop. 2.7.1]). In the statement, $\tau_A \sim \tau_B \circ \varphi$ means there is a continuous $\gamma : X_{\bar{A}} \rightarrow G$ such that $\tau_B(\varphi(x)) = \gamma^{-1}(x)\tau_A(x)\gamma(\sigma_A x)$. In the proposition we need only assume that G is a discrete group, not necessarily finite. In this case, any continuous function into G will then be locally constant.

Proposition A.1. *Let G be a discrete group. The following are equivalent for matrices A and B over \mathbb{Z}_+G .*

- (1) A and B are SSE over \mathbb{Z}_+G .
- (2) There is a homeomorphism $\varphi: X_{\overline{A}} \rightarrow X_{\overline{B}}$ such that $\varphi\sigma_{\overline{A}} = \sigma_{\overline{B}}\varphi$ and $\tau_A \sim \tau_B \circ \varphi$.
- (3) The G -SFTs $T_{\ell,A}$ and $T_{\ell,B}$ are G -conjugate.

Explanation for all this is in [11]– after correction of the following error.

Erratum A.2. In [11, Sec. 2.4], the group extensions (skew products) were defined as extensions for the right G action on $X \times G$. They should instead be extensions for the left G action on $X \times G$. Consequently two other changes should be made.

- (1) In paragraph 2 of [11, Sec. 2.7], “draw an edge from (g, i) to $(\ell(e)g, j)$ ” should be “draw an edge from (g, i) to $(g\ell(e), j)$ ”.
- (2) In the final sentence of paragraph 2 of [11, Sec. 2.7], “ $(h, j) \mapsto (hg, j)$ ” should be “ $(h, j) \mapsto (gh, j)$ ”.

Remark A.3. We record below some relations among matrices and extensions. We use A' to denote the transpose of a matrix A ; if A has entries in \mathbb{Z}_+G , we let $A^{\text{opp}} = A^\circ$ be the matrix defined by applying entrywise the map $\sum_g n_g g \mapsto \sum_g n_g g^{-1}$. (This map is an isomorphism from $\mathbb{Z}G$ to its opposite ring.)

- (1) $(T_{\ell,A})^{-1}$ and $T_{\ell,(A')^\circ}$ are conjugate G extensions.
- (2) The G extension $T_{r,A}$ is conjugated to the G extension T_{ℓ,A° , by the map $(x, h) \mapsto (x, h^{-1})$. (Note, $(x, hg) \mapsto (x, (hg)^{-1}) = (x, g^{-1}h^{-1})$.)
- (3) $T_{r,A}$ and $T_{r,B}$ are conjugate G extensions $\iff A^\circ$ and B° are SSE- \mathbb{Z}_+G .
- (4) A and B SSE- $\mathbb{Z}_+G \implies (A')^\circ$ and $(B')^\circ$ are SSE- \mathbb{Z}_+G .
(Note: $A = RS, B = SR \implies (A')^\circ = (S')^\circ(R')^\circ, (B')^\circ = (R')^\circ(S')^\circ$.)
- (5) For G nonabelian, for A and B SSE- \mathbb{Z}_+G :
 A' and B' need not be SSE- \mathbb{Z}_+G ; A° and B° need not be SSE- \mathbb{Z}_+G .
(See Example A.4).
- (6) For G nonabelian, for $T_{\ell,A}$ and $T_{\ell,B}$ conjugate G -extensions:
 $T_{r,A}$ and $T_{r,B}$ need not be conjugate G -extensions.

Example A.4. Let $A \sim B$ mean A and B are SSE- \mathbb{Z}_+G . We give an example here of $A \sim B$ with $A^{\text{opp}} \not\sim B^{\text{opp}}$ and $A' \not\sim B'$. We use G the group of permutations on $\{1, 2, 3, 4\}$, in which gh is defined by $(gh)(x) = g(h(x))$. Let $M[x, y, z]$ denote a matrix M with $M(1, 2) = x, M(2, 3) = y, M(3, 1) = z$ and $M = 0$ otherwise. In G , define $a = (143), b = (123), c = (12)(34), d = (13)(24)$; then $abc = e$ and $a^{-1}b^{-1}c^{-1} = d \neq e$. Set $A = M[a, b, c]$ and $B = M[e, e, e] = B^{\text{opp}}$. Then $A \sim M[e, e, abc] = B$, but $A^{\text{opp}} = M[a^{-1}, b^{-1}, c^{-1}] \sim M[e, e, a^{-1}b^{-1}c^{-1}] = M[e, e, d]$, and $M[e, e, d] \not\sim B$ (e.g. by Proposition B.3). Therefore $A^{\text{opp}} \not\sim B^{\text{opp}}$. Similarly, $B' \sim B$, and $A' \sim M[e, e, cab] = M[e, e, d] \not\sim B$.

APPENDIX B. G -PRIMITIVE MATRICES AND SHIFT EQUIVALENCE**Primitivity for matrices over $\mathbb{Z}G$.**

In this section, G is a finite group. We will spell out some basic facts around the regular representation of G , our use of the Perron Theorem and SE over \mathbb{Z}_+G .

Let $m = |G|$. Fix an enumeration of the elements of G , $G = \{g_1, \dots, g_m\}$, with $g_1 = e$, the identity element. If $x = \sum_i n_i g_i \in \mathbb{Z}G$, then its image under the augmentation map is $\bar{x} = \sum_i n_i$.

Definition B.1. For vectors v and matrices M over $\mathbb{Z}G[t]$ (perhaps over just $\mathbb{Z}G$), we define \bar{v} and \bar{M} by applying the augmentation map entrywise, $\sum_{s=0}^S \sum_g m_{sg} g t^s \mapsto \sum_{s=0}^S \sum_g m_{sg} t^s$.

Notational Convention B.2. Given a matrix A over $\mathbb{Z}G$, define $a_{ij} = A(i, j)$ and $a_{ijk} = A^k(i, j)$, and let $a_{ijk}g$ be the integers such that

$$A^k(i, j) = a_{ijk} = \sum_g a_{ijk}g g .$$

Define $\bar{a}_{ij} = \sum_g a_{ij1g}$, i.e., $\bar{A}(i, j) = \bar{a}_{ij}$. The uppercase - lowercase correspondence above producing a given A may be used for other letters as well.

Let e_i denote the size m column vector whose i th entry is 1 and whose other entries are zero. Define an isomorphism of additive groups $p : \mathbb{Z}G \rightarrow \mathbb{Z}^m$ by the rule $\sum_i n_i g_i \mapsto \sum_i n_i e_i$. We carry over the usual partial order on \mathbb{Z}^m : for $x = \sum_i n_i g_i$ we say $x \geq 0$ if $n_i \geq 0$ for all i , and we write $x \gg 0$ if $n_i > 0$ for all i . When we use an order relation for vectors or matrices, we mean that it holds entrywise. For example, $x \gg 0$ in $\mathbb{Z}G$ if and only if $p(x) > 0$ in \mathbb{Z}^m . We also carry over the usual notion of convergence in \mathbb{Z}^m : a sequence of elements $x^{(k)} = \sum_i n_i^{(k)} g_i$ converges to $x = \sum_i n_i g_i$ iff $\lim_k n_i^{(k)} = n_i$ for each i . Convergence of vectors or matrices over $\mathbb{Z}G$ is by definition entrywise convergence.

For $1 \leq r \leq m$, define $m \times m$ permutation matrices P_r, Q_r by the rules

$$\begin{aligned} P_r(i, j) &= 1 \quad \text{iff} \quad g_r g_j = g_i \\ Q_r(i, j) &= 1 \quad \text{iff} \quad g_j g_r = g_i . \end{aligned}$$

Then $P_r(p(g_j)) = p(g_r g_j)$ and $Q_r(p(g_j)) = p(g_j g_r)$. The map $g_r \mapsto P_r$ is the regular representation of G given by its action on itself by multiplication from the left; similarly for Q_r and right multiplication. For $x = \sum_j n_j g_j \in \mathbb{Z}G$, we similarly define $\rho(x)$ to be the $m \times m$ matrix over \mathbb{Z} which presents multiplication by x from the left. That is,

the following diagram commutes, with $\rho(x) = \sum_j n_j P_j$:

$$\begin{array}{ccc} \mathbb{Z}G & \xrightarrow{x} & \mathbb{Z}G & & y & \longrightarrow & xy \\ p \downarrow & & \downarrow p & & \downarrow & & \downarrow \\ \mathbb{Z}^m & \xrightarrow{\rho(x)} & \mathbb{Z}^m & & p(y) & \longrightarrow & \rho(x)p(y) \end{array}$$

Column 1 of the matrix $\rho(x)$ is $p(x) = \begin{pmatrix} n_1 \\ \vdots \\ n_m \end{pmatrix}$, since $p(x) = p(xe) = \rho(x)p(e) = \rho(x)e_1$.

For each j , column j of $\rho(x)$ is $Q_j \begin{pmatrix} n_1 \\ \vdots \\ n_m \end{pmatrix}$, since column j of $\rho(x)$ equals

$$\rho(x)e_j = \rho(x)p(g_j) = p(xg_j) = Q_j p(x) .$$

Now suppose A is $\ell \times n$ over $\mathbb{Z}G$. Define an $\ell m \times nm$ matrix \tilde{A} , with a block form of $m \times m$ blocks, in which the ij block is $\rho(a_{ij})$. If A, B over $\mathbb{Z}G$ have compatible sizes for matrix multiplication, then $\tilde{A}\tilde{B} = \tilde{A}B$. Letting κ be defined as in Definition 2.3, we pause to record some facts used in Section 2 to discuss the periodic data (2.8).

Proposition B.3. *Let G be a finite group, with $m = |G|$. Suppose A is an $n \times n$ matrix over $\mathbb{Z}G$. Let $\eta(t) \in \mathbb{Z}[t]$ be the characteristic polynomial of \tilde{A} . Then $\eta(A) = 0$, and*

- (1) *the finite sequence $(\text{tr}(A^k))_{1 \leq k \leq mn}$, determines $(\text{tr}(A^k))_{1 \leq k < \infty}$.*
- (2) *the finite sequence $(\kappa(\text{tr}(A^k)))_{1 \leq k \leq mn}$ determines $(\kappa(\text{tr}(A^k)))_{1 \leq k < \infty}$.*

If A and B are matrices SSE over $\mathbb{Z}G$, then $(\kappa(\text{tr}(A^k)))_{1 \leq k < \infty} = (\kappa(\text{tr}(B^k)))_{1 \leq k < \infty}$.

Proof. $\eta(A) = 0$ because $A \mapsto \tilde{A}$ defines an embedding of the ring of $n \times n$ matrices over $\mathbb{Z}G$ into the ring of $nm \times nm$ matrices over \mathbb{Z} . The coefficients of η are determined by the finite sequence $(\text{tr}(\tilde{A}^k))_{1 \leq k \leq mn}$, which equals $(m \sum_i a_{iike})_{1 \leq k \leq mn}$, which is determined by $(\kappa(\text{tr}(A^k)))_{1 \leq k \leq mn}$. The claims (1,2) then follow because $\eta(A) = 0$ gives integers c_1, \dots, c_{nm} such that $\text{tr}(A^k) = c_1 \text{tr}(A^{k-1}) + \dots + c_{nm} \text{tr}(A^{k-nm})$ for all $k > mn$. It suffices to prove the final claim in the case $A = RS, B = SR$ for matrices R, S over $\mathbb{Z}G$. Then

$$\text{tr}(RS) = \sum_{i,j,g,h} r_{ij1g} s_{ji1h} gh \quad \text{and} \quad \text{tr}(SR) = \sum_{i,j,g,h} s_{ji1h} r_{ij1g} hg .$$

Because $gh = h^{-1}(hg)h$, it follows that $\kappa(\text{tr}(RS)) = \kappa(\text{tr}(SR))$. For $k > 1$, we have $A^k = (A^{k-1}R)S$ and $B^k = S(A^{k-1}R)$. The conclusion follows. \square

Definition B.4. For a matrix A over $\mathbb{R}G$ (e.g., A over $\mathbb{Z}G$), we say A is G -primitive if A is square, $A \geq 0$ and, for some $k > 0$, $A^k \gg 0$.

Clearly A is G -primitive if and only if \tilde{A} is primitive, since $A \gg 0$ is equivalent to $\tilde{A} > 0$. (For an example, consider $G = \mathbb{Z}/2$, $g \neq e$ and $A = (5g)$, giving $\tilde{A} = \begin{pmatrix} 0 & 5 \\ 5 & 0 \end{pmatrix}$; here \tilde{A} is primitive but A is not G -primitive.) The spectral radius of a real matrix M is denoted λ_M . The matrices \bar{A} and \tilde{A} have the same spectral radius.

Definition B.5. Let G be a finite group. The spectral radius λ_A of a square matrix A over $\mathbb{Z}G$ is defined to be $\lambda_{\bar{A}} = \lambda_{\tilde{A}}$. The spectral radius λ_A of a square matrix A over $\mathbb{Z}G[t]$ is defined to be the spectral radius of A^\square .

Naturally, for A square over $\mathbb{Z}G$, we have $\lambda_A = \overline{\lim}_k \max_{i,j,g} |a_{ijk}|^{1/k}$.

Theorem B.6. *Suppose G is a finite group, $G = \{g_1, \dots, g_m\}$ with $g_1 = e$, the identity element of G . Suppose A is an $n \times n$ matrix over \mathbb{Z}_+G such that its augmentation \bar{A} is irreducible. Let $\lambda = \lambda_A$. For i in $\{1, \dots, n\}$, set $H_i = \cup_k \{g \in G : a_{iikg} > 0\}$. Then the following statements are true.*

- (1) *The sets H_i are conjugate subgroups of G .*
- (2) *The following are equivalent.*
 - (a) *\tilde{A} is primitive.*
 - (b) *A is G -primitive.*
 - (c) *Let $\bar{\ell}, \bar{r}$ denote positive left and right eigenvectors of \bar{A} such that $\bar{\ell}\bar{r} = (1)$ (these vectors exist because \bar{A} is irreducible). Then*

$$\lim_k \left(\frac{1}{\lambda} A \right)^k = (g_1 + \dots + g_m) \frac{1}{m} \bar{r} \bar{\ell}.$$

- (d) *With the notation $\text{tr}(A^k) = \sum_g \tau_{kg} g$, the following conditions hold:*

- (i) *There are relatively prime j, k such that $\tau_{ke} > 0$ and $\tau_{je} > 0$.*
- (ii) *There exists i such that $H_i = G$.*

- (3) *If G is abelian and \bar{A} is irreducible, then the polynomial $\det(I - tA)$ determines whether A is G -primitive.*

Remark B.7. It follows from the Perron theorem that the convergence in (3) above is exponentially fast.

Proof of Theorem B.6. (1) Given i , there exists a diagonal matrix D , with each diagonal entry an element of G , such that $D^{-1}AD$ has all entries in H_i [11, Proposition 4.4]. As in [11], it follows easily that the H_i are conjugate subgroups of G .

- (2) (a) \iff (b) This was part of the paragraph before the theorem.

(b) \implies (c) Let u denote $g_1 + \dots + g_m$. The augmentation matrix \bar{A} is primitive, because A is G -primitive. Therefore $((1/\lambda)\bar{A})^k$ converges to $\bar{r}\bar{\ell}$. Define size n vectors over \mathbb{R}_+G by setting $\ell = u\bar{\ell}$ and $r = u\bar{r}$. If $x = \sum_i n_i g_i \in \mathbb{Z}G$, then $xu = (\sum_i n_i)u =$

ux. Therefore

$$Ar = Au\bar{r} = u\bar{A}\bar{r} = u\lambda\bar{r} = \lambda r$$

and likewise $\ell A = u\lambda\bar{\ell} = \lambda\ell$. These eigenvectors lift to eigenvectors $\tilde{\ell}, \tilde{r}$ of \tilde{A} . Explicitly, $\tilde{\ell} = (\tilde{\ell}_1, \dots, \tilde{\ell}_n)$ in which $\tilde{\ell}_j$ is the size m row vector $p(u\bar{\ell}_j)$; every entry of $\tilde{\ell}_j$ equals $\bar{\ell}_j$. Likewise, every entry of \tilde{r}_j equals \bar{r}_j . We have $(\tilde{\ell}\tilde{r}) = m(\bar{\ell}\bar{r})$. Only now do we appeal to the primitivity of \tilde{A} , which guarantees

$$\lim_k \left(\frac{1}{\lambda} \tilde{A} \right)^k = \frac{1}{m} (\tilde{r}\tilde{\ell}) .$$

Translated back to A , this becomes

$$\lim_k \left(\frac{1}{\lambda} A \right)^k = (g_1 + \dots + g_m) \left(\frac{1}{m} \bar{r}\bar{\ell} \right) .$$

(c) \implies (d) Obvious.

(d) \implies (b) The subgroups H_i are conjugate, so (d) implies that $H_i = G$ for every i . Now suppose j, k are relatively prime with $\tau_{ke} > 0$ and $\tau_{je} > 0$. Pick indices y, z such that $(A^j)_{yye} > 0$ and $(A^k)_{zze} > 0$. If $y = z$ then for all large M we have $a_{yyMe} > 0$, and because $H_y = G$ we have for all g and all large M that $a_{yyMg} > 0$. It then easily follows from the irreducibility of \bar{A} that A is G -primitive.

So suppose $y \neq z$. Because \bar{A} is irreducible, we may choose integers s, s' such that $\bar{A}^s(y, z) > 0$ and $\bar{A}^{s'}(y, z) > 0$. There are corresponding paths π, π' in the labeled graph with adjacency matrix A , say with weights g and g' . Let π^* be a path from q to q with length k and weight e . The concatenation $\pi\pi'$ is a path of length $s + s'$ and weight gg' from y to y . Pick r such that $(gg')^r = e$. Then the path $(\pi\pi')^{jr-1}\pi\pi^*\pi'$ is a path from y to y of weight e and length $jr + k$, which is relatively prime to j . The argument of the last paragraph then applies to show A is G -primitive.

(3) Suppose G is abelian. In this case the conjugate groups H_i are equal and must equal $\cup_k \{g : \tau_{kg} > 0\}$. Thus A is G -primitive if and only if for some relatively prime j, k we have $\tau_{ke} \gg 0$ and $\tau_{je} \gg 0$. This is easily checked with $\det(I - tA)$, which constructively determines $(\text{tr}(A^k))_{k \in \mathbb{N}}$. □

Corollary B.8. *A matrix A over \mathbb{Z}_+G defines a mixing G -extension if and only if A is essentially G -primitive .*

Proof. The G extension defined by A is a SFT defined by \tilde{A} , and therefore is topologically mixing if and only if \tilde{A} is essentially primitive as a matrix over \mathbb{Z}_+ . Therefore the corollary follows from the equivalence of (1) and (2) in Proposition B.6. □

Polynomial matrices. Given A over $t\mathbb{Z}_+G[t]$, we have $\sum_n \text{tr}(A^n) = \sum_n \text{tr}((A^\square)^n)t^n$, and for G abelian $\det(I - A) = \det(I - tA^\square)$. By Theorem B.6, this data determines whether A is G -primitive.

We will need the following consequence of Theorem B.6.

Proposition B.9. *Suppose $A = (a)$ is a 1×1 matrix over $t\mathbb{Z}G_+[t]$ with A^\square G -primitive. Let (α_k) be the sequence of elements from $\mathbb{Z}G$ such that*

$$\sum_{j=1}^{\infty} a^j = \sum_{k=1}^{\infty} \alpha_k t^k .$$

Then there is a positive real number c such that

$$\lim_{k \rightarrow \infty} \frac{1}{(\lambda_A)^k} \alpha_k = c(g_1 + \cdots + g_m) .$$

Proof. The matrix A^\square is the adjacency matrix of a loop graph \mathcal{G} with base vertex 1. Let $a = \sum_{k,g} a_{kg} g t^k$, with the a_{kg} in \mathbb{Z}_+ . Then in \mathcal{G} , for every positive coefficient a_{kg} , there are a_{kg} first return loops to 1 of length k and weight g . The return loops to 1 are formed from all concatenations of first return loops. Under concatenation, lengths add and weights multiply. Consequently, for all k , $(A^\square)^k(1, 1) = \alpha_k$. The proposition is then a consequence of Theorem B.6. \square

In the rest of this section, we check that two standard results for SFTs carry over to G -SFTs. The main interest of the next proposition is (1) \iff (2). The proof is an adaptation of the proof of Kim and Roush in the \mathbb{Z} case (see [25, Section 7.5] or [20]).

Proposition B.10. *Suppose G is a finite group, $\mathcal{S} = \mathbb{Z}_+G$ or $\mathcal{S} = \mathbb{Z}G$, and A and B are square matrices over \mathcal{S} . Then the following are equivalent.*

- (1) A and B are SE over \mathcal{S} .
- (2) A^n and B^n are ESSE over \mathcal{S} for all large n .
- (3) A^n and B^n are SE over \mathcal{S} for all large n .
- (4) Let A be $n_1 \times n_1$ and let B be $n_2 \times n_2$. Let $n = \max\{n_1, n_2\}$ and let $m = |G|$. Then there exists k such that A^k, B^k are SE over \mathcal{S} and $k \equiv 1 \pmod{((mn)^2)!}$.

Proof. Clearly (1) \implies (2) \implies (3) \implies (4). Now, to show (4) \implies (1), assume (4). Then we have $\ell \in \mathbb{N}$, $k \equiv 1 \pmod{((mn)^2)!}$ and matrices U, V over \mathcal{S} such that the following hold:

$$(A^k)^\ell = UV , \quad (B^k)^\ell = VU , \quad A^k U = U B^k , \quad B^k V = V A^k .$$

For $i \geq n$ and $k \geq n$ define $U_i = A^i U$ and $V_j = B^j V$. Then

$$(A^k)^{\ell+i+j} = U_i V_j , \quad (B^k)^{\ell+i+j} = V_j U_i , \quad A^k U_i = U_i B^k , \quad B^k V_j = V_j A^k .$$

Via the map $\mathbb{Z}G \rightarrow \mathbb{Z}^m$ discussed earlier, this gives a shift equivalence of matrices over \mathcal{S} ,

$$(\tilde{A}^k)^{\ell+i+j} = \tilde{U}_i \tilde{V}_j, \quad (\tilde{B}^k)^{\ell+i+j} = \tilde{V}_j \tilde{U}_i, \quad \tilde{A}^k \tilde{U}_i = \tilde{U}_i \tilde{B}^k, \quad \tilde{B}^k \tilde{V}_j = \tilde{V}_j \tilde{A}^k.$$

Choose i such that $\ell + i + j \equiv 1 \pmod{((mn)^2)!}$. It suffices to show that the two intertwining equations then hold with k replaced by 1 (as this translates to the equations holding with the \sim decorations removed). Let $r = k(\ell + i + j)$.

Consider the intertwining equation for U_i . The matrix A is $mn_1 \times mn_1$, and \mathbb{C}^{mn_1} is the direct sum of the kernel K_A and the image W_A of A^{mn} . Because $i \geq mn$, restricted to K we have $\tilde{A}\tilde{U}_i = \tilde{U}_i\tilde{B} = 0$. Also, U_i maps W_A isomorphically to W_B , the image of B^{mn} . An invariant Jordan subspace of A for eigenvalue $\alpha \neq 0$ is mapped by U_i to an invariant Jordan subspace of B for eigenvalue $\beta \neq 0$, such that α/β is a root of unity ξ such that $\xi^r = 1$. Because ξ is in the number field generated by α and β , ξ is a q th root of unity with $q \leq (mn)^2$, and therefore q divides $((mn)^2)!$. Consequently $\xi^r = \xi$ and $\xi = 1$. It follows that $\tilde{A}\tilde{U}_i = \tilde{U}_i\tilde{B}$. The same argument works for the other intertwining equation. \square

Proposition B.11. *Suppose G is a finite group and A and B are square matrices over \mathbb{Z}_+G . Then the following are equivalent.*

- (1) *The G -SFTs σ_A, σ_B are eventually conjugate.*
- (2) *The matrices A, B are SE over \mathbb{Z}_+G .*

Proof. Clearly (2) \implies (1). Also, (2) implies A^n and B^n are SE over \mathbb{Z}_+G for all large n , and this implies (1) by Proposition B.10. \square

Proposition B.12. *Suppose A, B are G -primitive. Then the following are equivalent.*

- (1) *A and B are SE over \mathbb{Z}_+G .*
- (2) *A and B are SE over $\mathbb{Z}G$.*

Proof. Assuming (2), it suffices to prove (1). We have matrices U, V over $\mathbb{Z}G$ giving the assumed shift equivalence of A, B . Then \tilde{U}, \tilde{V} give a shift equivalence of \tilde{A}, \tilde{B} . Perhaps after replacing U, V with $-U, -V$ we have that U takes positive left/right eigenvectors of \tilde{A} to positive left/right eigenvectors for \tilde{B} , and likewise for V . It follows from the spectral gap given by primitivity that for large k , the matrices $\tilde{A}^k U$ and $\tilde{B}^k V$ are strictly positive. They give an SE over \mathbb{Z}_+ of \tilde{A}, \tilde{B} and consequently produce an SE over \mathbb{Z}_+G of A, B . \square

APPENDIX C. $NK_1(\mathbb{Z}G)$

Let \mathcal{R} be a ring (always assumed to be unital). In this appendix, we give background on the group $NK_1(\mathcal{R})$, especially for $\mathcal{R} = \mathbb{Z}G$, with G a finite group.

The first algebraic K group is defined by $K_1(\mathcal{R}) = \mathrm{GL}(\mathcal{R})/\mathrm{El}(\mathcal{R})$, where $\mathrm{GL}(\mathcal{R}) = \varinjlim \mathrm{GL}_n(\mathcal{R})$ and $\mathrm{El}(\mathcal{R}) = \varinjlim \mathrm{El}_n(\mathcal{R})$, $\mathrm{El}_n(\mathcal{R})$ the elementary matrices of size n . If \mathcal{R} is also commutative, then the determinant map $\det : \mathcal{R} \rightarrow \mathcal{R}^\times$ is a split surjection, and gives a decomposition $K_1(\mathcal{R}) \cong \mathrm{SK}_1(\mathcal{R}) \oplus \mathcal{R}^\times$, where $\mathrm{SK}_1(\mathcal{R}) = \ker(\det)$, and \mathcal{R}^\times denotes the group of units in \mathcal{R} .

The group $\mathrm{NK}_1(\mathcal{R})$ is defined to be $\ker(K_1(\mathcal{R}[t]) \xrightarrow{t \rightarrow 0} K_1(\mathcal{R}))$. The exact sequence $0 \rightarrow t\mathcal{R}[t] \rightarrow \mathcal{R}[t] \xrightarrow{t \rightarrow 0} \mathcal{R} \rightarrow 0$ is split on the right, giving a decomposition $K_1(\mathcal{R}[t]) \cong \mathrm{NK}_1(\mathcal{R}) \oplus K_1(\mathcal{R})$. Higman's trick shows that $\mathrm{NK}_1(\mathcal{R})$ is generated by elements of the form $[I - tN]$, with N nilpotent. If \mathcal{R} is reduced (has no non-trivial nilpotents), then one also has $\mathrm{NK}_1(\mathcal{R}) \subset \mathrm{SK}_1(\mathcal{R}[t])$.

For any ring \mathcal{R} , $\mathrm{NK}_1(\mathcal{R})$ either is trivial or is not finitely generated as a group. For many rings \mathcal{R} , $\mathrm{NK}_1(\mathcal{R}) = 0$. For any regular Noetherian ring \mathcal{R} , $\mathrm{NK}_1(\mathcal{R}) = 0$. For example, a polynomial ring $\mathcal{R}[x_1, \dots, x_n]$ is regular Noetherian if \mathcal{R} is a field, \mathbb{Z} , a Dedekind domain or any ring with finite global dimension. See [37, 47] for all this and more. However, if G is a non-trivial finite group, then $\mathbb{Z}G$ is not regular, and in general the computation of $\mathrm{NK}_1(\mathbb{Z}G)$ is difficult. If G is any finite group of square-free order, then $\mathrm{NK}_1(\mathbb{Z}G) = 0$ [17]. In [46], it is shown that $\mathrm{NK}_1(\mathbb{Z}[\mathbb{Z}/2 \oplus \mathbb{Z}/2])$, $\mathrm{NK}_1(\mathbb{Z}[\mathbb{Z}/4])$, and $\mathrm{NK}_1(\mathbb{Z}[D_4])$, where D_4 denotes the dihedral group of order 8, are all non-zero. In fact, both $\mathrm{NK}_1(\mathbb{Z}[\mathbb{Z}/2 \oplus \mathbb{Z}/2])$ and $\mathrm{NK}_1(\mathbb{Z}[\mathbb{Z}/4])$, as abelian groups, are isomorphic to a countably infinite direct sum of copies of $\mathbb{Z}/2$, while $\mathrm{NK}_1(\mathbb{Z}[D_4])$ is a quotient of a direct sum of a countably infinite free $\mathbb{Z}/4$ module and a countably infinite free $\mathbb{Z}/2$ module [46].

While the situation for $\mathbb{Z}[G]$ with G a general finite group is complicated, more is known for finite abelian groups. It follows from Theorem 3.12 in [29] together with Theorem 1.4 from [46] that $\mathrm{NK}_1(\mathbb{Z}[\mathbb{Z}/p^n]) \neq 0$ for $n \geq 2$ with p prime⁴. This taken together with Theorem 3.6 in [29] then implies that for a general finite abelian group $G = \bigoplus_{i=1}^n \mathbb{Z}/p_i^{k_i}$, $\mathrm{NK}_1(\mathbb{Z}[G])$ is non-zero if one of it's p -primary cyclic components has p -rank greater than 1, i.e. $k_i \geq 2$ for some $1 \leq i \leq n$.

For any ring \mathcal{R} and finite group G , $\mathrm{NK}_1(\mathcal{R}G)$ is a torsion group [15, 47]. In fact, [15, Theorem A] shows that the order of every element of $\mathrm{NK}_1(\mathcal{R}G)$ is some power of $|G|$, whenever $\mathrm{NK}_1(\mathcal{R}) = 0$. (For $\mathcal{R} = \mathbb{Z}$, and other rings, this is a result of Weibel.) In particular, if P is a finite p -group, then every element of $\mathrm{NK}_1(\mathbb{Z}P)$ has p -primary order [15].

Proposition C.1. *Suppose the ring \mathcal{R} is commutative and reduced (i.e., has no nonzero nilpotent element). Then the following hold.*

- (1) *Let N be a nilpotent matrix over \mathcal{R} . Then $\mathrm{tr}(N^k) = 0$ for all k in \mathbb{N} .*

⁴This is also proved in [41]

(2) $NK_1(\mathcal{R}) \subset SK_1(\mathcal{R}[t])$.

If G is a finitely generated abelian group, then $NK_1(\mathbb{Z}G) \subset SK_1(\mathbb{Z}G[t])$.

Proof. (1) Suppose N is nilpotent with $\text{tr}(N^\ell) = \alpha \neq 0$. Without loss of generality, suppose $\text{tr}(N^j) = 0$ for $j > \ell$. Set $M = N^\ell$ and suppose $M^J = 0$. Let $\det(I - tM) = 1 - c_1t - c_2t^2 - \dots$. Then $c_1 = \alpha$ and for $k > 1$,

$$\text{tr}(M^k) = kc_k + \sum_{1 \leq j < k} c_j \text{tr}(M^{k-j}) = kc_k + c_{k-1} \text{tr}(M) .$$

By induction, $(k!)c_k = (-1)^{k+1}\alpha^k$, for all k in \mathbb{N} . Since $\det(I - tM)$ is a polynomial, α is nilpotent, a contradiction.

(2) An element of $NK_1(\mathcal{R})$ contains a matrix of the form $I - tN$, where N is nilpotent over \mathcal{R} . Since $I - tN$ is invertible, $\det(I - tN)$ must be a unit in the polynomial ring $\mathcal{R}[t]$. Because \mathcal{R} is commutative and reduced, the only units in $\mathcal{R}[t]$ are degree zero polynomials, and therefore $\det(I - tN) = 1$.

For a finitely generated abelian group G , it follows from a theorem of Sehgal [42, page 176] that $\mathbb{Z}G$ has no nilpotent elements. \square

For a ring \mathcal{R} , the reduced nil group $\text{Nil}_0(\mathcal{R})$ is an abelian group which may be presented by generators and relations as follows. The generator set is the set of nilpotent matrices. The relations are $A = A \oplus 0$ (where 0 is any square zero matrix and A is nilpotent); $A = U^{-1}AU$ (A nilpotent, U invertible over \mathcal{R}); and for any block matrix with A, B square nilpotent,

$$A + B = \begin{pmatrix} A & X \\ 0 & B \end{pmatrix} .$$

An important correspondence in K-theory is that the map $N \mapsto I + tN$ (defined for N nilpotent) induces a well defined isomorphism from $\text{Nil}_0(\mathcal{R})$ to $NK_1(\mathcal{R})$.

Explicit examples over $\mathbb{Z}G$

Below we give some explicit examples of elements in NK_1 of certain integral group rings.

Example C.2. We give a 2×2 matrix M which represents a nontrivial element of $NK_1(\mathbb{Z}G)$, for the cyclic group $G = \mathbb{Z}/4\mathbb{Z}$. (The justification in [40] for the example is a nontrivial and computer-assisted exercise in K-theory.) We let σ be a generator of G and set

$$M = \begin{pmatrix} 1 - a & -b \\ -c & 1 - d \end{pmatrix}$$

with

$$a = (1 - \sigma^2)(x - 2x^2 + 2x^3 - \sigma + x\sigma + x^2\sigma)$$

$$b = (1 - \sigma^2)(1 + 2x - x^2 - x^3 - 2x^4 + \sigma - x\sigma - 2x^2\sigma - 3x^3\sigma + 2x^4\sigma)$$

$$c = (1 - \sigma^2)(-1 + 2x - 5x^2 + 7x^3 - 3x^4 + 2x^5 - \sigma + 2x\sigma - 2x^3\sigma + 3x^4\sigma - 2x^5\sigma)$$

$$d = (1 - \sigma^2)(2 + x - 2x^2 - 4x^4 - 2x^5 + \sigma - 3x\sigma - x^2\sigma - 4x^3\sigma + 6x^4\sigma - 4x^5\sigma + 4x^6\sigma) .$$

Because entries of the 2×2 matrix M have maximum degree 6, we can systematically produce from M a 12×12 nilpotent matrix N which is nontrivial in $\text{Nil}_0(\mathbb{Z}G)$. We could work harder to reduce the 12×12 size a bit, but we do not know how to produce a small nilpotent matrix nontrivial in $\text{Nil}_0(\mathbb{Z}G)$.

Example C.3. One could ask for an explicit example of two G -primitive matrices over \mathbb{Z}_+G with G abelian which are shift equivalent but not strong shift equivalent over \mathbb{Z}_+G (and thus present nonisomorphic mixing group extensions). We don't know small matrix examples for this, because we don't know small examples of nilpotents nontrivial in $\text{NK}_1(\mathbb{Z}G)$. We can do a bit better with polynomial matrix presentations. With $G = \mathbb{Z}/4\mathbb{Z}$ and a, b, c, d from Example C.2 and e, f elements of $\mathbb{Z}_+G[x]$, consider the 4×4 matrix

$$\begin{aligned} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e & f & 0 & 0 \\ e & f & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \\ & = \begin{pmatrix} e - 2f & f & f & f \\ e - 2f + (a + b + c + d) & f & f - (a + c) & f - (b + d) \\ e - (a + b) & 0 & f - c & f - d \\ e - (c + d) & 0 & f - a & f - b \end{pmatrix} := L . \end{aligned}$$

Let $K = \begin{pmatrix} e & f \\ e & f \end{pmatrix}$. Choosing f , and then e , with sufficiently large coefficients, one has K and L over $\mathbb{Z}_+G[t]$ such that K^\square and L^\square are G -primitive matrices. Because $I - K$ and $I - L$ are not $\text{El}(\mathbb{Z}G[t])$ equivalent, K^\square and L^\square are not SSE over $\mathbb{Z}G$, and therefore the associated group extensions cannot be isomorphic. However, K^\square and L^\square are shift equivalent over $\mathbb{Z}G$ and therefore (since they are G -primitive) shift equivalent over \mathbb{Z}_+G , by B.12.

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APPENDIX D. CORRECTIONS

The content preceding this appendix is essentially the content of the paper as accepted by Ergodic Theory and Dynamical Systems (doi:10.1017/etds.2015.87), before processing by the publisher. Two corrections should be made to the paper:

- The bijection of Theorem 2.2(2) is not to $NK_1(\mathcal{R})$, but to a certain quotient group $NK_1(\mathcal{R})/E(A, \mathcal{R})$.
- In Theorem 5.2 there should be added the hypothesis that the elementary stabilizer $E(A, \mathcal{R})$ is trivial. (This is known to hold if A is invertible over \mathcal{R} , and in some other cases [10].)

The “elementary stabilizer” $E(A, \mathcal{R})$ is defined to be the subgroup of elements $[U]$ in $NK_1(\mathcal{R})$ such that there exists an elementary matrix E such that $U(I-tA) = (I-tA)E$.

With these changes, the theorems and proofs remain correct. Note, the revised Theorem 5.2 still provides for every finite group G with nontrivial $NK_1(\mathbb{Z}G)$ many cases in which the answer to Parry’s question is decisively no.

We give now a little more context.

Theorem 2.2 states a result claimed in the version 1 arXiv post of [10]; this quoted result was corrected in the version 2 post of [10] (and in the paper itself, to appear in *Crelle’s Journal*), where we proved for every commutative ring \mathcal{R} that

$$\bigcup_A E(A, \mathcal{R}) = NSK_1(\mathcal{R}) .$$