STRONG SHIFT EQUIVALENCE AND ALGEBRAIC K-THEORY

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Abstract. For a semiring $\mathcal{R}$, the relations of shift equivalence over $\mathcal{R}$ (SE-$\mathcal{R}$) and strong shift equivalence over $\mathcal{R}$ (SSE-$\mathcal{R}$) are natural equivalence relations on square matrices over $\mathcal{R}$, important for symbolic dynamics. When $\mathcal{R}$ is a ring, we prove that the refinement of SE-$\mathcal{R}$ by SSE-$\mathcal{R}$, in the SE-$\mathcal{R}$ class of a matrix A, is classified by the quotient $NK_1(\mathcal{R})/E(A,\mathcal{R})$ of the algebraic K-theory group $NK_1(\mathcal{R})$. Here, $E(A,\mathcal{R})$ is a certain stabilizer group, which we prove must vanish if A is nilpotent or invertible. For this, we first show for any square matrix A over $\mathcal{R}$ that the refinement of its SE-$\mathcal{R}$ class into SSE-$\mathcal{R}$ classes corresponds precisely to the refinement of the $\mathrm{GL}(\mathcal{R}[t])$ equivalence class of $I - tA$ into $\mathrm{El}(\mathcal{R}[t])$ equivalence classes. We then show this refinement is in bijective correspondence with $NK_1(\mathcal{R})/E(A,\mathcal{R})$. For a general ring $\mathcal{R}$ and A invertible, the proof that $E(A,\mathcal{R})$ is trivial rests on a theorem of Neeman and Ranicki on the K-theory of noncommutative localizations. For $\mathcal{R}$ commutative, we show $\cup_A E(A,\mathcal{R}) = NSK_1(\mathcal{R})$; the proof rests on Nenashev's presentation of K_1 of an exact category.

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1. Introduction

Let $\mathcal{R}$ (always assumed to contain 0 and 1) be a subset of a ring. Let A, B be square matrices over $\mathcal{R}$ (not necessarily of equal size). Matrices A and B over $\mathcal{R}$ are *elementary strong shift equivalent* over $\mathcal{R}$ (ESSE-$\mathcal{R}$) if there exist matrices U, V over $\mathcal{R}$ such that $A = UV$ and $B = VU$. A and B are *strong shift equivalent* over $\mathcal{R}$ (SSE-$\mathcal{R}$) if they are connected by a chain of elementary strong shift equivalences. A and B are *shift equivalent* over $\mathcal{R}$ (SE-$\mathcal{R}$) if there exist matrices U, V over $\mathcal{R}$ and ℓ in $\mathbb{N}$ such that the following hold:

$$\begin{aligned} A^\ell &= UV \qquad & B^\ell &= VU \\ AU &= UB \qquad & VA &= BV \quad . \end{aligned}$$

If A, B are SSE-$\mathcal{R}$, then they are SE-$\mathcal{R}$.

For symbolic dynamics, these are central relations, introduced by Williams [23, 42]; they may be familiar from other settings. For example, idempotent matrices p, q over a unital C^*-algebra $\mathcal{A}$ are Murray-von Neumann equivalent if and only if p and q are SSE-$\mathcal{R}$ (this can be deduced from [19, Lemma A.4.4]). We give background and motivation in Section 2. Briefly: shift equivalence is very useful for symbolic dynamics and reasonably tractable, with several algebraic characterizations when $\mathcal{R}$ is a ring (see Theorem 6.2). Strong shift equivalence is a more fundamental and mysterious relation.

There is an obvious basic question: assuming $\mathcal{R}$ is a ring, does SE $-\mathcal{R}$ imply SSE $-\mathcal{R}$? The answer was shown to be yes for $\mathcal{R} = \mathbb{Z}$ (see Williams' proof in [43] on his work from the 70s); for $\mathcal{R}$ a principal ideal domain (Effros, 1981, [12]); and for $\mathcal{R}$ a Dedekind domain (Boyle-Handelman, 1993 [5]). There were no counterexamples, and no results after [5]. In his 1999 Bulletin AMS survey, Wagoner formally posed the "Algebraic Shift Equivalence Problem" [34, Problem 2.14]: for what rings Λ does SE over Λ imply SSE over Λ? We will show that for $\mathcal{R}$ a ring, in a given SE $-\mathcal{R}$ class the refinement of SE $-\mathcal{R}$ by SSE $-\mathcal{R}$ is captured exactly by a certain quotient group of the algebraic K-theory group $NK_1(\mathcal{R})$.

From here, let $\mathcal{R}$ be a ring, and $\mathfrak{M}_n(\mathcal{R})$ the $n \times n$ matrices over $\mathcal{R}$. With the maps $p_n \colon \mathfrak{M}_n(\mathcal{R}) \to \mathfrak{M}_{n+1}(\mathcal{R})$ defined by $M \mapsto M \oplus 1$, we form a direct limit of semigroups $\mathfrak{M}(\mathcal{R})$, with a finite matrix M sent to M_{st1} in $\mathfrak{M}(\mathcal{R})$. The maps p_n are the maps which construct $\mathrm{GL}(\mathcal{R})$ and the elementary group $\mathrm{El}(\mathcal{R})$ as direct limits. A $\mathrm{GL}_n(\mathcal{R})$ equivalence $UMV = M'$ gives a $\mathrm{GL}_{n+1}(\mathcal{R})$ equivalence $p_n(U)p_n(M)p_n(V) = p_n(M')$, so $\mathrm{GL}(\mathcal{R})$ equivalence and $\mathrm{El}(\mathcal{R})$ equivalence of the objects M_{st1} is well defined. When we say that two finite matrices M and M' are $\mathrm{GL}(\mathcal{R})$ equivalent or $\mathrm{El}(\mathcal{R})$ equivalent, we mean that the relation holds for M_{st1} and $(M')_{\mathrm{st1}}$, i.e. $UM_{\mathrm{st1}}V = (M')_{\mathrm{st1}}$ for $U, V \in GL(\mathcal{R})$ or $U, V \in \mathrm{El}(\mathcal{R})$. It is natural to identify M_{st1} with an $\mathbb{N} \times \mathbb{N}$ matrix (see Sec. 2).

For finite square matrices A, B over $\mathcal{R}$, we will show

$$A \text{ and } B \text{ are SE-}\mathcal{R} \iff I - tA \text{ and } I - tB \text{ are } \mathrm{GL}(\mathcal{R}[t]) \text{ equivalent} \tag{1.1}$$

$$A \text{ and } B \text{ are SSE-}\mathcal{R} \iff I - tA \text{ and } I - tB \text{ are } \mathrm{El}(\mathcal{R}[t]) \text{ equivalent} \tag{1.2}$$

The proof of (1.1) in Section 6 uses an old stabilization result of Fitting, following Warfield (see Theorem 5.11). In Section 7, (1.2) is proved. The formulation of the correspondence in Theorem 7.2 as induced by a map $I - A \mapsto \mathcal{A}^{\square}$ is simple and natural. The matrix arguments of the proof, however, are nonstandard for K-theory, and a K-theorist may find the details barbaric: nonfunctorial, complicated and (worst of all?) bereft of exact sequences. For better and for worse, this is the proof we have.

Given a ring $\mathcal{R}$ and a square matrix M over $\mathcal{R}$, we define associated sets of square matrices over $\mathcal{R}$:

$$\mathrm{Orb}_{\mathrm{GL}(\mathcal{R})}(M) = \{M' : M' \text{ is } \mathrm{GL}(\mathcal{R}) \text{ equivalent to } M\}$$
$$\mathrm{Orb}_{\mathrm{EL}(\mathcal{R})}(M) = \{M' : M' \text{ is } \mathrm{El}(\mathcal{R}) \text{ equivalent to } M\}$$

Now suppose A is any square matrix over $\mathcal{R}$. Then $\mathrm{Orb}_{\mathrm{GL}(\mathcal{R}[t])}(I - tA)$ is a disjoint union of the sets $\mathrm{Orb}_{\mathrm{EL}(\mathcal{R}[t])}(I - tB)$ such that $I - tB$ is $\mathrm{GL}(\mathcal{R}[t])$ equivalent to $I - tA$ and B has entries in $\mathcal{R}$. Define the elementary stabilizer

$$E(A, \mathcal{R}) = \{U \in \mathrm{GL}(\mathcal{R}[t]) \colon U\mathrm{Orb}_{\mathrm{El}(\mathcal{R}[t])}(I - tA) \subset \mathrm{Orb}_{\mathrm{El}(\mathcal{R}[t])}(I - tA)\} \ .$$

Because $\mathrm{El}(\mathcal{R}[t]) \subset E(A, \mathcal{R})$, we may also regard $E(A, \mathcal{R})$ as a subgroup of $K_1(\mathcal{R}[t])$; there, $E(A, \mathcal{R}) \subset NK_1(\mathcal{R})$. (We will recall definitions in Section 2.)

We will show that there is a bijection

$$\begin{aligned} NK_1(\mathcal{R})/E(A, \mathcal{R}) &\to \{\mathrm{Orb}_{\mathrm{EL}(\mathcal{R}[t])}(I - tB) : I - tB \in \mathrm{Orb}_{\mathrm{GL}(\mathcal{R}[t])}(I - tA)\} \\ [I - tN] &\mapsto \mathrm{Orb}_{\mathrm{EL}(\mathcal{R}[t])}(I - t(A \oplus N)) \ . \end{aligned} \tag{1.3}$$

In (1.3), B is a square matrix over $\mathcal{R}$; N is a nilpotent matrix over $\mathcal{R}$; and $[I - tN]$ is the class in $NK_1(\mathcal{R})$ containing $I - tN$.

For a square matrix B over $\mathcal{R}$, let $[B]_{SSE-\mathcal{R}}$ denote the set of matrices SSE-$\mathcal{R}$ to B; similarly define $[B]_{SE-\mathcal{R}}$. From (1.1) ,(1.2) and (1.3), for any square matrix A over $\mathcal{R}$ we get a well-defined bijection (Theorem 6.6),

$$\begin{aligned} NK_1(\mathcal{R})/E(A, \mathcal{R}) &\to \{[B]_{SSE-\mathcal{R}} \mid [A]_{SE-\mathcal{R}} = [B]_{SE-\mathcal{R}}\} \\ [I - tN] &\mapsto [A \oplus N]_{SSE-\mathcal{R}} \ . \end{aligned} \tag{1.4}$$

It is easy to check $E(A, \mathcal{R})$ is trivial if A is nilpotent. There are rings with nontrivial $NK_1(\mathcal{R})$ such that $E(A, \mathcal{R})$ is trivial for every A (Remark 5.4). We will show $E(A, \mathcal{R})$ is trivial if A is SE-$\mathcal{R}$ to a matrix which is invertible or idempotent (Theorem 4.7), and in some other cases when $\mathcal{R}$ is the integral group ring of a finite abelian group (Cor. 4.10).

The key to the triviality of $E(A,\mathcal{R})$ for invertible or idempotent A (important for applications) is Theorem 3.1, which shows that the map $K_1(\mathcal{R}[t]) \to K_1(\Omega_+^{-1}\mathcal{R}[t])$ induced by a certain Cohn localization $\mathcal{R}[t] \to \Omega_+^{-1}\mathcal{R}[t]$ is injective. In the case $\mathcal{R}$ is commutative, this can be handled with a standard localization exact sequence. But for the generality of all rings $\mathcal{R}$, the proof depends on the work of Neeman and Ranicki on the K-theory of noncommutative localization. For general $\mathcal{R}$, they extended a localization finite exact sequence of Schofield by a single term (see Theorem 3.8). We need that extra term to prove Theorem 3.1.

The elementary stabilizer $E(A,\mathcal{R})$ is not always trivial. For $\mathcal{R}$ commutative, we show (Theorem 5.1) that

$$\bigcup_{A\in\mathcal{M}(\mathcal{R})} E(A,\mathcal{R}) = NSK_1(\mathcal{R})$$

where $NSK_1(\mathcal{R}) = \{[M] \in NK_1(\mathcal{R})\colon \det(M) = 1\}$. If $\mathcal{R}$ is a reduced ring (one with no nonzero nilpotent element), then $NSK_1(\mathcal{R}) = NK_1(\mathcal{R})$. The proof uses Fitting's stabilization result (Theorem 5.11); Quillen's localization sequence in K-theory for the localization of $\mathcal{R}[t]$ at the reverse monic polynomials; and Nenashev's characterization of K_1 of an exact category. We leave open the problem of finding a more complete understanding of the elementary stabilizer (see Conjecture 5.20 and Problem 5.21).

At the end of Section 4, we provide some context for the statement and proof of Theorem 4.7. In Section 8, we note that for nilpotent matrices N, N' over $\mathcal{R}$, $[N] = [N']$ in $\mathrm{Nil}_0(\mathcal{R})$ if and only if N and N' are SSE-$\mathcal{R}$.

This paper is entirely about matrices over rings and related K-theory. However, strong motivation for the paper comes from symbolic dynamics (where the paper already has a serious application [7]), as indicated in the last two subsections of Section 2. The original arXiv post [9] of our paper contained an error (see Remark 5.2). The implications of that error for the applications is discussed in Remark 2.4.

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2. Background and applications

In this section, we give basic definitions we need for K-theory, shift equivalence and strong shift equivalence. Then we give a little background from symbolic dynamics (not needed for proofs), and summarize motivations and applications.

Notational convention 2.1. Let M_{st1} be defined as in the introduction from a finite square matrix M. We regard M_{st1} as an $\mathbb{N}\times\mathbb{N}$ matrix which has M as its upper left corner and is otherwise equal to the identity matrix. In the set of $\mathbb{N}\times\mathbb{N}$ matrices, I denotes the infinite identity matrix. Thus the direct limit semigroup $\mathfrak{M}(\mathcal{R})$ may be identified with the set of all $\mathbb{N}\times\mathbb{N}$ matrices over $\mathcal{R}$ equal to I outside finitely many entries. To avoid a heavier notation, we sometimes suppress the subscript $_{\mathrm{st1}}$. For example, if M is a finite square matrix and U in $\mathrm{GL}(\mathcal{R})$, then UM means UM_{st1}. When we say finite square matrices M, M' are $\mathrm{GL}(\mathcal{R})$ equivalent, we mean there are U, V in $\mathrm{GL}(\mathcal{R})$ such that $UM_{\mathrm{st1}}V = (M')_{\mathrm{st1}}$.

Remark 2.2. If in the introduction for p_n we used $M \mapsto M\oplus 0$ rather than $M \mapsto M\oplus 1$, we would produce a more standard stable version of M, which we denote M_{st0}. Consistent with the matrix interpretation of M_{st1}, we regard M_{st0} as an $\mathbb{N}\times\mathbb{N}$ matrix which has upper left corner M and has other entries zero. With this interpretation, $(I_n - A)_{\mathrm{st1}} = I - A_{\mathrm{st0}}$.

Some basic K-theory. Throughout this paper, a ring means a ring with unit. Unless mentioned otherwise, for $\mathcal{R}$ a ring, an $\mathcal{R}$-module M is a right $\mathcal{R}$-module ($r : m \mapsto mr$), and matrix multiplication of vectors is multiplication of column vectors. Everything in the paper would remain true if instead we used left $\mathcal{R}$ modules and multiplication of row vectors.

We briefly review some definitions and notation. We recommend the books [30, 41] for an introduction to algebraic K-theory.

Let $\mathcal{R}$ be a ring. The group $K_1(\mathcal{R})$ is defined by $K_1(\mathcal{R}) = \mathrm{GL}(\mathcal{R})/\mathrm{El}(\mathcal{R})$, where $\mathrm{GL}(\mathcal{R}) = \varinjlim \mathrm{GL}_n(\mathcal{R})$ and $\mathrm{El}(\mathcal{R}) = \varinjlim \mathrm{El}_n(\mathcal{R})$, with $\mathrm{El}_n(\mathcal{R})$ the group generated by basic elementary matrices of size n (those equal to I except possibly in a single offdiagonal entry). If $\mathcal{R}$ is commutative, then $\mathrm{El}(\mathcal{R}) \subset \mathrm{SL}(\mathcal{R}) := \varinjlim \mathrm{SL}_n(\mathcal{R})$, and $SK_1(\mathcal{R})$ denotes $\{[M] \in K_1(\mathcal{R}) : \det M = 1\}$. As above, we use $\mathbb{N}\times\mathbb{N}$ matrices as a notation for these direct limits. The group $NK_1(\mathcal{R})$ is the kernel of the homomorphism $K_1(\mathcal{R}[t]) \to K_1(\mathcal{R})$ induced by the ring homomorphism $\mathcal{R}[t] \overset{t\to 0}{\to} \mathcal{R}$. The exact sequence $0 \to t\mathcal{R}[t] \to \mathcal{R}[t] \overset{t\to 0}{\to} \mathcal{R} \to 0$ is split on the right, giving a decomposition $K_1(\mathcal{R}[t]) \cong NK_1(\mathcal{R}) \oplus K_1(\mathcal{R})$.

For a category $\mathcal{P}$ with exact sequences and small skeleton $\mathcal{P}_0$, $K_0(\mathcal{P})$ is defined to be the free abelian group on $Obj(\mathcal{P}_0)$, modulo the relations:
(1) $[P_1] = [P_2]$ if P_1 and P_2 are isomorphic in $\mathcal{P}$.
(2) $[P] = [P_1] + [P_2]$ if there is a short exact sequence in $\mathcal{P}$

$$0 \to P_1 \to P \to P_2 \to 0$$

For a ring $\mathcal{R}$, the nil category $\mathbf{Nil}(\mathcal{R})$ is the exact category whose objects are pairs (P, f), where P is an object in $\mathbf{Proj}(\mathcal{R})$, the category of finitely generated projective $\mathcal{R}$-modules, and f is a nilpotent endomorphism of P. A morphism $h\colon (P, f) \to (Q, g)$ in $\mathbf{Nil}(\mathcal{R})$ is a morphism $h\colon P \to Q$ in $\mathbf{Proj}(\mathcal{R})$ such that

$$\begin{array}{ccc} P & \xrightarrow{h} & Q \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ P & \xrightarrow{h} & Q \end{array}$$

commutes. There is a split surjective functor $\mathbf{Nil}(\mathcal{R}) \to \mathbf{Proj}\ \mathcal{R}$ defined by sending (P, f) to P, and we let $\mathrm{Nil}_0(\mathcal{R})$ denote the kernel of $K_0(\mathbf{Nil}(\mathcal{R})) \to K_0(\mathcal{R})$, giving a decomposition $K_0(\mathbf{Nil}(\mathcal{R})) = K_0(\mathcal{R}) \oplus \mathrm{Nil}_0(\mathcal{R})$.

Every element of $NK_1(\mathcal{R})$ contains a matrix of the form $I - tN$, with N a nilpotent matrix with entries in $\mathcal{R}$. It is a classic result that the map $[I - tN] \to [N]$ defines an isomorphism $NK_1(\mathcal{R}) \to \mathrm{Nil}_0(\mathcal{R})$. A theorem of Farrell [13] shows that when $NK_1(\mathcal{R}) \neq 0$, $NK_1(\mathcal{R})$ is not finitely generated as a group. If G is a finite group of order n, then $NK_1(\mathbb{Z}G)$ is trivial if n is square-free [18], but in general may not vanish [39].

To appreciate that $NK_1(\mathcal{R})$ is often trivial, recall that a (left) Noetherian ring is regular if every finitely generated (left) $\mathcal{R}$-module M has a finite-type projective resolution, i.e. there exists an exact sequence

$$0 \to P_n \to \cdots \to P_0 \to M \to 0$$

with P_i projective for all i. These Noetherian regular rings form a large class, containing rings of finite global dimension (fields, principal ideal domains, Dedekind domains ...). If $\mathcal{R}$ is regular, then the polynomial ring $\mathcal{R}[x_1, \ldots, x_n]$ is regular. If $\mathcal{R}$ is a Noetherian regular ring, then $NK_1(\mathcal{R})$ is trivial.

Cohn Localization. Cohn localization is a fundamental tool for the study of non-commutative rings.

Let Σ be a collection of matrices over a ring $\mathcal{R}$, $\Sigma = \{A_i\}$. The Cohn localization of $\mathcal{R}$ with respect to Σ consists of a ring (denoted $\Sigma^{-1}\mathcal{R}$) with a ring homomorphism $\phi\colon \mathcal{R} \to \Sigma^{-1}\mathcal{R}$ satisfying two properties:

(1) For every matrix A in Σ, $\phi(A)$ is invertible in $\Sigma^{-1}\mathcal{R}$.
(2) If $\gamma\colon \mathcal{R} \to S$ is any other ring homomorphism such that $\gamma(A)$ is invertible over S for all $A \in \Sigma$, then there is a (unique) ring homomorphism $\delta\colon \Sigma^{-1}\mathcal{R} \to S$ such that $\gamma = \phi \circ \delta$.

The ring $\Sigma^{-1}\mathcal{R}$ is thus a universal Σ-inverting ring. With the usual nontriviality assumption for a ring, $0 \neq 1$, there might be no ring over which the matrices in Σ become invertible. Therefore, so that $\Sigma^{-1}\mathcal{R}$ is always defined, the degenerate possibility $\Sigma^{-1}\mathcal{R} = \{0\}$ is allowed. Then $\Sigma^{-1}\mathcal{R}$ exists and is essentially unique (see [31] or [11]).

The Cohn localization can also be constructed given a collection of morphisms between finitely generated projective $\mathcal{R}$-modules in an analogous fashion. Given such a collection Σ, call a ring morphism $\mathcal{R} \to \mathcal{S}$ Σ-inverting if $\sigma \otimes 1\colon P \otimes_{\mathcal{R}} \mathcal{S} \to Q \otimes_{\mathcal{R}} \mathcal{S}$ is an $\mathcal{S}$-module isomorphism for every $\sigma\colon P \to Q$ in Σ. Then the noncommutative localization is a ring $\Sigma^{-1}\mathcal{R}$ with a Σ-inverting map $\mathcal{R} \to \Sigma^{-1}\mathcal{R}$ such that $\Sigma^{-1}\mathcal{R}$ is universal with respect to Σ-inverting maps, analogous to (2) above.

More details regarding the general construction of $\Sigma^{-1}\mathcal{R}$ may be found in 7.2 of [11].

Given $\mathcal{R}$, define Ω_+ to be the collection of $\mathcal{R}[t]$-module homomorphisms satisfying the following:

(1) Each $f \in \Omega_+$ is an $\mathcal{R}[t]$-module homomorphism $f\colon P \to Q$ between some finitely generated $\mathcal{R}[t]$-modules P, Q.
(2) For every $f \in \Omega_+$, f is injective, and $\operatorname{coker}(f)$ is a finitely generated projective $\mathcal{R}$-module.

Following [29], we refer to Ω_+ as the set of Fredholm homomorphisms. The localization $\Omega_+^{-1}\mathcal{R}[t]$ has the property that the map $\mathcal{R}[t] \to \Omega_+^{-1}\mathcal{R}[t]$ is injective [29, Prop. 10.7].

One can alternatively construct the Fredholm localization using matrices. Let Ω_+^{mat} denote the set of matrices A over $\mathcal{R}[t]$ such that (with A $m \times n$) the induced map on free $\mathcal{R}[t]$-modules $\mathcal{R}[t]^n \xrightarrow{A} \mathcal{R}[t]^m$ is injective and $\operatorname{coker}(A)$ is a finitely generated projective $\mathcal{R}$-module. We refer to Ω_+^{mat} as the set of Fredholm matrices. That the localizations $\Omega_+^{-1}\mathcal{R}[t]$ and $(\Omega_+^{\mathrm{mat}})^{-1}\mathcal{R}[t]$ coincide is easy to check. We may occasionally abuse notation and write Ω_+ in place of Ω_+^{mat} when it is clear that matrices are being considered.

An alternative construction of $\Omega_+^{-1}\mathcal{R}[t]$ may be described as follows. Let Ω_M denote the set of monic matrices over $\mathcal{R}[t]$, i.e. the square matrices $A = \sum_{i=0}^{d} A_i t^i$ with the A_i matrices over $\mathcal{R}$ such that A_d is the identity matrix. Note that $\Omega_M \subset \Omega_+$. In fact, the two localizations coincide [29, Prop. 10.7]: $\Omega_+^{-1}\mathcal{R}[t] = \Omega_M^{-1}\mathcal{R}[t]$.

Shift equivalence Two square matrices A, B over $\mathcal{R}$ are called shift equivalent over $\mathcal{R}$ (SE-$\mathcal{R}$) if there exists a positive integer l (the lag) and matrices R, S over $\mathcal{R}$ such that

$$RS = A^l, SR = B^l, RB = AR, BS = SA.$$

While shift equivalence is an equivalence relation, lag one shift equivalence is not. The transitive closure of lag one shift equivalence is called strong shift equivalence, so two square matrices A, B over $\mathcal{R}$ are strong shift equivalent over $\mathcal{R}$ (SSE-$\mathcal{R}$) if there is a chain of lag one shift equivalences between them.

Strong shift equivalence Let $\mathcal{R}$ be a ring. The nature of SSE-$\mathcal{R}$ as a kind of stabilized version of similarity over $\mathcal{R}$ is shown by the following characterization from [24]. The relation SSE-$\mathcal{R}$ is generated by two relations:
(1) Similarity over $\mathcal{R}$: $A = U^{-1}BU$.
(2) "Zero extensions":

$$\begin{pmatrix} A & U \\ 0 & 0 \end{pmatrix} \sim A \sim \begin{pmatrix} A & 0 \\ U & 0 \end{pmatrix}$$

Similarity over $\mathcal{R}$ implies SSE-$\mathcal{R}$, since $A = U^{-1}BU$ gives $A = VU$, $B = UV$ with $V = U^{-1}B$. Each type of zero extension respects SSE-$\mathcal{R}$, because

$$A = \begin{pmatrix} A & U \end{pmatrix} \begin{pmatrix} I \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} I \\ 0 \end{pmatrix} \begin{pmatrix} A & U \end{pmatrix} = \begin{pmatrix} A & U \\ 0 & 0 \end{pmatrix}$$
$$A = \begin{pmatrix} I & 0 \end{pmatrix} \begin{pmatrix} A \\ U \end{pmatrix}, \qquad \begin{pmatrix} A \\ U \end{pmatrix} \begin{pmatrix} I & 0 \end{pmatrix} = \begin{pmatrix} A & 0 \\ U & 0 \end{pmatrix}.$$

Conversely, given $A = UV,\ B = VU$ we have a similarity:

$$\begin{pmatrix} I & 0 \\ V & I \end{pmatrix} \begin{pmatrix} A & U \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & U \\ 0 & B \end{pmatrix} \begin{pmatrix} I & 0 \\ V & I \end{pmatrix} \tag{2.3}$$

Antecedents. The connection between $\mathrm{Nil}_0(\mathcal{R})$ and SSE $-\mathcal{R}$ grew for us out of the "positive K-theory" [4, 3] approach to classification problems in symbolic dynamics. That approach grew out of earlier work, especially [2, 20, 21], and Wagoner's background in algebraic K-theory. Some classification problems in symbolic dynamics can be presented, for a suitable ordered ring $\mathcal{R}$, as the problem of classifying square matrices A, B over $\mathcal{R}$ up to SSE $-\mathcal{R}_+$. In the most important example, for the classification of shifts of finite type, Williams used $\mathcal{R} = \mathbb{Z}_+$ [42]. For the classification of group extensions of shifts of finite type by a finite group G for example, Parry used $\mathcal{R} = \mathbb{Z}_+G$ [10, 8]. For a group ring $\mathcal{R} = \mathbb{Z}G$, the relation SSE $-\mathbb{Z}_+G$ of A and B is equivalent to "positive" equivalence of the matrices $I - tA$ and $I - tB$ [3, Theorem 7.2]. Here a positive equivalence is a certain type of $\mathrm{El}(\mathbb{Z}G[t])$ equivalence $U(I - tA)V = I - tB$

(see [4, 3, 8] for definitions and explanation). This by analogy raises the question for rings answered by (1.2).

The elementary stabilizer as a subgroup of $K_1(\mathcal{R})$ appeared in a related context in [10] (see Remark 4.12).

Motivation and applications. The results in this paper have been used to answer (in the negative) a question of Parry [28, Sec. 4.4] about a possible extension of Livšic theory to finite group extensions of shifts of finite type, and have significantly clarified the structure of their algebraic invariants [8]. They have also been used to show that two old conjectures about the algebraic structure of nonnegative matrices are equivalent [7].

Remark 2.4. The papers [7, 8] appealed to the incorrect claim in our original arXiv post [9] that $E(A, \mathcal{R})$ is always trivial (see Remark 5.2). However, the arguments of [7] go through unchanged, with appropriate reference to Theorem 6.3 in place of [7, Theorem 2.1]. In [8], after replacing Theorem 2.2(2) with a reference to Theorem 6.3 below, the theorems and proofs remain correct, with one amendment: in Theorem 6.4 of [8], there should be added the assumption that the elementary stabilizer $E(A, \mathbb{Z}G)$ (see (Defn. 4.2)) is trivial. By Theorem 4.7, for every finite group G, $E(A, \mathbb{Z}G)$ is trivial for many matrices A, e.g. for every A invertible over $\mathbb{Z}G$ (also note Cor. 4.10). Thus the revised Theorem 6.4 still provides for every finite group G with nontrivial $NK_1(\mathbb{Z}G)$ many cases in which the answer to Parry's question is decisively no.

In [6], a three part program for understanding SSE for positive real matrices was proposed. One part, understanding the refinement of SSE by SE for subrings of $\mathbb{R}$, is addressed by the current paper.

One "application" of a result describing the refinement of SE by SSE is that one acquires constraints on what proofs might possibly work. For example, the main result of [6] had a hypothesis of SSE (not SE) of two matrices over a subring of $\mathbb{R}$. We now know that hypothesis is not an artifact of the proof.

The classification problem for shifts of finite type is a central open problem for symbolic dynamics. Wagoner used K_2 of the dual numbers as an ingredient for producing a counterexample to Williams' conjecture that SE-$\mathbb{Z}_+$ implies SSE-$\mathbb{Z}_+$, and suggested further possible connection between the classification problem and algebraic K-theory [35, 36]. The current paper is, we hope, a step toward understanding that connection.

3. $K_1(\mathcal{R}[t]) \to K_1(\Omega_+^{-1}\mathcal{R}[t])$ is injective

The main purpose of this section is to prove Theorem 3.1, which we need to prove Theorem 4.6.

Theorem 3.1. *Let Ω_+ denote the set of Fredholm homomorphisms of finitely generated projective modules over $\mathcal{R}[t]$. Then the natural map*

$$K_1(\mathcal{R}[t]) \to K_1(\Omega_+^{-1}\mathcal{R}[t])$$

induced by $\mathcal{R}[t] \to \Omega_+^{-1}\mathcal{R}[t]$ is injective.

The proof of Theorem 3.1 for general $\mathcal{R}$ requires us to delve into the proofs behind the Neeman and Ranicki results on the K-theory of Cohn localizations. Before going to that more difficult work, we'll give the (shorter) proof for the case that $\mathcal{R}$ is commutative. The proof for this case uses the standard K-theory localization exact sequence (3.3) with claims appealing to standard references. After that, we will be better positioned to understand (and appreciate) how the work of Neeman and Ranicki fits in. We provide more explanation and reference than experts might need, in an effort to make the material more widely accessible and easily checked.

The Commutative Case

In this subsection, $\mathcal{R}$ is assumed to be commutative.

Definition 3.2. For a ring $\mathcal{R}$, we consider the following exact categories:

1. $\mathcal{H}_1(\mathcal{R})$ is the exact category whose objects are $\mathcal{R}$-modules which have a resolution of length ≤ 1 by finitely generated projective $\mathcal{R}$-modules, and whose morphisms are the $\mathcal{R}$-module homomorphisms between them.
2. Given a multiplicatively closed set $S \subset \mathcal{R}$ of non-zero divisors, $\mathcal{H}_{1,S}(\mathcal{R})$ denotes the full subcategory of $\mathcal{H}_1(\mathcal{R})$ whose objects are the objects of $\mathcal{H}_1(\mathcal{R})$ which are S-torsion modules (i.e. $sM = 0$ for some $s \in S$).

Our use of the term exact category matches the standard one, as in [41, Definition II.7.0]. The notation $\mathcal{H}_1(\mathcal{R})$ was chosen to match Weibel's K-Book [41, Definition II.7.7]. It follows from the Resolution Theorem [41, V.3.1] that the inclusion of Proj$\mathcal{R}$ into $\mathcal{H}_1(\mathcal{R})$ induces an isomorphism $\rho\colon K_1(\mathcal{R}) \to K_1(\mathcal{H}_1(\mathcal{R}))$. The category $\mathcal{H}_{1,S}(\mathcal{R})$ appears in the standard long exact sequence [41, V.7.1]

$$\cdots \to K_n(\mathcal{H}_{1,S}(\mathcal{R})) \to K_n(\mathcal{R}) \to K_n(S^{-1}\mathcal{R}) \to \cdots \tag{3.3}$$

which holds for the localization of a commutative ring $\mathcal{R}$ at a multiplicatively closed set S of central non-zero divisors.

Let S_+ denote the collection of monic polynomials in $\mathcal{R}[t]$, i.e. polynomials of the form $p(t) = \sum_{i=0}^{n} a_i t^i$ with $a_n = 1$. The set S_+ is a multiplicatively closed set of non-zero divisors. Replacing $\mathcal{R}$ and S in (3.3) with $\mathcal{R}[t]$ and S_+, we get the exact sequence

$$\cdots \to K_n(\mathcal{H}_{1,S_+}(\mathcal{R}[t])) \to K_n(\mathcal{R}[t]) \to K_n(S_+^{-1}\mathcal{R}[t]) \to \cdots \tag{3.4}$$

To prove Theorem 3.1 for $\mathcal{R}$ commutative, it is now sufficient to show that the map $\alpha\colon K_1(\mathcal{H}_{1,S_+}(\mathcal{R}[t])) \to K_1(\mathcal{R}[t])$ in (3.4) is the zero map. This map factors through the map induced by the inclusion functor (see the proof of [41, V.7.1]) $j\colon \mathcal{H}_{1,S_+}(\mathcal{R}[t]) \to \mathcal{H}_1(\mathcal{R}[t])$, giving a diagram

$$\begin{array}{ccc} K_1(\mathcal{H}_{1,S_+}(\mathcal{R}[t])) & \xrightarrow{K_1(j)} & K_1(\mathcal{H}_1(\mathcal{R}[t])) \\ & \searrow^{\alpha} & \downarrow \rho^{-1} \\ & & K_1(\mathcal{R}[t]) \end{array}$$

in which the vertical map is the inverse to the isomorphism $K_1(\mathrm{Proj}\mathcal{R}[t]) \to K_1(\mathcal{H}_1(\mathcal{R}[t]))$ given by the Resolution Theorem. It suffices then to show the map

$$K_1(j)\colon K_1(\mathcal{H}_{1,S_+}(\mathcal{R}[t])) \to K_1(\mathcal{H}_1(\mathcal{R}[t]))$$

is the zero map.

For M in $\mathcal{H}_{1,S_+}(\mathcal{R}[t])$, define $\eta(M) = M \otimes_{\mathcal{R}} \mathcal{R}[t]$. The right $\mathcal{R}[t]$-module $\eta(M)$ carries no memory of the original action of t on M; as an $\mathcal{R}$-module, it is isomorphic to a direct sum of countably many copies of M. A well known argument [17, p. 441] shows that every object M in $\mathcal{H}_{1,S_+}(\mathcal{R}[t])$ is finitely generated projective as an $\mathcal{R}$-module. For M in $\mathcal{H}_{1,S_+}(\mathcal{R}[t])$, it follows that $\eta(M)$ is a finitely generated projective $\mathcal{R}[t]$-module, and hence lies in $\mathcal{H}_1(\mathcal{R}[t])$. Let η also denote the functor $\mathcal{H}_{1,S_+}(\mathcal{R}[t]) \to \mathcal{H}_1(\mathcal{R}[t])$ which is $M \mapsto \eta(M)$ on objects and $f \mapsto f \otimes_{\mathcal{R}} \mathrm{id}$ on morphisms. The functor η is exact, since $\mathcal{R}[t]$ is free as an $\mathcal{R}$-module.

Given $M \in \mathcal{H}_{1,S_+}(\mathcal{R}[t])$, let f_M denote the endomorphism of M induced by the $\mathcal{R}[t]$-module structure of M (so, $f_M(x) = x \cdot t$). Let $\pi_M\colon \eta(M) \to M$ be the $\mathcal{R}[t]$ module homomorphism such that $\pi_M\colon x \otimes t^i \mapsto (f_M)^i(x)$, for i in $\mathbb{Z}_+$. Recall $j\colon \mathcal{H}_{1,S_+}(\mathcal{R}[t]) \to \mathcal{H}_1(\mathcal{R}[t])$ denotes the inclusion functor. For morphisms $\psi\colon A \to B$ in $\mathcal{H}_{1,S_+}(\mathcal{R}[t])$, we define transformations of functors, $\mathcal{F}\colon \eta \mapsto \eta$ and $\mathcal{G}\colon \eta \mapsto j$, by the following commutative diagrams of $\mathcal{R}[t]$-module homomorphisms,

$$\begin{array}{ccc} \eta(A) & \xrightarrow{\eta(\psi)} & \eta(B) \\ \mathcal{F}(A)\downarrow & & \downarrow\mathcal{F}(B) \\ \eta(A) & \xrightarrow{\eta(\psi)} & \eta(B) \end{array} \quad = \quad \begin{array}{ccc} A \otimes_{\mathcal{R}} \mathcal{R}[t] & \xrightarrow{\psi\otimes\mathrm{id}} & B \otimes_{\mathcal{R}} \mathcal{R}[t] \\ \mathrm{id}\otimes t - f_A\otimes\mathrm{id}\downarrow & & \downarrow\mathrm{id}\otimes t - f_B\otimes\mathrm{id} \\ A \otimes_{\mathcal{R}} \mathcal{R}[t] & \xrightarrow{\psi\otimes\mathrm{id}} & B \otimes_{\mathcal{R}} \mathcal{R}[t] \end{array}$$

and

$$\begin{array}{ccc} \eta(A) & \xrightarrow{\eta(\psi)} & \eta(B) \\ \mathcal{G}(A)\downarrow & & \downarrow\mathcal{G}(B) \\ j(A) & \xrightarrow{j(\psi)} & j(B) \end{array} \quad = \quad \begin{array}{ccc} A\otimes_R \mathcal{R}[t] & \xrightarrow{\psi\otimes \mathrm{id}} & B\otimes_{\mathcal{R}} \mathcal{R}[t] \\ \pi_A\downarrow & & \downarrow\pi_B \\ A & \xrightarrow{\psi} & B \end{array}$$

Because the vertical arrows do not depend on ψ, $\mathcal{F}$ and $\mathcal{G}$ are natural transformations. Also $\eta \overset{\mathcal{F}}{\rightarrowtail} \eta \overset{\mathcal{G}}{\twoheadrightarrow} j$ is a short exact sequence of functors since for any $M \in \mathcal{H}_{1,S_+}(\mathcal{R}[t])$, the sequence

$$0 \longrightarrow M\otimes_{\mathcal{R}} \mathcal{R}[t] \xrightarrow{t-f_M} M\otimes_{\mathcal{R}} \mathcal{R}[t] \xrightarrow{\pi_M} M \longrightarrow 0$$

(with $t - f_M \colon x\otimes t^i \mapsto x\otimes t^{i+1} - f_M(x)\otimes t^i$) is exact (see e.g. [1, p. 630]). Let $K_1(\eta), K_1(j)$ denote the corresponding maps on K-theory. Because $\eta \overset{\mathcal{F}}{\rightarrowtail} \eta \overset{\mathcal{G}}{\twoheadrightarrow} j$ is a short exact sequence of exact functors of exact categories, it follows from the Additivity Theorem [41, V.1.2] that $K_1(\eta) = K_1(\eta) + K_1(j)$. Thus $K_1(j)$ is the zero map. This concludes the proof of Theorem 3.1 in the case $\mathcal{R}$ is commutative.

Remark 3.5. In the commutative case, the injectivity of $K_1(\mathcal{R}[t]) \to K_1(S_+^{-1}\mathcal{R}[t])$ may also be deduced using an argument of Grayson, found in [17, Corollary 6]. As described in [17, Corollary 6], one constructs a Mayer-Vietoris sequence that splits up, analogous to the proof of the Fundamental Theorem concerning $K_1(\mathcal{R}[t,t^{-1}])$ as found in [16, p.20].

The General Case

From here on, we do not assume the ring $\mathcal{R}$ is commutative. Before proving the general case of Theorem 3.1, we present the necessary material from [25, 26].

Definition 3.6. Let $\Sigma = \{\sigma_i\}$ be a collection of monomorphisms between finitely generated projective $\mathcal{R}$-modules. The exact category $\mathcal{E} = \mathcal{E}(\Sigma)$ is defined to be the full subcategory of $\mathcal{H}_1(\mathcal{R})$ determined by the following conditions:

(1) For every $\sigma \in \Sigma$, coker(σ) lies in $\mathcal{E}$.
(2) If $0 \to M_1 \to M_2 \to M_3 \to 0$ is a short exact sequence of objects in $\mathcal{H}_1(\mathcal{R})$ such that two of the objects M_1, M_2, M_3 lie in $\mathcal{E}$, then so does the third.
(3) $\mathcal{E}$ contains all direct summands of its objects.
(4) $\mathcal{E}$ is minimal, subject to (1),(2) and (3).

Following [25], we refer to the objects in the category $\mathcal{E}(\Sigma)$ as $(\mathcal{R},\Sigma)$-torsion modules. When the collection Σ is clear, we may simply refer to $\mathcal{E}$ instead of $\mathcal{E}(\Sigma)$. Note that in Definition 3.6 we have used $\mathcal{H}_1(\mathcal{R})$ in place of the category of all finitely presented

$\mathcal{R}$-modules of projective dimension ≤ 1 in [25]. The two definitions are equivalent, because the category $\mathcal{H}_1(\mathcal{R})$ and the category of finitely presented modules of projective dimension ≤ 1 coincide: given a finitely presented module M of projective dimension less than or equal to one, one may always construct a resolution of length one or less by finitely generated projective modules [41, 4.1.6].

The next theorem will not be used directly, but helps provide context for the torsion category $\mathcal{E}$ defined above, so we include it.

Theorem 3.7. [25, Proposition 0.7] *Assume for all $\sigma \in \Sigma$ that σ is a monomorphism, and let $\mathcal{E} = \mathcal{E}(\Sigma)$ be as in Definition 3.6. Then an $\mathcal{R}$-module M belongs to $\mathcal{E}$ iff*
(i) M is finitely presented with projective dimension ≤ 1, and
(ii) $\{\Sigma^{-1}\mathcal{R}\} \otimes_{\mathcal{R}} M$ and $Tor_1^{\mathcal{R}}(\Sigma^{-1}\mathcal{R}, M)$ both vanish.

When $\mathcal{R}$ is commutative and $S \subset \mathcal{R}$ is a multiplicatively closed set of non-zero-divisors, we let Ω_S denote the collection of all homomorphisms $f_s \colon \mathcal{R} \to \mathcal{R}$ given by $f_s \colon x \mapsto xs$, with $s \in S$. In this case the Cohn localization $\Omega_S^{-1}\mathcal{R}$ coincides with the standard commutative localization $S^{-1}\mathcal{R}$, and $\mathcal{E}(\Omega_S)$ agrees with $\mathcal{H}_{1,S}(\mathcal{R})$. Indeed, in the commutative case $S^{-1}\mathcal{R}$ is flat, so we always have $Tor_1^{\mathcal{R}}(S^{-1}\mathcal{R}, M) = 0$, and for a nontrivial finitely generated $\mathcal{R}$-module M, $S^{-1}\mathcal{R} \otimes_{\mathcal{R}} M = 0$ iff there exists $s \in S$ such that $Ms = 0$.

The following theorem is the main tool we use to prove the injectivity of the map $K_1(\mathcal{R}[t]) \to K_1(\Omega_+^{-1}\mathcal{R}[t])$. The sequence 3.9, without the leftmost map, was established by Schofield in [31]. The extension to include the term $K_1(\mathcal{E}) \to K_1(\mathcal{R})$, which is critical for our application, is due to Neeman and Ranicki; Theorem 3.8 is a combination of [25, Theorem 0.5] and the result stated as Theorem 3.14 below.

Theorem 3.8. [25, p. 789] *Let $\mathcal{R}$ be a ring, and Σ be a collection of monomorphisms between finitely-generated projective $\mathcal{R}$-modules. Let $\mathcal{E} = \mathcal{E}(\Sigma)$ denote the torsion category of Definition 3.2. Then there is an exact sequence*

$$K_1(\mathcal{E}) \to K_1(\mathcal{R}) \to K_1(\Sigma^{-1}\mathcal{R}) \to K_0(\mathcal{E}) \to K_0(\mathcal{R}) \to K_0(\Sigma^{-1}\mathcal{R}) \tag{3.9}$$

Remark 3.10. Neeman and Ranicki [26] extended (3.9) to

$$\cdots \to K_n(\mathcal{E}) \to K_n(\mathcal{R}) \to K_n(\Sigma^{-1}\mathcal{R}) \to K_{n-1}(\mathcal{E}) \to \cdots$$

for all $n > 1$ under the hypothesis that the localization $\Sigma^{-1}\mathcal{R}$ is *stably flat*: for all $n \geq 1$ the group $\mathrm{Tor}_n^{\mathcal{R}}(\Sigma^{-1}\mathcal{R}, \Sigma^{-1}\mathcal{R})$ vanishes. The six term version (3.9) has no stably flat requirement. We have no need of the full long exact in the present paper.

By Theorem 3.8, to prove the injectivity of $K_1(\mathcal{R}[t]) \to K_1(\Omega_+^{-1}\mathcal{R}[t])$ it is sufficient to show the map $K_1(\mathcal{E}) \to K_1(\mathcal{R}[t])$ in (3.9) is zero. For this, we will need a more detailed examination of the original sequence from [26, Corollary 4.9]. Definitions of maps in (3.9) involve identifications of various groups, and we take care to track through these identifications. We do this for general Σ at first, specializing to the case of interest ($\Sigma = \Omega_+$, the Fredholms) at a later point.

Recall that a Waldhausen category consists of a category with a subcategory of morphisms called cofibrations, along with a distinguished family of morphisms called weak equivalences, satisfying some axioms, which may be found in [41, Definition II.9.1.1]. We let $C_b(\text{Proj}\mathcal{R})$ denote the following Waldhausen category:

(1) The objects are bounded chain complexes of finitely generated projective $\mathcal{R}$-modules
(2) The morphisms are chain maps
(3) The cofibrations are degree-wise split monomorphisms
(4) The weak equivalences are the quasi-isomorphisms, i.e. the chain maps inducing an isomorphism on homology in every degree.

The only Waldhausen categories which will be considered in this article are full subcategories of the category of chain complexes over some exact category, where the morphisms are chain maps, the cofibrations are degree-wise split monomorphisms, and the weak equivalences are quasi-isomorphisms.

For an exact category $\mathcal{A}$ or Waldhausen category $\mathcal{B}$, we let $K(\mathcal{A})$ and $K(\mathcal{B})$ denote the corresponding K-theory spaces, as in [41, IV.6.3 and IV.8.4]. For a topological space X, let $\pi_n(X)$ denote the nth homotopy group. By definition, $K_n(\mathcal{A}) = \pi_n(K(\mathcal{A}))$, and $K_n(\mathcal{B}) = \pi_n(K(\mathcal{B}))$. Since the definitions agree in the case $\mathcal{B}$ is exact [41, IV.8.6], we do not distinguish, and use the same $K(\mathcal{A})$ and $K(\mathcal{B})$ for both.

We will make use of the following theorem.

Theorem 3.11 (Gillet-Waldhausen)**.** *Let $\mathcal{A}$ be an exact category, closed under taking kernels of surjections. Then the exact monomorphism $\mathcal{A} \hookrightarrow C_b(\mathcal{A})$, taking an object M to the chain complex which is M in degree 0 and is zero elsewhere, induces a homotopy equivalence $K(\mathcal{A}) \xrightarrow{\sim} K(C_b(\mathcal{A}))$, and hence isomorphisms $K_n(\mathcal{A}) \xrightarrow{\cong} K_n(C_b(\mathcal{A}))$.*

A proof of Theorem 3.11 may be found in [41, V.2.2, II.9.2.2].

Let $\Sigma = \{\sigma_i\}$ denote a collection of morphisms between finitely generated projective $\mathcal{R}$-modules. Note that each $\sigma \in \Sigma$ may be considered in $C_b(\text{Proj}\mathcal{R})$ as the complex

$$\cdots \to 0 \to P \xrightarrow{\sigma} Q \to 0 \cdots \tag{3.12}$$

with P, Q in degrees $0, 1$ and modules in all other degrees zero.

By a Waldhausen subcategory $\mathcal{A} \subset \mathcal{B}$ of a Waldhausen category $\mathcal{B}$ we mean a subcategory $\mathcal{A} \subset \mathcal{B}$ which is also a Waldhausen category, satisfying:

(1) the inclusion functor $\mathcal{A} \to \mathcal{B}$ is exact, i.e. preserves all of the following: zero, cofibrations, weak equivalences, and pushouts along cofibrations,
(2) the cofibrations in $\mathcal{A}$ are the maps in $\mathcal{A}$ which are cofibrations in $\mathcal{B}$ and whose cokernels lie in $\mathcal{A}$,
(3) the weak equivalences in $\mathcal{A}$ are the weak equivalences of $\mathcal{B}$ which lie in $\mathcal{A}$.

Define a Waldhausen category as follows:

Definition 3.13. The category $\mathbf{R}$ is the smallest subcategory of $C_b(\mathrm{Proj}\mathcal{R})$ which:
(i) contains the complex (3.12) as defined above, for all $\sigma \in \Sigma$,
(ii) contains all acyclic complexes,
(iii) is closed under the formation of mapping cones and suspensions,
(iv) contains any direct summand of any of its objects.

The following theorem is a combination of [26, Corollary 4.9] and [25, Theorem 0.10].

Theorem 3.14. [25, p.789] *Let $\mathcal{R}$ be a ring, and Σ a collection of homomorphisms between finitely generated projective $\mathcal{R}$-modules. There is an exact sequence*
(3.15)
$$K_1(\boldsymbol{R}) \to K_1(C_b(\mathrm{Proj}\mathcal{R})) \to K_1(\Sigma^{-1}\mathcal{R}) \to K_0(\boldsymbol{R}) \to K_0(C_b(\mathrm{Proj}\mathcal{R})) \to K_0(\Sigma^{-1}\mathcal{R})$$

In Theorem 3.14, $\mathcal{R}$ is general and there is no requirement that Σ consists of monomorphisms. The maps $K_i(\mathbf{R}) \to K_i(C_b(\mathrm{Proj}\mathcal{R}))$ are induced by the inclusion $\mathbf{R} \to C_b(\mathrm{Proj}\mathcal{R})$. Upon replacing $C_b(\mathrm{Proj}\mathcal{R})$ in Theorem 3.14 with $\mathcal{R}$ using Gillet-Waldhausen, the maps $K_i(\mathcal{R}) \to K_i(\Sigma^{-1}\mathcal{R})$ coincide with the maps $K_i(\mathcal{R}) \to K_i(\Sigma^{-1}\mathcal{R})$ induced by the ring homomorphism $\mathcal{R} \to \Sigma^{-1}\mathcal{R}$ (see the discussion following Theorem 0.10 in [25]).

Let $C_b(\mathcal{H}_1(\mathcal{R}))$ denote the Waldhausen category of bounded chain complexes of finitely presented $\mathcal{R}$-modules of projective dimension ≤ 1. Given Σ a collection of monomorphisms and $\mathcal{E} = \mathcal{E}(\Sigma)$ as in Definition 3.6, we let $C_b(\mathcal{E})$ denote the Waldhausen category of bounded chain complexes of objects of $\mathcal{E}$. For both $C_b(\mathcal{H}_1(\mathcal{R}))$ and $C_b(\mathcal{E})$, the cofibrations consist of the chain maps which are degree-wise split monomorphisms, and the weak equivalences are the quasi-isomorphisms.

Lemma 3.16. [25, Theorem 2.7] *There is a Waldhausen subcategory $\boldsymbol{R}' \subset C_b(\mathcal{H}_1(\mathcal{R}))$ and inclusions $\boldsymbol{R} \to \boldsymbol{R}'$, $C_b(\mathcal{E}) \to \boldsymbol{R}'$ that induce homotopy equivalences*

$$\begin{array}{ccc} K(\boldsymbol{R}) & & \\ & \searrow^{\simeq} & \\ & & K(\boldsymbol{R}') \\ & \nearrow_{\simeq} & \\ K(C_b(\mathcal{E})) & & \end{array}$$

Remark 3.17. The subcategory $\mathbf{R}'$ of Lemma 3.16 defined in [25, Theorem 2.7] is the full Waldhausen subcategory of $C_b(\mathcal{H}_1(\mathcal{R}))$ consisting of all objects which become isomorphic in $D(C_b(\mathcal{H}_1(\mathcal{R})))$ to objects in the image of $D(\mathbf{R})$, the derived category of $\mathcal{R}$. Details regarding $\mathbf{R}'$ are not important for the present article, and may be found in the proof of Theorem 2.7 in [25].

One consequence of 3.16 is that, by the Gillet-Waldhausen theorem, we have $K(\mathbf{R}) \simeq K(\mathcal{E})$, which gives one of the identifications made when passing between 3.8 and 3.14.

We now specialize to the case of interest, in order to prove the main result of the section. *For the remainder of the section, we let $\Sigma = \Omega_+$ denote the collection of Fredholm homomorphisms of finitely generated projective $\mathcal{R}[t]$-modules.*

Proposition 3.18. *Consider a polynomial ring $\mathcal{R}[t]$, with Ω_+ the collection of Fredholm homomorphisms, and $\boldsymbol{R}$ as defined in Definition 3.8. Then the maps*

$$K_n(i)\colon K_n(\boldsymbol{R}) \to K_n(C_b(\mathrm{Proj}\mathcal{R}[t]))$$

are zero, for all n, where $K_n(i)$ is the map induced by the inclusion $\boldsymbol{R} \to C_b(\mathrm{Proj}\mathcal{R}[t])$.

Since the maps $K_n(\mathbf{R}) \to K_n(C_b(\mathrm{Proj}\mathcal{R}[t]))$ in Theorem 3.14 are induced by the inclusion $\mathbf{R} \to C_b(\mathrm{Proj}\mathcal{R}[t])$, Theorem 3.1 will follow from Proposition 3.18.

Proof of Proposition 3.18: Consider the diagram of inclusions

$$\begin{array}{ccc} & & C_b(\mathcal{H}_1(\mathcal{R}[t])) \\ & \nearrow & \uparrow \\ \mathbf{R} & \longrightarrow & C_b(\mathrm{Proj}\mathcal{R}[t]) \end{array}$$

By the Resolution Theorem (see [41, V.3.1]) we have $K(\mathrm{Proj}\mathcal{R}[t]) \simeq K(\mathcal{H}_1(\mathcal{R}[t]))$, so combined with Gillet-Waldhausen, the vertical functor on the right induces a homotopy equivalence

$$K(C_b(\mathrm{Proj}\mathcal{R}[t])) \xrightarrow{\simeq} K(C_b(\mathcal{H}_1(\mathcal{R}[t])))$$

and therefore isomorphisms $K_n(C_b(\mathrm{Proj}\mathcal{R}[t])) \to K_n(C_b(\mathcal{H}_1(\mathcal{R}[t])))$ for all n. Furthermore, Lemma (3.16) shows that the images of the homomorphisms

$$K_n(\mathbf{R}) \to K_n(C_b(\mathcal{H}_1(\mathcal{R}[t])))$$
$$K_n(C_b(\mathcal{E})) \to K_n(C_b(\mathcal{H}_1(\mathcal{R}[t])))$$

coincide. We claim that the map $K_n(C_b(\mathcal{E})) \to K_n(C_b(\mathcal{H}_1(\mathcal{R}[t])))$ is zero for all n. This will prove that $K_1(\mathcal{R}[t]) \to K_1(\Omega_+^{-1}\mathcal{R}[t])$ is injective, in light of Theorem 3.8. We have a diagram

$$\begin{array}{ccc} C_b(\mathcal{E}) & \longrightarrow & C_b(\mathcal{H}_1(\mathcal{R}[t])) \\ \uparrow & & \uparrow \\ \mathcal{E} & \longrightarrow & \mathcal{H}_1(\mathcal{R}[t]) \end{array}$$

in which by Gillet-Waldhausen the vertical arrows induce homotopy equivalences in K,

$$\begin{array}{ccc} K(C_b(\mathcal{E})) & \longrightarrow & K(C_b(\mathcal{H}_1(\mathcal{R}[t]))) \\ \simeq\uparrow & & \uparrow\simeq \\ K(\mathcal{E}) & \longrightarrow & K(\mathcal{H}_1(\mathcal{R}[t])) \end{array}$$

Thus it suffices to show that the maps $K_n(\mathcal{E}) \to K_n(\mathcal{H}_1(\mathcal{R}[t]))$, induced by the inclusion functor $j\colon \mathcal{E} \to \mathcal{H}_1(\mathcal{R}[t]))$, are zero for all n.

Let X be the full subcategory of $\mathcal{H}_1(\mathcal{R}[t])$ whose objects are the modules M in (i.e. the objects M of) $\mathcal{H}_1(\mathcal{R}[t])$ such that $\eta(M) := M \otimes_{\mathcal{R}} \mathcal{R}[t]$ is in $\mathcal{H}_1(\mathcal{R}[t])$. (For example, $\mathcal{R}[t]$ is in $\mathcal{H}_1(\mathcal{R}[t])$ but is not in X, because $\mathcal{R}[t]$ is not finitely generated as an $\mathcal{R}$-module.) We claim that $\mathcal{E}$ is contained in X. Consider each of the following:

(1) If $\sigma \in \Omega_+$, then $\mathrm{coker}(\sigma)$ is finitely generated projective as an $\mathcal{R}$-module, since Ω_+ consists of Fredholm morphisms. It follows that $\mathrm{coker}(\sigma) \otimes_{\mathcal{R}} \mathcal{R}[t]$ lies in $\mathrm{Proj}\mathcal{R}[t] \subset \mathcal{H}_1(\mathcal{R}[t])$, so X contains the cokernels of all morphisms $\sigma \in \Omega_+$.

(2) Now suppose

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is exact in $\mathcal{H}_1(\mathcal{R}[t])$.

Tensoring this sequence with $\mathcal{R}[t]$ gives

$$0 \to M_1 \otimes_{\mathcal{R}} \mathcal{R}[t] \to M_2 \otimes_{\mathcal{R}} \mathcal{R}[t] \to M_3 \otimes_{\mathcal{R}} \mathcal{R}[t] \to 0 \tag{3.19}$$

which is an exact sequence of $\mathcal{R}[t]$-modules, since $\mathcal{R}[t]$ is free over $\mathcal{R}$. We claim that if two of M_1, M_2, M_3 lie in X, then so does the third.

(a) Suppose M_2 and M_3 lie in X. Then $\eta(M_2)$ and $\eta(M_3)$ lie in $\mathcal{H}_1(\mathcal{R}[t])$. Since $\mathcal{H}_1(\mathcal{R}[t])$ is closed under kernels of surjections (see [41, II.7.7.1]), the exactness of (3.19) shows that M_1 lies in X.

(b) Suppose M_1 and M_3 lie in X. Then the exactness of (3.19) along with the fact that $\mathcal{H}_1(\mathcal{R}[t])$ is closed under extensions (see [40, 2.2.8]) implies M_2 lies in X as well.

(c) Suppose M_1 and M_2 lie in X. Then the exactness of (3.19) shows that $\eta(M_3)$ is finitely presented, being a quotient of two finitely presented modules. But it is clear that $\eta(M_3)$ is also of homological dimension ≤ 1, so M_3 is in X as well.

(3) X contains all direct summands of its objects, since $\mathcal{H}_1(\mathcal{R}[t])$ is closed under direct summands.

Since $\mathcal{E}$ is the minimal subcategory of $\mathcal{H}_1(\mathcal{R}[t])$ satisfying the corresponding properties (1,2,3) in Definition 3.6, we have $\mathcal{E} \subset X$, as desired.

The remainder of the proof closely follows that of the commutative case given earlier. Given $M \in \mathcal{E}$, let f_M denote the endomorphism of M induced by the $\mathcal{R}[t]$-module structure of M (so $f_M(x) = t \cdot x$). From the discussion above we have the exact functor $\eta \colon \mathcal{E}(\Omega_+) \to \mathcal{H}_1(\mathcal{R}[t])$, and we denote by $\mathcal{F}$ the natural transformation $\mathcal{F} \colon \eta \mapsto \eta$ defined by $\mathcal{F}(M) \colon \eta(M) \overset{t-f_M}{\to} \eta(M)$. Recall $j \colon \mathcal{E} \to \mathcal{H}_1(\mathcal{R}[t])$ denotes the inclusion functor. Define the natural transformation $\mathcal{G} \colon \eta \mapsto j$ by $\mathcal{G} \colon \eta(M) \overset{\pi}{\to} M$, where $\pi(p(t) \otimes x) = p(f_M)(x)$. Then $\eta \overset{\mathcal{F}}{\rightarrowtail} \eta \overset{\mathcal{G}}{\twoheadrightarrow} j$ is an exact sequence of functors, since for any $M \in \mathcal{E}$, the sequence

$$0 \to M \otimes_{\mathcal{R}} \mathcal{R}[t] \overset{t-f_m}{\rightarrowtail} M \otimes_{\mathcal{R}} \mathcal{R}[t] \overset{\pi}{\twoheadrightarrow} M \to 0$$

is exact (see [1, p. 630]). Letting $K_n(\eta), K_n(j)$ denote the corresponding maps on K-theory, the Additivity Theorem (V.1.2 in [41]) now implies that, for all n, $K_n(\eta) = K_n(\eta) + K_n(j)$. Thus $K_n(j)$ is the zero map, for all n. This finishes the proof of Theorem 3.1.

4. The elementary stabilizer

Recall our notational conventions (2.1, 2.2). In particular, $\mathfrak{M}(\mathcal{R})$ is the set of $\mathbb{N} \times \mathbb{N}$ matrices over the ring $\mathcal{R}$ equal to the identity except in finitely many entries, with $\mathrm{El}(\mathcal{R}) \subset \mathrm{GL}(\mathcal{R}) \subset \mathfrak{M}(\mathcal{R})$. Given $\mathcal{R}$ and M in $\mathfrak{M}(\mathcal{R})$, the elementary stabilizer of M

is defined to be

$$\mathrm{ElSt}_{\mathcal{R}}(M) = \{U \in \mathrm{GL}(\mathcal{R}) : U\mathrm{Orb}_{\mathrm{El}(\mathcal{R})}(M) \subset \mathrm{Orb}_{\mathrm{El}(\mathcal{R})}(M)\} \ . \tag{4.1}$$

Because $\mathrm{El}(\mathcal{R})$ is a subgroup of $\mathrm{ElSt}_{\mathcal{R}}(M)$, $\{[U] \in K_1(\mathcal{R})\colon U \in \mathrm{ElSt}_{\mathcal{R}}(M)\}$ is a subgroup of $K_1(\mathcal{R})$, which by abuse of notation we also denote by $\mathrm{ElSt}_{\mathcal{R}}(M)$. We give a shorter notation for the elementary stabilizer which is our main interest. Given an $n \times n$ matrix A over $\mathcal{R}$, let $I - tA$ denote $(I_n - tA)_{\mathrm{st1}} = I - tA_{\mathrm{st0}}$ (as in 2.2) and define

$$E(A, \mathcal{R}) := \mathrm{ElSt}_{\mathcal{R}[t]}(I - tA) \ . \tag{4.2}$$

If $U \in E(A, \mathcal{R})$, then there are E, F from $\mathrm{El}(\mathcal{R})$ such that $U(I - tA) = E(I - tA)F$. Evaluating at $t = 0$, we see that $E(A, \mathcal{R})$, considered as a subgroup of $K_1(\mathcal{R}[t])$, satisfies

$$E(A, \mathcal{R}) \subset NK_1(\mathcal{R}) \ . \tag{4.3}$$

Proposition 4.4. *Suppose $\mathcal{R}$ is a ring and $A \in \mathfrak{M}(\mathcal{R})$. Then there is a bijection*

$$\begin{aligned} K_1(\mathcal{R})/\mathrm{ElSt}_{\mathcal{R}}(A) &\to \{\mathrm{Orb}_{\mathrm{El}(\mathcal{R})}(B) : B \in \mathrm{Orb}_{\mathrm{GL}(\mathcal{R})}(A)\} \\ [U] \ &\mapsto \ U\mathrm{Orb}_{\mathrm{El}(\mathcal{R})}(A) \ . \end{aligned}$$

If $B \in \mathrm{Orb}_{\mathrm{GL}(\mathcal{R})}(A)$, then $\mathrm{ElSt}_{\mathcal{R}}(B) = \mathrm{ElSt}_{\mathcal{R}}(A)$.
Also,

$$\bigcup_{A \in \mathcal{M}(\mathcal{R})} E(A, \mathcal{R}) = \bigcup_{C \in \mathrm{GL}(\mathcal{R}[t])\colon C_0 \in \mathrm{GL}(\mathcal{R})} \mathrm{ElSt}_{\mathcal{R}[t]}(C) \ . \tag{4.5}$$

Proof. For $B \in \mathrm{Orb}_{\mathrm{GL}(\mathcal{R})}(A)$, let $\mathcal{O}_B = \mathrm{Orb}_{\mathrm{El}(\mathcal{R})}(B)$. Then $U\mathcal{O}_B = \mathcal{O}_{UB} = \mathcal{O}_{BU} = \mathcal{O}_B U$, for all U in $\mathrm{GL}(\mathcal{R})$ and $B \in \mathrm{Orb}_{\mathrm{GL}(\mathcal{R})}(A)$. Therefore the rule $U\colon \mathcal{O} \mapsto U\mathcal{O}$ gives a well defined action of $\mathrm{GL}(\mathcal{R})$ on $\{\mathcal{O}_B\colon B \in \mathrm{Orb}_{\mathrm{GL}(\mathcal{R})}(A)\}$. The isotropy group of an element $\mathcal{O}_B$ under this action is $\mathrm{ElSt}_{\mathcal{R}}(B)$, which contains $\mathrm{El}(\mathcal{R})$. Therefore given $B \in \mathrm{Orb}_{\mathrm{GL}(\mathcal{R})}(A)$ we have well defined bijections

$$\begin{array}{ccccc} K_1(\mathcal{R})/\mathrm{ElSt}_{\mathcal{R}}(B) & \to & \mathrm{GL}(\mathcal{R})/\mathrm{ElSt}_{\mathcal{R}}(B) & \to & \{\mathcal{O}_C\colon C \in \mathrm{Orb}_{\mathrm{GL}(\mathcal{R})}(A)\} \\ [U] & \mapsto & [U] & \mapsto & U\mathcal{O}_C \qquad . \end{array}$$

For $B \in \mathrm{Orb}_{\mathrm{GL}(\mathcal{R})}(A)$, the isotropy groups $\mathrm{ElSt}_{\mathcal{R}}(A)$ and $\mathrm{ElSt}_{\mathcal{R}}(B)$ are conjugate in $\mathrm{GL}(\mathcal{R})$, and therefore equal, as $\mathrm{ElSt}_{\mathcal{R}}(A)$ contains $\mathrm{El}(\mathcal{R})$, the commutator subgroup of $\mathrm{GL}(\mathcal{R})$.

To prove “Also”, it now suffices, given a square matrix C over $\mathcal{R}[t]$ with $C(0)$ in $\mathrm{GL}(\mathcal{R})$, to note that the $\mathrm{GL}(\mathcal{R}[t])$ orbit of C contains a matrix of the form $I - tA$ with A over $\mathcal{R}$. This holds by application of Higman's trick to $C_0^{-1}C$. □

The next result, a key fact for us, follows directly from Theorem 3.1. For its statement, recall that by our notational convention, the elementary stabilizer of a finite matrix $I-A$ means the elementary stabilizer of $(I-A)_{\mathrm{st1}}$. Recall that the map $i:\mathcal{R}[t]\to\Omega_+^{-1}\mathcal{R}[t]$ denotes the standard map coming from the definition of the localization.

Theorem 4.6. *Let $\mathcal{R}[t]$ be a polynomial ring, with coefficient ring $\mathcal{R}$. If A is a square matrix over $\mathcal{R}[t]$ such that $I-A\in\Omega_+(\mathcal{R}[t])$, then $\mathrm{ElSt}_{\mathcal{R}[t]}(I-A)$ is trivial in $K_1(\mathcal{R}[t])$.*

Proof. If $I-A\in\Omega_+(\mathcal{R}[t])$ and $U\in\mathrm{ElSt}_{\mathcal{R}[t]}(I-A)$, then $[U]$ is in the kernel of the map $i_*\colon K_1(\mathcal{R}[t])\to K_1(\Omega_+^{-1}\mathcal{R}[t])$ of Theorem 3.1. By Theorem 3.1, this implies $[U]=0$. □

Theorem 4.7. *Let A be a square matrix over $t\mathcal{R}[t]$ such that $A=\sum_{i=1}^{d}t^iA_i$. Suppose A satisfies any of the following:*

(1) A_d is nilpotent and $A_i=0$ for $1\le i<d$.
(2) A_d is invertible over $\mathcal{R}$.
(3) A_d is idempotent and $A_i=0$ for $1\le i<d$.

Then $\mathrm{ElSt}_{\mathcal{R}[t]}(I-A)$ is trivial in $K_1(\mathcal{R}[t])$.

In the statement of Theorem 4.7, if $d=1$ then $\mathrm{ElSt}_{\mathcal{R}[t]}(I-A)=E(A_1,\mathcal{R})$.

Proof of Theorem 4.7. The claim for case (1) is clear, because $I-A\in\mathrm{GL}(\mathcal{R}[t])$. For the remaining cases, by Theorem 4.6 it suffices to show that the matrix $I-A$ is invertible over $\Omega_+(\mathcal{R}[t])$. For case (2), the matrix $(I-A)A_d^{-1}$ is monic, and hence invertible over $\Omega_+^{-1}\mathcal{R}[t]$.

For case (3), we first note that if P is an $n\times n$ idempotent matrix over $\mathcal{R}$, then $cok(I-tP)$ is a finitely generated projective $\mathcal{R}$-module. Let J denote $P(\mathcal{R}^n)$, the image of the $\mathcal{R}$-module endomorphism $\mathcal{R}^n\xrightarrow{P}\mathcal{R}^n$ given by multiplication by P. The finitely generated $\mathcal{R}$-module J is projective, since P is idempotent. Letting $x_0,\dots,x_{d-1}$ denote elements of $\mathcal{R}^n$, we have an isomorphism of $\mathcal{R}$-modules $cok(I-tP)\to J^d$ given by $[\sum_{i=0}^{d-1}t^{i+d}Px_i]\mapsto(Px_0,\dots,Px_{d-1})$. It follows that the matrix $I-t^dP$ belongs to $\Omega_+^{mat}(\mathcal{R}[t])$, i.e. is Fredholm. Thus, for the map $i\colon\mathcal{R}[t]\to\Omega_+^{-1}\mathcal{R}[t]$ given by the localization, the matrix $i(I-t^dP)$ is invertible over $\Omega_+^{-1}\mathcal{R}[t]$. □

In the case $\mathcal{R}$ is commutative, localization techniques allow us to make further statements regarding $\mathrm{ElSt}_{\mathcal{R}[t]}(I-A)$ for certain matrices A which satisfy none of the sufficent conditions (1)-(3) of Theorem 4.7.Our main tool for this will be the following result of Vorst (see [33, 1.7, Remark 1.12]). For a an element $r\in\mathcal{R}$ we let $\mathcal{R}_r$ denote the localization of $\mathcal{R}$ at the multiplicative subset $\{r^i\}$. Recall that a collection of elements $\{f_1, \ldots, f_k\} \subset \mathcal{R}$ is called a unimodular row if the ideal $(f_1, \ldots, f_k)$ generated by the collection is $\mathcal{R}$ itself.

Theorem 4.8. [33, Corollary 1.7] *Let $\mathcal{R}$ be a commutative ring, and let $f_1, \ldots, f_k \in \mathcal{R}$ be a unimodular row over $\mathcal{R}$. Then the map $NK_1(\mathcal{R}) \to \prod_{i=1}^{r} NK_1(\mathcal{R}_{f_i})$ is injective.*

Proposition 4.9. *Let $\mathcal{R}$ be a commutative ring, and let A be a square matrix over $\mathcal{R}$ such that $0 \neq \det(A)$ and $\det(A)$ is not a zero-divisor. Suppose there exists j unimodular rows*

$$\{\det(A), f_{1,1}, \ldots, f_{k_1,1}\}, \ldots, \{\det(A), f_{1,j}, \ldots, f_{k_j,j}\}$$

each containing $\det(A)$ such that

$$\bigcup_{i=1}^{j} \bigcap_{n=1}^{k_i} \ker(NK_1(\mathcal{R}) \to NK_1(\mathcal{R}_{f_{n,i}})) = NK_1(\mathcal{R})$$

Then $\mathrm{ElSt}_{\mathcal{R}[t]}(I - A) = 0$.

Proof. Suppose $G \in \mathrm{ElSt}_{\mathcal{R}[t]}(I - A)$, and let $[G]$ denote its class in $NK_1(\mathcal{R})$. The assumptions give an i such that $[G] \in \bigcap_{n=1}^{k_i} \ker(NK_1(\mathcal{R}) \to NK_1(\mathcal{R}_{f_{n,i}}))$. Since A is invertible over $\mathcal{R}_{\det(A)}$, Theorem 4.7 implies $[G] \in \ker(NK_1(\mathcal{R}) \to NK_1(\mathcal{R}_{\det(A)}))$ as well, and hence by Theorem 4.8 we must have $[G] = 0$. □

Proposition 4.9 can be used to show that for certain matrices A over $\mathbb{Z}G$, the elementary stabilizer $E(A, \mathcal{R}) := \mathrm{ElSt}_{\mathcal{R}[t]}(I - tA)$ must vanish, as follows.

Corollary 4.10. *Let G be a finite abelian group of order $|G|$, and let $\mathbb{Z}G$ denote the integral group ring. Let A be a square matrix over $\mathbb{Z}G$ such that $0 \neq \det(A) = a \in \mathbb{Z}$ and $(a, |G|) = 1$ (so a and the order of G are relatively prime). Then $= 0$.*

Proof. The collection $\{a, |G|\}$ forms a unimodular row over $\mathbb{Z}G$. However, by [38, 6.5, pg. 490], $\ker(NK_1(\mathbb{Z}G) \to NK_1((\mathbb{Z}G)_a)) = NK_1(\mathbb{Z}G)$, so Proposition 4.9 implies the claim. □

Remark 4.11. The technique of using localization to prove $\mathrm{ElSt}_{\mathcal{R}[t]}(I - A)$ is trivial, as in the proof of Corollary 4.10, has its limits. For example, if G is a finite group G, then the map $NK_1(\mathbb{Z}G) \to NK_1((\mathbb{Z}G)_{|G|})$ is the zero map.

Remark 4.12. For G a finite group and A a matrix over $\mathbb{Z}G$, the group $K_1(\mathbb{Z}G)/\mathrm{ElSt}_{\mathbb{Z}G}(I - A)$ appeared in [10] as the primary invariant for the classification up to equivariant flow equivalence of certain symbolic dynamical systems: irreducible shifts of finite type with a free continuous shift-commuting G-action.

Limits to generalizations

Theorem 4.7 applies to a rather special class of matrices and its proof appeals to the sophisticated algebraic K-theory of Neeman and Ranicki [25, 26]. It is natural to ask if there is an easier proof. It is also natural to hope the conclusion of Theorem 4.7 might hold for a more general class of matrices. We'll note next that some candidate improvements cannot work.

Remark 4.13. With an eye to an easier proof of Theorem 4.7, one might note for A over $t\mathcal{R}[t]$ that $(I-A)$ also inverts over the familiar ring of formal power series $\mathcal{R}[[t]]$, and ask if $\mathcal{R}[[t]]$ could play the role of $\Omega_+^{-1}\mathcal{R}[t]$ in Theorem 3.1. We thank Wolfgang Steimle for showing us this fails: the natural map $i^*\colon K_1(\mathcal{R}[t]) \to K_1(\mathcal{R}[[t]])$ induced by the inclusion $i\colon \mathcal{R}[t] \to \mathcal{R}[[t]]$ need not be injective. For example, if $\mathcal{R}$ is commutative, then there is a straightforward decomposition of $K_1(\mathcal{R}[[t]])$ given by

$$0 \to K_1(\mathcal{R}) \to K_1(\mathcal{R}[[t]]) \xrightarrow{d} \hat{W}(\mathcal{R}) \to 0$$

where $\hat{W}(\mathcal{R}) = \{1 + \sum_{i=1}^{\infty} a_i t^i\} \in \mathcal{R}[[t]]$ is the group of Witt vectors. The map d is given by $d(M) = \det(M_0^{-1}M)$, where $M = \sum_{i=0}^{\infty} M_i t^i$ (as in e.g. [29, 14.6]). Thus, if $\mathcal{R}$ is a commutative ring (for example, an integral domain) such that $\det(I - tN) = 1$ for all nilpotent matrices N, then the kernel of the map $K_1(\mathcal{R}[t]) \to K_1(\mathcal{R}[[t]])$ induced by the inclusion $\mathcal{R}[t] \to \mathcal{R}[[t]]$ will always contain $NK_1(\mathcal{R})$. Indeed, $d(I - tN) = \det(I - tN) = 1$, so $NK_1(\mathcal{R})$ maps into the kernel of d, which is generated by the image of $K_1(\mathcal{R})$; but the only class of the form $[I - tN]$ which lies in the image of $K_1(\mathcal{R})$ is the class [1]. Since there are integral domains $\mathcal{R}$ with $NK_1(\mathcal{R}) \neq 0$ (e.g. [7, Example 3.5]) the map $NK_1(\mathcal{R}[t]) \to K_1(\mathcal{R}[[t]])$ need not be injective.

Similarly, one could hope to prove in place of Theorem 3.1 that the map $i_*\colon K_1(\mathcal{R}[t]) \to K_1(S_{RM}^{-1}\mathcal{R}[t])$ is injective, where Σ_{RM} is the set of reverse monic matrices (those of the form $A = I + \sum_{i=1}^{n} A_i t^i$). But this map need not be injective. In the case $\mathcal{R}$ is commutative, localizing at Σ_{RM} is equivalent to localizing at $S_{RMP} = \{p(t) = 1 + \sum_{i=1}^{n} a_i t^i\}$, the set of reverse monic polynomials. There is an exact sequence

$$0 \to K_1(\mathcal{R}) \to K_1(S_{RMP}^{-1}\mathcal{R}[t]) \xrightarrow{d} 1 + tS_{RMP}^{-1}\mathcal{R}[t] \to 0$$

which can be found by examining [17, Corollary 3], or [41, III.2.4(2)]. As in the previous paragraph, the map $i_*\colon NK_1(\mathcal{R}) \to K_1(S_{RMP}^{-1}\mathcal{R}[t])$ will fail to be injective for an integral domain $\mathcal{R}$ with $NK_1(\mathcal{R})$ nontrivial.

With regard to generalizing the result, Corollary 4.15 of Proposition 4.14 below shows Theorem 4.7 already fails badly for the more general class of matrices $I - A$ which are injective (in the statement of Corollary 4.15, $\mathcal{R}$ could be a polynomial ring). The rest of this section is devoted to establishing that corollary. We thank David Handelman for showing us the embedding argument which produces the nonderogatory matrix $V = UE$ in the reduction step of Prop. 4.14 below.

Proposition 4.14. *Suppose $\mathcal{R}$ is an integral domain of characteristic zero which does not embed into $\mathbb{Z}[i]$ or $\mathbb{Z}[e^{i2\pi/3}]$, and U is in $\mathrm{SL}(n,\mathcal{R})$. Then there is an $n \times n$ matrix A over $\mathcal{R}$ such that $I-A$ is injective and U is in the elementary stabilizer $\mathrm{ElSt}_{\mathcal{R}}(I-A)$.*

Proof. Case I: For this case, we assume there is a matrix B over the field of fractions $\mathbb{F}$ of $\mathcal{R}$ such that $B^{-1}UB = C$, with C a companion matrix. Without loss of generality, we then assume B has all entries in $\mathcal{R}$. Because C must be the companion matrix of the characteristic polynomial of U, the entries of C must lie in $\mathcal{R}$. From the companion matrix form and $\det C = 1$, we have $C \in \mathrm{El}(n,\mathcal{R})$. Now $UB = BC$; defining $A = I - B$, we have that U is in $\mathrm{ElSt}(I-A)$. Clearly $I-A$ is injective.

For the reduction to Case I, it suffices to show that there is a matrix $E \in \mathrm{El}(n,\mathcal{R})$ such that the matrix $V = UE$ has no repeated eigenvalue (and therefore is similar over $\mathbb{F}$ to its companion matrix). After passing if needed to a subring containing the entries of U and still satisfying the nonembeddability hypothesis, we may assume $\mathcal{R}$ is finitely generated. Then $\mathbb{F}$ is isomorphic to an algebraic extension of a subfield of $\mathbb{R}$ (generated by $\mathbb{Q}$ and a set of algebraically independent elements). Thus after embedding $\mathbb{F}$ into $\mathbb{R}$ or $\mathbb{C}$, we have the closure $\overline{\mathbb{F}}$ equal to $\mathbb{R}$ or $\mathbb{C}$. In either case, except under the very special conditions which are excluded in the hypotheses (and are not of interest to us now), the ring $\mathcal{R}$ will likewise be dense in $\overline{\mathbb{F}}$, and consequently $\mathrm{El}(n,\mathcal{R})$ will be dense in $\mathrm{El}(n,\overline{\mathbb{F}}) = \mathrm{Sl}(n,\overline{\mathbb{F}})$. Let W be a matrix in $\mathrm{SL}(n,\mathbb{Z})$ without repeated eigenvalues. The matrices over $\mathbb{F}$ without repeated eigenvalues form a dense open set. Consequently the matrix $U^{-1}W$ in $\mathrm{SL}(n,\mathbb{F})$ can be perturbed to a matrix E in $\mathrm{El}(n,\mathcal{R})$ such that UE has no repeated eigenvalues. □

In the next statement, $E_{A,\mathcal{R}[t]}$ denotes $\{[U] \in K_1(\mathcal{R}[t]) : U \in \mathrm{ElSt}(I-A)\}$.

Corollary 4.15. *Suppose $\mathcal{R}$ is a characteristic zero integral domain which is not generated by three elements as an additive group, and $NK_1(\mathcal{R})$ nontrivial. (Such domains exist.) In the class of injective matrices $(I-A)$ over $\mathcal{R}[t]$, the elementary stabilizer $E_{A,\mathcal{R}[t]}$ is not independent of A. If H is a finitely generated subgroup of $NK_1(\mathcal{R})$, then there exists an injective $(I-A)$ such that $E_{A,\mathcal{R}[t]}$ contains H.*

Proof. For $I-A$ invertible over $\mathcal{R}[t]$, $E_{A,\mathcal{R}[t]}$ is trivial in $NK_1(\mathcal{R})$. Now choose U_k in $\mathrm{GL}(\mathcal{R}[t])$ for $1 \le k \le K$ with $[U_k] \in NK_1(\mathcal{R})$. Because $\mathcal{R}$ is an integral domain, the U_k lie in $SL(\mathcal{R}[t])$. Proposition 4.14 then gives finite matrices $I-A_k$ over $\mathcal{R}[t]$ with $I-A_k$ injective such that for $A = A_k$, $E_{A,\mathcal{R}[t]}$ contains U_k. If $A = \oplus_{k=1}^{K} A_k$, then $E_{A,\mathcal{R}[t]}$ contains all of the U_k. For an explicit example of an integral domain $\mathcal{R}$ which embeds into $\mathbb{R}$ and has $NK_1(\mathcal{R}) \neq 0$ see [7, Example 3.5].

□

5. The union of the elementary stabilizers

The purpose of this section is to prove Theorem 5.1. Throughout, we will use the following notational conventions

- $\mathcal{R}$ denotes a commutative ring (except in Conjecture 5.20).
- $NSK_1(\mathcal{R})$ denotes the subgroup of $NK_1(\mathcal{R})$ defined by
$$\{[G] \in NK_1(\mathcal{R}) \,|\, \det(G) = 1\}.$$
- S_{RMP} denotes the collection of polynomials in $\mathcal{R}[t]$ whose constant term is 1, i.e. $S_{RMP} = \{1 + a_1 t + \cdots + a_n t^n\}$ (the "reverse monic" polynomials).
- Given a matrix C in $\mathcal{M}(\mathcal{R})[t])$, $C_0, \dots C_n$ are the matrices over $\mathcal{R}$ such that $C = \sum_{i=0}^{n} C_i t^i$. (For $\mathcal{M}(\mathcal{R}[t])$, recall the notation (2.1).)
- $E(A, \mathcal{R}) = \mathrm{ElSt}_{\mathcal{R}[t]}(I - tA)$ (recall Definitions 4.1, 4.2).

Theorem 5.1. *For a commutative ring $\mathcal{R}$, we have*
$$\bigcup_{A \in \mathcal{M}(\mathcal{R})} E(A, \mathcal{R}) = NSK_1(\mathcal{R})$$
If $\mathcal{R}$ is reduced, then $NSK_1(\mathcal{R}) = NK_1(\mathcal{R})$.

Remark 5.2. Version 1 of our arXiv post [9] claimed that for every $\mathcal{R}$ and every A, $E(A, \mathcal{R})$ is trivial. Theorem 5.1 corrects that statement. The error in the proof in [9] is that [9, Corollary 3.20] is not true. The error in the proof of [9, Corollary 3.20] is the claim of existence of the map f_1. Under j, a monic matrix is carried to a reverse monic matrix, which need not be invertible in $\Omega_+^{-1}\mathcal{R}[t]$; so we cannot apply the universal property of the Cohn localization to produce f_1.

Proof of Theorem 5.1. The last statement of Theorem 5.1 recapitulates for reference a well known fact. (A ring is reduced if it contains no nontrivial nilpotent element; for a nilpotent matrix N over a commutative and reduced ring $\mathcal{R}$, the unit $\det(I - tN)$ in $\mathcal{R}[t]$ must equal 1.) So, we only need to prove the claim in Theorem 5.1 for $NSK_1(\mathcal{R})$.

Let $j \colon \mathcal{R}[t] \to S_{RMP}^{-1}\mathcal{R}[t]$ be the localization map, with induced map $j_* \colon K_1(\mathcal{R}[t]) \to K_1(S_{RMP}^{-1}\mathcal{R}[t])$. If $A \in \mathcal{M}(\mathcal{R})$ and $U \in E(A, \mathcal{R})$, then $[U] \in \ker(j_*)$, because $I - tA$ is invertible over $S_{RMP}^{-1}\mathcal{R}[t]$. Therefore $E(A, \mathcal{R}) \subset NK_1(\mathcal{R})$ follows from $\ker(j_*) \subset NSK_1(\mathcal{R})$, which is part of the next proposition.

Proposition 5.3. $NSK_1(\mathcal{R}) = \ker(j_*)$.

Proof. Let $\tau_1 \colon \mathcal{R}[t] \to \mathcal{R}$ and $\tau_2 \colon S_{RMP}^{-1}\mathcal{R}[t] \to \mathcal{R}$ be the maps induced by $t \mapsto 0$. Then $\tau_1 = \tau_2 j$ (because $t \mapsto 0$ sends elements of S_{RMP} to 1), so $j_*(x) = 0$ implies $x \in \ker((\tau_1)_*) = NK_1(\mathcal{R})$. Also, if $j_*(x) = 0$ and U is a matrix such that $x = [U]$, then $\det(U) = 1$. Therefore $\ker(j_*) \subset NSK_1(\mathcal{R})$.

Now suppose $x \in NSK_1(\mathcal{R})$. By Higman's trick [41, III.3.5.1], $x = [I - tN]$ for some nilpotent matrix N over $\mathcal{R}$. The matrix $I - tN$ has the property that all diagonal entries are units in $S_{RMP}^{-1}\mathcal{R}[t]$, and all off-diagonal entries lie in $t\mathcal{R}[t]$. It follows that $I - tN$ is elementary equivalent over $S_{RMP}^{-1}\mathcal{R}[t]$ to a 1×1 matrix. Thus $j_*([I - tN]) = [(\det(I - tN))] = [(1)] = [I]$, and $NSK_1(\mathcal{R}) \subset \ker(j_*)$. □

Before continuing the proof of Theorem 5.1, we pause to note there are rings for which NK_1 is nontrivial, but the elementary stabilizer is always trivial.

Remark 5.4. Suppose $\mathcal{R}$ is a commutative ring for which the embedding $SK_1(\mathcal{R}) \to SK_1(\mathcal{R}[t])$ induced by the inclusion $\mathcal{R} \to \mathcal{R}[t]$ is surjective. (For example, $\mathcal{R} = \mathcal{S}[x]/(x^N)$, with $N > 1$ and $\mathcal{S}$ a commutative regular ring [41, Example III.3.8.1].) Then $NSK_1(\mathcal{R}) = \{0\} \neq NK_1$, and $E(A, \mathcal{R})$ is trivial for every matrix A over $\mathcal{R}$.

To finish the proof of Theorem 5.1, it suffices given $x \in NK_1(\mathcal{R})$ to find A in $\mathcal{M}(\mathcal{R})$ such that $E(A, \mathcal{R})$ contains x. Moreover, by (4.5), its suffices to find A in $\mathcal{M}(\mathcal{R})$ such that $\mathrm{ElSt}_{\mathcal{R}[t]}(A)$ contains x and

$$A_0 \in \mathrm{GL}(\mathcal{R}). \tag{5.5}$$

Recall $\mathcal{H}_1(\mathcal{R}[t])$ denotes the category of finitely generated $\mathcal{R}[t]$-modules of projective dimension ≤ 1. For a multiplicatively closed set of non-zero-divisors $S \subset \mathcal{R}[t]$ we let $\mathcal{H}_1(\mathcal{R}[t], S)$ denote the full subcategory of $\mathcal{H}_1(\mathcal{R}[t])$ whose objects are also S-torsion modules, i.e. modules M such that $S^{-1}M = S^{-1}\mathcal{R}[t] \otimes M = 0$. The categories $\mathcal{H}_1(\mathcal{R}[t])$ and $\mathcal{H}_1(\mathcal{R}[t], S)$ are exact categories. The elements of the multiplicative set S_{RMP} are non-zero-divisors in $\mathcal{R}[t]$.

Let $\mathcal{A}$ be an exact category. By a *double short exact sequence (d.s.e.s.)* we mean a pair of short exact sequences in $\mathcal{A}$ on the same objects

$$0 \to A \xrightarrow{f_1} B \xrightarrow{g_1} C \to 0$$

$$0 \to A \xrightarrow{f_2} B \xrightarrow{g_2} C \to 0$$

which we denote by $A \underset{f_2}{\overset{f_1}{\rightrightarrows}} B \underset{g_2}{\overset{g_1}{\rightrightarrows}} C$. In [27], Nenashev defined the following group.

Definition 5.6. The group $D(\mathcal{A})$ is defined to be the abelian group with generators $\langle \ell \rangle$ for all double short exact sequences ℓ subject to the following relations:

(i) The class of any double short exact sequence of the form $A \underset{f}{\overset{f}{\rightrightarrows}} B \underset{g}{\overset{g}{\rightrightarrows}} C$ is zero. (A d.s.e.s. of this form is called *diagonal*.)

(ii) Suppose we have a diagram consisting of six double short exact sequences of the form

$$\begin{array}{ccccc} A_0 & \rightrightarrows & A_1 & \rightrightarrows & A_2 \\ \downdownarrows & & \downdownarrows & & \downdownarrows \\ B_0 & \rightrightarrows & B_1 & \rightrightarrows & B_2 \\ \downdownarrows & & \downdownarrows & & \downdownarrows \\ C_0 & \rightrightarrows & C_1 & \rightrightarrows & C_2 \end{array}$$

satisfying the following commutativity conditions: all arrows on top commute with all arrows on the left, and all arrows on bottom commute with all arrows on the right. (We will a diagram of this form a *Nenashev diagram.*) Then, letting r_i denote the ith row in the diagram, c_i the ith column in the diagram, we have the relation

$$[c_0] - [c_1] + [c_2] = [r_0] - [r_1] + [r_2]$$

Nenashev proved in [27] the following. Here $K_1(\mathcal{A})$ refers to Quillen's K_1 group.

Theorem 5.7 ([27, Nenashev])**.** *$K_1(\mathcal{A}) \cong D(\mathcal{A})$.*

Nenashev's Theorem (Theorem 5.7) is based on the Gillet-Grayson construction $G.\mathcal{A}$ associated to $\mathcal{A}$, whereby $K_n(\mathcal{A})$ is presented as $\pi_n(G.\mathcal{A})$. To a d.s.e.s. $A \underset{f_2}{\overset{f_1}{\rightrightarrows}} B \underset{g_2}{\overset{g_1}{\rightrightarrows}} C$ one may associate a loop in $\pi_1(G.\mathcal{A}) = K_1(\mathcal{A})$, and this produces a map $m\colon D(\mathcal{A}) \to \pi_1(G.\mathcal{A})$ enacting the isomorphism in Nenashev's Theorem.

The Bass K_1 group of $\mathcal{A}$, denoted $K_1^{\text{det}}(\mathcal{A})$, is the group $K_0(\mathbf{Aut}\mathcal{A})/R$, where R is the subgroup generated by elements of the form $[(M,\alpha)] + [(M,\beta)] - [(M,\alpha\beta)]$. There is a homomorphism $\eta_{\mathcal{A}}\colon K_1^{\text{det}}(\mathcal{A}) \to K_1(\mathcal{A})$, often called the Gersten-Sherman map, defined in [15, Section 5], [32, Section 3]. In fact, η defines a natural transformation between the functors K_1^{det} and K_1. In general, $\eta_{\mathcal{A}}$ is neither surjective [15, Prop. 5.1] nor injective [15, Prop. 5.2]. In the case $\mathcal{A} = \mathbf{Proj}\mathcal{R}$ is the category of finitely generated projective $\mathcal{R}$-modules, the map $\eta_{\mathcal{A}}$ is an isomorphism [32, Section 3].

The map $\eta_{\mathcal{A}}\colon K_1^{\text{det}}(\mathcal{A}) \to K_1(\mathcal{A})$ factors as $K_1^{\text{det}}(\mathcal{A}) \xrightarrow{\gamma} D(\mathcal{A}) \xrightarrow{m} K_1(\mathcal{A})$, where the map γ is determined as follows [27, pg.198]: for an automorphism $\alpha\colon M \to M$ in $\mathcal{A}$ and its corresponding class $[M,\alpha]$ in $K_1^{\text{det}}(\mathcal{A})$, $\gamma\colon [M,\alpha] \mapsto \big\langle 0 \rightrightarrows M \underset{\alpha}{\overset{1}{\rightrightarrows}} M \big\rangle$.

Remark 5.8. Using the relations for $D(\mathcal{A})$, one may show the following (as in [27]):

(1) If $\alpha\colon M \to M, \beta\colon M \to M$ are automorphisms in $\mathcal{A}$, then

$$\left\langle M \underset{\beta\alpha}{\overset{1}{\rightrightarrows}} M \rightrightarrows 0 \right\rangle = \left\langle M \underset{\alpha}{\overset{1}{\rightrightarrows}} M \rightrightarrows 0 \right\rangle + \left\langle M \underset{\beta}{\overset{1}{\rightrightarrows}} M \rightrightarrows 0 \right\rangle$$

(2) Given two d.s.e.s.'s

$$\left\langle M_1 \underset{f_2}{\overset{f_1}{\rightrightarrows}} M_2 \underset{g_2}{\overset{g_1}{\rightrightarrows}} M_3 \right\rangle, \left\langle M_1' \underset{f_2'}{\overset{f_1'}{\rightrightarrows}} M_2' \underset{g_2'}{\overset{g_1'}{\rightrightarrows}} M_3' \right\rangle$$

we have

$$\left\langle M_1 \underset{f_2}{\overset{f_1}{\rightrightarrows}} M_2 \underset{g_2}{\overset{g_1}{\rightrightarrows}} M_3 \right\rangle + \left\langle M_1' \underset{f_2'}{\overset{f_1'}{\rightrightarrows}} M_2' \underset{g_2'}{\overset{g_1'}{\rightrightarrows}} M_3' \right\rangle$$

$$= \left\langle M_1 \oplus M_1' \underset{f_2 \oplus f_2'}{\overset{f_1 \oplus f_1'}{\rightrightarrows}} M_2 \oplus M_2' \underset{g_2 \oplus g_2'}{\overset{g_1 \oplus g_1'}{\rightrightarrows}} M_3 \oplus M_3' \right\rangle$$

Quillen's localization sequence (see [41, V.7.1]) in K-theory for the localization map $j\colon R[t] \to S_{RMP}^{-1}R[t]$ has the form

$$\cdots \to K_1(H_1(\mathcal{R}[t], S_{RMP})) \xrightarrow{i_*} K_1(\mathcal{R}[t]) \xrightarrow{j_*} K_1(S_{RMP}^{-1}\mathcal{R}[t]) \to \cdots \tag{5.9}$$

in which the map i_* factors as

$$K_1(\mathcal{H}_1(\mathcal{R}[t], S)) \xrightarrow{I_{1,*}} K_1(\mathcal{H}_1(\mathcal{R}[t])) \xrightarrow{I_{2,*}^{-1}} K_1(\mathbf{Proj}\mathcal{R}[t])$$

where $I_{1,*}$ is induced by the inclusion functor $I_1\colon \mathcal{H}_1(\mathcal{R}[t], S) \to \mathcal{H}_1(\mathcal{R}[t])$, and $I_{2,*}^{-1}$ is the inverse of the isomorphism $I_{2,*}\colon K_1(\mathbf{Proj}\mathcal{R}[t]) \to K_1(\mathcal{H}_1(\mathcal{R}[t]))$ induced by the inclusion $I_2\colon \mathbf{Proj}\mathcal{R}[t] \to \mathcal{H}_1(\mathcal{R}[t])$ (see the proof of [41, V.7.1]). That the map $I_{2,*}$ is an isomorphism follows from Quillen's Resolution Theorem.

Lemma 5.10. *Given $M \in \mathcal{H}_1(\mathcal{R}[t], S_{RMP})$ there exists n, $M' \in \mathcal{H}_1(\mathcal{R}[t], S_{RMP})$, and a map $h\colon \mathcal{R}[t]^n \to \mathcal{R}[t]^n$ such that $M \oplus M' \cong \operatorname{coker}(h\colon \mathcal{R}[t]^n \to \mathcal{R}[t]^n)$.*

Proof. Let $0 \to P_1 \xrightarrow{f} P_0 \to M \to 0$ be a projective resolution for M. Since f is invertible over $S_{RMP}^{-1}\mathcal{R}[t]$, we may find $s \in S_{RMP}$ such that sf^{-1} is isomorphic to a map $g\colon P_0 \to P_1$. Since $P_1 \oplus P_0$ is projective we may choose Q such that $P_1 \oplus P_0 \oplus Q \cong \mathcal{R}[t]^n$ for some n. Letting $M' = \operatorname{coker}(g)$ we have that

$$0 \to P_1 \oplus P_0 \oplus Q \xrightarrow{f \oplus g \oplus 1} P_0 \oplus P_1 \oplus Q \to M \oplus M' \to 0$$

is exact. $\square$

Notation: The notation $\langle \ell \rangle$ refers to the class of a double short exact sequence ℓ in $D(\mathcal{A})$. From here on, we will abuse notation and also use $\langle \ell \rangle$ to refer to the image of $\langle \ell \rangle$ under Nenashev's isomorphism $m\colon D(\mathcal{A}) \to K_1(\mathcal{A})$.

Now suppose $x \in NSK_1(\mathcal{R})$. We will construct the required matrix A (A is $\alpha_2 \oplus 1$ below) such that $E(A, \mathcal{R})$ contains x. By Proposition 5.3 and the exactness of (5.9), we may fix $y \in K_1(\mathcal{H}_1(\mathcal{R}[t], S_{RMP}))$ such that $i_*(y) = x$, and by Nenashev's Theorem we may represent y in $K_1(\mathcal{H}_1(\mathcal{R}[t], S_{RMP}))$ by a d.s.e.s. in $\mathcal{H}_1(\mathcal{R}[t], S_{RMP})$

$$y = \Big\langle N_1 \underset{k_2}{\overset{k_1}{\rightrightarrows}} N_2 \underset{l_2}{\overset{l_1}{\rightrightarrows}} N_3 \Big\rangle$$

By Lemma 5.10 we may choose N_1', N_3' and endomorphisms $\alpha_1\colon \mathcal{R}[t]^n \to \mathcal{R}[t]^n$, $\alpha_3\colon \mathcal{R}[t]^m \to \mathcal{R}[t]^m$ such that $N_1 \oplus N_1' \cong \operatorname{coker}(\alpha_1)$, $N_3 \oplus N_3' \cong \operatorname{coker}(\alpha_3)$. Let $M_1 = N_1 \oplus N_1'$, $M_2 = N_2 \oplus N_1' \oplus N_3'$, $M_3 = N_3 \oplus N_3'$, and define

$$f_1 = \begin{pmatrix} k_1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} k_2 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad g_1 = \begin{pmatrix} l_1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} l_2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

It follows from part 2 of Remark 5.8 that

$$y = \Big\langle M_1 \underset{f_2}{\overset{f_1}{\rightrightarrows}} M_2 \underset{g_2}{\overset{g_1}{\rightrightarrows}} M_3 \Big\rangle$$

By construction, we have free resolutions for M_1 and M_3

$$0 \to \mathcal{R}[t]^n \xrightarrow{\alpha_1} \mathcal{R}[t]^n \xrightarrow{\pi_1} M_1 \to 0$$
$$0 \to \mathcal{R}[t]^m \xrightarrow{\alpha_3} \mathcal{R}[t]^m \xrightarrow{\pi_3} M_3 \to 0$$

giving the following diagram in $\mathcal{H}_1(\mathcal{R}[t])$

$$\begin{array}{ccccc}
\mathcal{R}[t]^n & & & & \mathcal{R}[t]^m \\
{\scriptstyle \alpha_1}\downarrow\downarrow{\scriptstyle \alpha_1} & & & & {\scriptstyle \alpha_3}\downarrow\downarrow{\scriptstyle \alpha_3} \\
\mathcal{R}[t]^n & & & & \mathcal{R}[t]^m \\
{\scriptstyle \pi_1}\downarrow\downarrow{\scriptstyle \pi_1} & & & & {\scriptstyle \pi_3}\downarrow\downarrow{\scriptstyle \pi_3} \\
M_1 & \underset{f_2}{\overset{f_1}{\rightrightarrows}} & M_2 & \underset{g_2}{\overset{g_1}{\rightrightarrows}} & M_3
\end{array}$$

Using the Horseshoe Lemma ([40, 2.2.8]) we may fill this in to get the Nenashev diagram

$$\begin{array}{ccccc}
\mathcal{R}[t]^n & \rightrightarrows & \mathcal{R}[t]^n \oplus \mathcal{R}[t]^m & \rightrightarrows & \mathcal{R}[t]^m \\
{\scriptstyle \alpha_1}\downarrow\downarrow{\scriptstyle \alpha_1} & & {\scriptstyle \alpha_2}\downarrow\downarrow{\scriptstyle \alpha_2} & & {\scriptstyle \alpha_3}\downarrow\downarrow{\scriptstyle \alpha_3} \\
\mathcal{R}[t]^n & \rightrightarrows & \mathcal{R}[t]^n \oplus \mathcal{R}[t]^m & \rightrightarrows & \mathcal{R}[t]^m \\
{\scriptstyle \pi_1}\downarrow\downarrow{\scriptstyle \pi_1} & & {\scriptstyle \pi_2'}\downarrow\downarrow{\scriptstyle \pi_2} & & {\scriptstyle \pi_3}\downarrow\downarrow{\scriptstyle \pi_3} \\
M_1 & \underset{f_2}{\overset{f_1}{\rightrightarrows}} & M_2 & \underset{g_2}{\overset{g_1}{\rightrightarrows}} & M_3
\end{array}$$

Here $\pi_2' = f_1\pi_1 \oplus \pi_3^{(g_1)}$ and $\pi_2 = f_2\pi_1 \oplus \pi_3^{(g_2)}$, where $\pi_3^{(g_1)} \colon \mathcal{R}[t]^m \to M_2$ is a lift of π_3 along g_1, and $\pi_3^{(g_2)} \colon \mathcal{R}[t]^m \to M_2$ is a lift of π_3 along g_2. The horizontal maps in the first and second row are the canonical inclusions and projections.

It follows from Nenashev's Theorem that in $K_1(\mathcal{H}_1(\mathcal{R}[t]))$ we have

$$\begin{aligned}
I_{1,*}(y) = \big\langle M_1 \underset{f_2}{\overset{f_1}{\rightrightarrows}} M_2 \underset{g_2}{\overset{g_1}{\rightrightarrows}} M_3 \big\rangle &= -\big\langle \mathcal{R}[t]^n \oplus \mathcal{R}[t]^m \underset{\alpha_2}{\overset{\alpha_2}{\rightrightarrows}} \mathcal{R}[t]^n \oplus \mathcal{R}[t]^m \underset{\pi_2}{\overset{\pi_2'}{\rightrightarrows}} M_2 \big\rangle \\
&= \big\langle \mathcal{R}[t]^n \oplus \mathcal{R}[t]^m \underset{\alpha_2}{\overset{\alpha_2}{\rightrightarrows}} \mathcal{R}[t]^n \oplus \mathcal{R}[t]^m \underset{\pi_2'}{\overset{\pi_2}{\rightrightarrows}} M_2 \big\rangle
\end{aligned}$$

A proof for the second equality follows that of [27, Lemma 3.4].

We will use the following result, extracted from Warfield's presentation [37, pp. 1816-1817] of some results from a 1936 paper of Fitting [14] (which were generalized by Warfield). For the convenience of the reader we include a proof. Below and later, we identify M and $M \oplus \{0\}$.

Theorem 5.11. [14, 37] *Let $\mathcal{S}$ be a ring, and suppose $\pi_1 \colon P \to M$, $\pi_2 \colon Q \to M$ are cokernels for injective $\mathcal{S}$-module maps $h_1 \colon P \to P$, $h_2 \colon Q \to Q$ respectively, with P, Q f.g. projective $\mathcal{S}$-modules. Then there exist S-module isomorphisms ϕ_1, ϕ_2 making the following diagram commute*

$$\begin{array}{ccccc}
P \oplus Q & \xrightarrow{h_1 \oplus 1} & P \oplus Q & \xrightarrow{\pi_1 \oplus 0} & M \\
{\scriptstyle \phi_2}\downarrow & & {\scriptstyle \phi_1}\downarrow & & \downarrow{\scriptstyle 1} \\
P \oplus Q & \xrightarrow{1 \oplus h_2} & P \oplus Q & \xrightarrow{0 \oplus \pi_2} & M
\end{array}$$

Moreover, ϕ_1 can be chosen such that $[P \oplus Q, \phi_1] = 0$ in $K_1^{\det}(\mathcal{S})$, and hence $\eta_{\boldsymbol{Proj}\mathcal{S}}([P \oplus Q, \phi_1]) = 0$ in $K_1(\boldsymbol{Proj}\mathcal{A})$. Equivalently, upon choosing a projective module Q' such *that $P \oplus Q \oplus Q'$ is free and a matrix M_{ϕ_1} to represent the map $\phi_1 \oplus 1 \colon P \oplus Q \oplus Q' \to P \oplus Q \oplus Q'$, M_{ϕ_1} is elementary.*

Proof. In the proof, we will use a matrix formalism to denote maps from $P \oplus Q$. For example, (π_1, π_2) denotes the map $P \oplus Q \to M$ which sends a pair (p, q) (viewed as a column vector) to $\pi_1(p) + \pi_2(q)$. Because P is projective and π_2 is surjective, we may choose $\rho \colon P \to Q$ such that $\pi_1 = \pi_2 \rho$; likewise we have $\psi \colon Q \to P$ such that $\pi_2 = \pi_1 \psi$. Then

$$(\pi_1, 0) \begin{pmatrix} 1 & \psi \\ 0 & 1 \end{pmatrix} = (\pi_1, \pi_2) = (0, \pi_2) \begin{pmatrix} 1 & 0 \\ \rho & 1 \end{pmatrix} .$$

Define $\phi_1 \colon P \oplus Q \to P \oplus Q$ to be $\left(\begin{smallmatrix} 1 & 0 \\ \rho & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & -\psi \\ 0 & 1 \end{smallmatrix}\right)$. Because we identify M and $M \oplus \{0\}$, $(\pi_1, 0) = (0, \pi_2)\phi_1$ means $\pi_1 \oplus 0 = (0 \oplus \pi_2)\phi_1$, as required. From the triangular forms, we see ϕ_1 is trivial as an element of $K_1(S)$.

We have

$$\begin{aligned} &\text{image}(h_1 \oplus 1) = \text{image}(h_1) \oplus Q = \ker((\pi_1, 0)) \quad \text{and} \\ &\text{image}(1 \oplus h_2) = P \oplus \text{image}(h_2) = \ker((0, \pi_2)) \ . \end{aligned}$$

Because ϕ_1 takes $\ker((0, \pi_1))$ onto $\ker((\pi_2, 0))$, it follows that ϕ_1 maps $\text{image}(h_1 \oplus 1)$ onto $\text{image}(1 \oplus h_2)$. Because π_1 and π_2 are injective, the maps $h_1 \oplus 1$ and $1 \oplus h_2$ are isomorphisms onto their images. Therefore, for the given isomorphism ϕ_1 there is a unique isomorphism $\phi_2 \colon P \oplus Q \to P \oplus Q$ such that $(h_1 \oplus 1)\phi_1 = \phi_2(1 \oplus h_2)$. □

Resuming the proof of Theorem 5.1, we have

$$x = I_{1,*}(y) = \left\langle \mathcal{R}[t]^n \oplus \mathcal{R}[t]^m \underset{\alpha_2}{\overset{\alpha_2}{\rightrightarrows}} \mathcal{R}[t]^n \oplus \mathcal{R}[t]^m \underset{\pi_2'}{\overset{\pi_2}{\rightrightarrows}} M_2 \right\rangle \tag{5.12}$$

We apply Theorem 5.11 to the top and bottom short exact sequence of (5.12) to get

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{R}[t]^{n+m} \oplus \mathcal{R}[t]^{n+m} & \xrightarrow{\alpha_2 \oplus 1} & \mathcal{R}[t]^{n+m} \oplus \mathcal{R}[t]^{n+m} & \xrightarrow{\pi_2 \oplus 0} & M_2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle 1} & & \\ 0 & \longrightarrow & \mathcal{R}[t]^{n+m} \oplus \mathcal{R}[t]^{n+m} & \xrightarrow{1 \oplus \alpha_2} & \mathcal{R}[t]^{n+m} \oplus \mathcal{R}[t]^{n+m} & \xrightarrow{0 \oplus \pi_2'} & M_2 & \longrightarrow & 0 \end{array} \tag{5.13}$$

with $[\phi_1]$ trivial in $K_1(\mathcal{R}[t])$.

We may regard the morphisms of (5.13) as matrices. Because $\phi_1 \in \text{El}(\mathcal{R}[t])$ and $\alpha_2 \oplus 1$ and $1 \oplus \alpha_2$ are $\text{El}(\mathcal{R}[t])$ equivalent, we have $\phi_2 \in \text{ElSt}_{\mathcal{R}[t]}(\alpha_2 \oplus 1)$. Because $M_2 \cong \text{coker}(\alpha_2)$ is a S_{RMP}-torsion module, there is some q in S_{RMP} such that $q\mathcal{R}[t]^{n+m} \subset \text{image}(\alpha_2 \oplus 1)$; therefore the injective map $\alpha_2 \oplus 1$ defines an automorphism of $S_{RMP}^{-1}\mathcal{R}[t]^{n+m}$, and the image of $\alpha_2 \oplus 1$ under $t \mapsto 0$ lies in $\text{GL}(\mathcal{R})$, as required (recall the condition (5.5)). So, to finish the proof of Theorem 5.1 it suffices to show $[\phi_2] = x$.

Using the pair (5.13) of short exact sequences as column one, we get the following Nenashev diagram in $\mathcal{H}_1(\mathcal{R}[t])$

$$
\begin{array}{ccccc}
\mathcal{R}[t]^{n+m} \oplus \mathcal{R}[t]^{n+m} & \overset{\phi_2}{\underset{1}{\rightrightarrows}} & \mathcal{R}[t]^{n+m} \oplus \mathcal{R}[t]^{n+m} & \rightrightarrows & 0 \\
{\scriptstyle \alpha_2\oplus 1}\downarrow\downarrow{\scriptstyle 1\oplus\alpha_2} & & {\scriptstyle 1\oplus\alpha_2}\downarrow\downarrow{\scriptstyle 1\oplus\alpha_2} & & \downarrow\downarrow \\
\mathcal{R}[t]^{n+m} \oplus \mathcal{R}[t]^{n+m} & \overset{\phi_1}{\underset{1}{\rightrightarrows}} & \mathcal{R}[t]^{n+m} \oplus \mathcal{R}[t]^{n+m} & \rightrightarrows & 0 \\
{\scriptstyle \pi_2\oplus 0}\downarrow\downarrow{\scriptstyle 0\oplus\pi_2'} & & {\scriptstyle 0\oplus\pi_2'}\downarrow\downarrow{\scriptstyle 0\oplus\pi_2'} & & \downarrow\downarrow \\
M_2 & \overset{1}{\underset{1}{\rightrightarrows}} & M_2 & \rightrightarrows & 0
\end{array} \tag{5.14}
$$

Letting c_i, r_i denote the ith column, row, respectively, of diagram (5.14), we have

$$\langle c_2 \rangle = \langle c_3 \rangle = \langle r_3 \rangle = 0$$

Define $\langle l \rangle = \langle c_1 \rangle$. The diagram (5.14), together with Nenashev's relations, implies

$$\langle l \rangle = \langle r_1 \rangle - \langle r_2 \rangle \tag{5.15}$$

We claim that

$$\langle l \rangle = \Big\langle \mathcal{R}[t]^n \oplus \mathcal{R}[t]^m \overset{\alpha_2}{\underset{\alpha_2}{\rightrightarrows}} \mathcal{R}[t]^n \oplus \mathcal{R}[t]^m \overset{\pi_2}{\underset{\pi_2'}{\rightrightarrows}} M_2 \Big\rangle = I_{1,*}(y)$$

To see this, let $E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and consider the Nenashev diagram

$$
\begin{array}{ccccc}
\mathcal{R}[t]^{n+m} \oplus \mathcal{R}[t]^{n+m} & \overset{1}{\underset{E}{\rightrightarrows}} & \mathcal{R}[t]^{n+m} \oplus \mathcal{R}[t]^{n+m} & \rightrightarrows & 0 \\
{\scriptstyle \alpha_2\oplus 1}\downarrow\downarrow{\scriptstyle \alpha_2\oplus 1} & & {\scriptstyle \alpha_2\oplus 1}\downarrow\downarrow{\scriptstyle 1\oplus\alpha_2} & & \downarrow\downarrow \\
\mathcal{R}[t]^{n+m} \oplus \mathcal{R}[t]^{n+m} & \overset{1}{\underset{E}{\rightrightarrows}} & \mathcal{R}[t]^{n+m} \oplus \mathcal{R}[t]^{n+m} & \rightrightarrows & 0 \\
{\scriptstyle \pi_2\oplus 0}\downarrow\downarrow{\scriptstyle \pi_2'\oplus 0} & & {\scriptstyle \pi_2\oplus 0}\downarrow\downarrow{\scriptstyle 0\oplus\pi_2'} & & \downarrow\downarrow \\
M_2 & \overset{1}{\underset{1}{\rightrightarrows}} & M_2 & \rightrightarrows & 0
\end{array} \tag{5.16}
$$

Using Nenashev's relations on (5.16) to justify (5.18), we have

$$\langle l\rangle := \Big\langle \mathcal{R}[t]^{n+m} \oplus \mathcal{R}[t]^{n+m} \underset{1\oplus\alpha_2}{\overset{\alpha_2\oplus 1}{\rightrightarrows}} \mathcal{R}[t]^{n+m} \oplus \mathcal{R}[t]^{n+m} \underset{0\oplus\pi_2'}{\overset{\pi_2\oplus 0}{\rightrightarrows}} M_2\Big\rangle \tag{5.17}$$

$$= \Big\langle \mathcal{R}[t]^{n+m} \oplus \mathcal{R}[t]^{n+m} \underset{\alpha_2\oplus 1}{\overset{\alpha_2\oplus 1}{\rightrightarrows}} \mathcal{R}[t]^{n+m} \oplus \mathcal{R}[t]^{n+m} \underset{\pi_2'\oplus 0}{\overset{\pi_2\oplus 0}{\rightrightarrows}} M_2\Big\rangle \tag{5.18}$$

$$= \Big\langle \mathcal{R}[t]^{n} \oplus \mathcal{R}[t]^{m} \underset{\alpha_2}{\overset{\alpha_2}{\rightrightarrows}} \mathcal{R}[t]^{n} \oplus \mathcal{R}[t]^{m} \underset{\pi_2'}{\overset{\pi_2}{\rightrightarrows}} M_2\Big\rangle$$

$$+ \Big\langle \mathcal{R}[t]^{n+m} \oplus \mathcal{R}[t]^{n+m} \underset{1}{\overset{1}{\rightrightarrows}} \mathcal{R}[t]^{n+m} \oplus \mathcal{R}[t]^{n+m} \underset{0}{\overset{0}{\rightrightarrows}} M_2\Big\rangle\ , \quad \text{by 5.6(ii)},$$

$$= \Big\langle \mathcal{R}[t]^{n} \oplus \mathcal{R}[t]^{m} \underset{\alpha_2}{\overset{\alpha_2}{\rightrightarrows}} \mathcal{R}[t]^{n} \oplus \mathcal{R}[t]^{m} \underset{\pi_2'}{\overset{\pi_2}{\rightrightarrows}} M_2\Big\rangle + 0\ , \quad \text{by 5.8(1)},$$

$$= \Big\langle \mathcal{R}[t]^{n} \oplus \mathcal{R}[t]^{m} \underset{\alpha_2}{\overset{\alpha_2}{\rightrightarrows}} \mathcal{R}[t]^{n} \oplus \mathcal{R}[t]^{m} \underset{\pi_2'}{\overset{\pi_2}{\rightrightarrows}} M_2\Big\rangle\ , \quad \text{by (5.12)}.$$

Therefore

$$I_{1,*}(y) := \ \big\langle \ell \big\rangle = \big\langle r_1\big\rangle - \big\langle r_2\big\rangle\ , \quad \text{by (5.15)},$$

$$= \Big\langle \mathcal{R}[t]^{n+m} \oplus \mathcal{R}[t]^{n+m} \underset{1}{\overset{\phi_2}{\rightrightarrows}} \mathcal{R}[t]^{n+m} \oplus \mathcal{R}[t]^{n+m} \rightrightarrows 0\Big\rangle$$

$$- \Big\langle \mathcal{R}[t]^{n+m} \oplus \mathcal{R}[t]^{n+m} \underset{1}{\overset{\phi_1}{\rightrightarrows}} \mathcal{R}[t]^{n+m} \oplus \mathcal{R}[t]^{n+m} \rightrightarrows 0\Big\rangle$$

$$= \Big\langle \mathcal{R}[t]^{n+m} \oplus \mathcal{R}[t]^{n+m} \underset{1}{\overset{\phi_2\phi_1^{-1}}{\rightrightarrows}} \mathcal{R}[t]^{n+m} \oplus \mathcal{R}[t]^{n+m} \rightrightarrows 0\Big\rangle\ , \quad \text{by 5.8}(i).$$

Applying the map $I_{2,*}^{-1}\colon K_1(\mathcal{H}_1(\mathcal{R}[t])) \to K_1(\mathcal{R}[t])$, we get

$$I_{2,*}^{-1}\colon \Big\langle \mathcal{R}[t]^{n+m} \oplus \mathcal{R}[t]^{n+m} \underset{1}{\overset{\phi_2\phi_1^{-1}}{\rightrightarrows}} \mathcal{R}[t]^{n+m} \oplus \mathcal{R}[t]^{n+m} \rightrightarrows 0\Big\rangle \mapsto [\phi_2\phi_1^{-1}] = [\phi_2] \in K_1(\mathcal{R}[t])$$

where the last equality comes from the fact that ϕ_1 is elementary. Altogether,

$$x = i_*(y) = I_{2,*}^{-1} I_{1,*}(y) = [\phi_2]$$

This finishes the proof of Theorem 5.1. □

Corollary 5.19. *For any finitely generated subgroup $H \subset NSK_1(\mathcal{R})$, there exists a matrix A_H over $\mathcal{R}$ such that $H \subset \mathrm{ElSt}_{\mathcal{R}[t]}(A_H)$.*

Proof. This follows from Theorem 5.1, along with the fact that if $x \in \mathrm{ElSt}_{\mathcal{R}[t]}(A)$ and $y \in \mathrm{ElSt}_{\mathcal{R}[t]}(B)$, then $\{x, y\} \subset \mathrm{ElSt}_{\mathcal{R}[t]}(A \oplus B)$. □

Naturally, one asks what statement would replace Theorem 5.1 if $\mathcal{R}$ is not assumed to be commutative.

Conjecture 5.20. *For a ring $\mathcal{R}$,*

$$\bigcup_{A\in\mathcal{R}} E(A,\mathcal{R}) = \ker\Big(K_1(\mathcal{R}[t]) \xrightarrow{j_*} K_1(\Sigma_{RMP}^{-1}\mathcal{R}[t])\Big)$$

where Σ_{RM} is the set of reverse monic matrices (those of the form $A = I + \sum_{i=1}^{n} A_i t^i$) and j_* is the map on K_1 induced by the localization map j from $\mathcal{R}[t]$ into its Cohn localization with respect to Σ_{RM}.

The conjecture is true when $\mathcal{R}$ is commutative, where the Cohn localization with respect to Σ_{RMP} can be identified with the standard localization.

Finally, we leave the following problem regarding the structure of the elementary stabilizer groups in general.

Elementary Stabilizer Problem 5.21. For a square matrix A over a ring $\mathcal{R}$, find a satisfactory description of the elementary stabilizer $E(A,\mathcal{R})$. In particular, when is $E(A,\mathcal{R})$ trivial?

6. SSE/SE$(A,\mathcal{R})$ = NK$_1(\mathcal{R})/E(A,\mathcal{R})$

In this section we prove one of our main results, Theorem 6.3, assuming the main result of the next section, Theorem 7.2.

To begin we state a matrix version of Theorem 5.11. A slightly different formulation of Theorem 6.1 is given in [10, Lemma 9.1], with further commentary. We say a $k\times k$ matrix A over $\mathcal{R}$ is injective if matrix multiplication $x\mapsto Ax$ defines an injective map $\mathcal{R}^k \to \mathcal{R}^k$.

Theorem 6.1. [14] *Suppose A and B are square injective matrices over a ring $\mathcal{R}$ and the $\mathcal{R}$-modules* $\mathrm{coker}(A)$ *and* $\mathrm{coker}(B)$ *are isomorphic. Then there are identity matrices I_m, I_n; $k\in\mathbb{N}$; U in* $\mathrm{GL}(k,\mathcal{R})$*; and V in* $\mathrm{El}(k,\mathcal{R})$ *such that $U(A\oplus I_m)V = B\oplus I_n$.*

Next, we compile some characterizations of shift equivalence as a theorem. The equivalence of (1), (2) and (3) below is well known. The equivalence of (1) and (4) is what we need for Theorem 6.3. For an $n\times n$ matrix A over a ring $\mathcal{R}$, the $\mathcal{R}[t]$ module $\overline{\mathcal{R}_A}$ is direct limit $\mathcal{R}$-module $\mathcal{R}^n \xrightarrow{A} \mathcal{R}^n \xrightarrow{A} \mathcal{R}^n \xrightarrow{A} \cdots$, with t acting by $[v,i]\mapsto[v,i+1]$ (inverse to $[v,i]\mapsto[Av,i]$).

For $n\in\mathbb{N}$, 0_n and I_n denote the $n\times n$ zero and identity matrices. For a square matrix A over $\mathcal{R}$, $E(A,\mathcal{R})$ denotes $\mathrm{ElSt}_{\mathcal{R}[t]}(I-tA)$, as in (4.2).

Theorem 6.2. *Suppose A and B are square matrices over a ring $\mathcal{R}$. Then the following are equivalent.*

1. *A and B are shift equivalent over $\mathcal{R}$.*
2. *$\overline{\mathcal{R}_A}$ and $\overline{\mathcal{R}_B}$ are isomorphic $\mathcal{R}[t]$ modules.*
3. *$\operatorname{coker}(I-tA)$ and $\operatorname{coker}(I-tB)$ are isomorphic $\mathcal{R}[t]$ modules.*
4. *There are $k,m,n \in \mathbb{N}$ and U,V in $\mathrm{GL}(k,\mathcal{R}[t])$ such that*
   $U\big((I-tA)\oplus I_m\big)V = \big((I-tB)\oplus I_n\big)$ *, i.e.,*
   $U\big((I-t(A\oplus 0_m)\big)V = \big(I-t(B\oplus 0_n)\big)$ *.*

If A and B are shift equivalent over $\mathcal{R}$, then $E(A,\mathcal{R}) = E(B,\mathcal{R})$.

Proof. (1) $\iff$ (2) See [5, p.122]. This connection is due to Krieger; the result for $\mathcal{R} = \mathbb{Z}$ was a piece of his introduction of dimension groups to symbolic dynamics [22]. Another proof for the case $\mathcal{R} = \mathbb{Z}$ can be found in [23, 7.5.6–7.5.7].

(2) $\iff$ (3) The map $[v,i] \mapsto [t^i v]$ defines an $\mathcal{R}[t]$-module isomorphism $\overline{\mathcal{R}_A} \to \operatorname{coker}(I-tA)$. This connection was introduced by Kim, Roush and Wagoner [20], for $\mathcal{R} = \mathbb{Z}$.

(4) $\implies$ (3) Clear.

(3) $\implies$ (4) $I-tA$ and $I-tB$ are injective matrices over $\mathcal{R}[t]$, so (4) follows by Theorem 6.1.

Because (1) implies (4), the final claim of the theorem follows from the final claim of Proposition 4.4. $\square$

We let $\sim$ denote $\mathrm{El}(\mathcal{R}[t])$ equivalence.

Theorem 6.3. *Let $\mathcal{R}$ be a ring, and A a square matrix over $\mathcal{R}$. The following hold.*

1. *If B is shift equivalent over $\mathcal{R}$ to A, then there is a nilpotent matrix N over $\mathcal{R}$ such that B is SSE over $\mathcal{R}$ to the matrix $A \oplus N$.*
2. *For nilpotent matrices N_1, N_2 over $\mathcal{R}$, the matrices $A\oplus N_1$ and $A\oplus N_2$ are SSE over $\mathcal{R}$ iff $I-tN_1$ and $I-tN_2$ are the same element in $NK_1(\mathcal{R})/E(A,\mathcal{R})$.*
3. *If A is shift equivalent over $\mathcal{R}$ to a matrix which is nilpotent, invertible or idempotent, then $E(A,\mathcal{R})$ is the trivial group.*

Proof. For the proof of (1), suppose B is shift equivalent over $\mathcal{R}$ to A. Let k,m,n,U,V be as in (4) of Theorem 6.2. After replacing A with $A\oplus 0_m$ and B with $B\oplus 0_n$ (which is harmless), we have $(I-tB) = U(I-tA)V$.

Because $\begin{pmatrix} VU & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} I-tA & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} V & 0 \\ 0 & V^{-1} \end{pmatrix}\begin{pmatrix} U(I-tA)V & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} V^{-1} & 0 \\ 0 & V \end{pmatrix}$, we have $I-tB \sim W(I-tA)$, where $W = VU$. Setting $t=0$, we see W represents an element of $NK_1(\mathcal{R})$. So, for some j, after replacing W with $W\oplus I_j$ there exists N nilpotent over $\mathcal{R}$ and E and F elementary over $\mathcal{R}[t]$ such that $EWF = I-tN$. After replacing A with $A\oplus 0_j$, we have

$$I - tB \sim W(I-tA) \sim (I-tA) \oplus W \sim \begin{pmatrix} I & 0 \\ 0 & E \end{pmatrix} \begin{pmatrix} I-tA & 0 \\ 0 & W \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & F \end{pmatrix}$$
$$= (I-tA) \oplus (I-tN) = I - t(A \oplus N) \ .$$

Now Theorem 7.2 implies B is strong shift equivalent over $\mathcal{R}$ to $A \oplus N$. This proves (1).

For (2), suppose N_1, N_2 are nilpotent matrices over $\mathcal{R}$. By Theorem 7.2, the matrices $A \oplus N_1$ and $A \oplus N_2$ are SSE over $\mathcal{R}$ iff $(I - t(A \oplus N_1)) \sim (I - t(A \oplus N_2))$. For N nilpotent, $(I - t(A \oplus N) \sim (I - tN)(I - tA)$. Therefore $A \oplus N_1$ and $A \oplus N_2$ are SSE over $\mathcal{R}$ iff $(I - tN_1)(I - tA) \sim (I - tN_2)(I - tA)$. By Proposition 4.4, this holds iff $(I - tN_1)^{-1}(I - tN_2) \in \mathrm{ElSt}(I - tA)$. By Theorem 4.7, this inclusion holds iff $I - tN_1 = I - tN_2$ in $K_1(\mathcal{R}[t])/E(A, \mathcal{R})$ (equivalently, in $NK_1(\mathcal{R})/E(A, \mathcal{R})$). This proves (2).

(3) holds by Theorem 4.7 and the final claim of Theorem 6.2. Note, the nilpotent matrices form the shift equivalence class of the zero matrices. □

Corollary 6.4. *Suppose $NK_1(\mathcal{R})$ is trivial (for example, when $\mathcal{R}$ is a Noetherian regular ring). Then SE-$\mathcal{R}$ implies SSE-$\mathcal{R}$.*

Corollary 6.4 answers in the affirmative a question of Wagoner [34, Sec. 9, Problem Number 3]: does SE-$\mathcal{R}$ implies SSE-$\mathcal{R}$ when $\mathcal{R}$ is a commutative regular ring?

Given $\mathcal{R}$ and a square matrix B over $\mathcal{R}$, let $[B]_{SSE}$ denote the SSE-$\mathcal{R}$ class of B and let $[B]_{SE}$ denote the SE-$\mathcal{R}$ class. For a square matrix A over $\mathcal{R}$, define

$$\text{(6.5)} \qquad \mathrm{SSE/SE}(A, \mathcal{R}) = \{[B]_{SSE} : \ [A]_{SE} = [B]_{SE}\} \ .$$

We can now give a short summary of the correspondence provided by Theorem 6.3.

Theorem 6.6. *Let N range over nilpotent matrices over $\mathcal{R}$. Then for any square matrix A over $\mathcal{R}$, the map $[I - tN] \to [A \oplus N]_{SSE}$ is a well-defined bijection*

$$NK_1(\mathcal{R})/E(A, \mathcal{R}) \to \mathit{SSE/SE}(A, \mathcal{R})$$

Equivalently, the map $[N] \to [A \oplus N]_{SSE}$ is a well defined bijection

$$\mathrm{Nil}_0(\mathcal{R})/E_{\mathrm{Nil}}(A, \mathcal{R}) \to \mathit{SSE/SE}(A, \mathcal{R})$$

with $E_{\mathrm{Nil}}(A, \mathcal{R}) = \{[N] \in \mathrm{Nil}_0(\mathcal{R}) : [I - tN] \in E(A, \mathcal{R})\}$.

Using Theorems 6.2 and 7.2, we record a restatement of Theorem 6.3.

Theorem 6.7. *Let $\mathcal{R}$ be a ring. Then the following hold.*

(1) *If A, B are square matrices over $\mathcal{R}$ such that the $\mathcal{R}[t]$-modules* $\text{coker}(I - tA)$, $\text{coker}(I - tB)$ *are isomorphic, then there is a nilpotent matrix N over $\mathcal{R}$ such that $I - tB \sim I - t(A \oplus N)$.*

(2) *Suppose N_1, N_2 are nilpotent matrices over $\mathcal{R}$. Then*

$$I - t(A \oplus N_1) \sim I - t(A \oplus N_2)$$

iff $[I - tN_1]$ and $[I - tN_2]$ are the same element in $NK_1(\mathcal{R})/E(A, \mathcal{R})$.

7. SSE as elementary equivalence

The purpose of this section is to prove Theorem 7.2, our central result for connecting strong shift equivalence and algebraic K-theory. To prepare for its statement, we give some definitions.

Definition 7.1. Given $A \in t\mathcal{R}[t]$, choose $n \in \mathbb{N}$ and $k \in \mathbb{N}$ such that $A_1, \dots A_k$ are $n \times n$ matrices over $\mathcal{R}$ such that

$$A = \sum_{i=1}^{k} t^i A_i$$

and define a finite matrix $\mathcal{A}^{\square} = \mathcal{A}^{\square(k,n)}$ over $\mathcal{R}$ by the following block form, in which every block is $n \times n$:

$$\mathcal{A}^{\square} = \begin{pmatrix} A_1 & A_2 & A_3 & \dots & A_{k-2} & A_{k-1} & A_k \\ I & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & I & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & I & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & I & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & I & 0 \end{pmatrix}.$$

In the definition, there is some freedom in the choice of $\mathcal{A}^{\square}$: k can be increased by using zero matrices, and n can be increased by filling additional entries of the A_i with zero. These choices do not affect the SSE-$\mathcal{R}$ class of $\mathcal{A}^{\square}$.

With $\sim$ denoting $\text{El}(\mathcal{R}[t])$ equivalence, recall that for finite matrices $I - A$ and $I - B$, $I - A \sim I - B$ by definition means $(I - A)_{\text{st1}} \sim (I - B)_{\text{st1}}$.

Theorem 7.2. *Let $\mathcal{R}$ be a ring. Then there is a bijection between the following sets:*

- *the set of* $\text{El}(\mathcal{R}[t])$ *equivalence classes of square matrices $I - A$ with A over $t\mathcal{R}[t]$*
- *the set of SSE-$\mathcal{R}$ classes of square matrices over $\mathcal{R}$.*

The map to SSE-$\mathcal{R}$ classes is induced by the map $I - A \mapsto \mathcal{A}^\square$. The inverse map (from the set of SSE-$\mathcal{R}$ classes) is induced by the map sending A over $\mathcal{R}$ to the matrix $I - tA$.

Proof. We will first show that when A and B are SSE over $\mathcal{R}$, it follows that the matrices $I-tA$ and $I-tB$ are $\mathrm{El}(\mathcal{R}[t])$ equivalent. It suffices to do this for an elementary strong shift equivalence. Suppose U, V are matrices over $\mathcal{R}$ such that $A = UV$ and $B = VU$. Then (as pointed out by Maller and Shub [24]),

$$\begin{pmatrix} I & 0 \\ V & I \end{pmatrix} \begin{pmatrix} A & U \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & U \\ 0 & B \end{pmatrix} \begin{pmatrix} I & 0 \\ V & I \end{pmatrix}$$

and therefore

$$\begin{pmatrix} I & 0 \\ V & I \end{pmatrix} \begin{pmatrix} I - tA & -tU \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & -tU \\ 0 & I - tB \end{pmatrix} \begin{pmatrix} I & 0 \\ V & I \end{pmatrix} .$$

Also,

$$\begin{pmatrix} I - tA & -tU \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & -tU \\ 0 & I \end{pmatrix} \begin{pmatrix} I - tA & 0 \\ 0 & I \end{pmatrix} \quad \text{and}$$
$$\begin{pmatrix} I & -tU \\ 0 & I - tB \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I - tB \end{pmatrix} \begin{pmatrix} I & -tU \\ 0 & I \end{pmatrix} .$$

Therefore $I - tA$ and $I - tB$ are $\mathrm{El}(\mathcal{R}[t])$ equivalent.

Now suppose that A and B are matrices over $t\mathcal{R}[t]$ such that $I - A$ and $I - B$ are $\mathrm{El}(\mathcal{R}[t])$ equivalent. We will show that $\mathcal{A}^\square$ and $\mathcal{B}^\square$ are SSE over $\mathcal{R}$.

There are basic elementary matrices $E_1, \dots, E_j$ and $F_1, \dots, F_k$, in each of which the single nonzero offdiagonal term has the form rt^ℓ, with $r \in \mathcal{R}$ and $\ell \geq 0$, such that

$$E_j \cdots E_2 E_1 (I - A) = (I - B) F_1 F_2 \cdots F_k .$$

Choose the block size n for $A^\square$ and $B^\square$ large enough that each E_i and F_j equals I outside the principal submatrix on indices $\{1, \dots, n\} \times \{1, \dots, n\}$. Let G_i denote the image of E_i in $\mathrm{El}(\mathcal{R})$ under the map induced by $t \mapsto 0$. Recursively, for $0 < i \leq j$, given A_{i-1} we will define A_i over $\mathcal{R}[t]$ such that $(A_i)^\square$ is SSE over $\mathcal{R}$ to $(A_{i-1})^\square$ and also

$$\begin{aligned} E_j \cdots E_{i+1}(I - A_i) &= (I - B)(F_1 F_2 \cdots F_k)(G_1)^{-1} \cdots (G_i)^{-1} \quad \text{if } i < j \\ (I - A_i) &= (I - B)(F_1 F_2 \cdots F_k)(G_1)^{-1} \cdots (G_i)^{-1} \quad \text{if } i = j . \end{aligned} \tag{7.3}$$

There are two cases.

Case 1: The offdiagonal entry of E_i has the form rt^ℓ with $\ell > 0$. In this case, define A_i by the equation $I - A_i = E_i(I - A_{i-1})$. By Lemma 7.4, $(A_i)^\square$ is SSE over $\mathcal{R}$ to $\mathcal{A}^\square$. Equation (7.3) holds because $G_i = I$.

Case 2: E_i has all entries in $\mathcal{R}$. Then define A_i over $t\mathcal{R}[t]$ by the equation $I - A_i = E_i(I - A_{i-1})(E_i)^{-1}$. Equation (7.3) holds because $G_i = E_i$, so for this case it remains to check the strong shift equivalence. Let E_i also denote the restriction of E_i to the finite principal submatrix on indices $\{1, \dots, n\} \times \{1, \dots, n\}$, define D to be the block diagonal matrix with k diagonal blocks, each equal to E_i. Then $A_i^\square = D^{-1}A_{i-1}^\square D$, and therefore $A_i^\square$ is SSE over $\mathcal{R}$ to A_{i-1}.

Define $G = G_j \cdots G_2G_1 \in \mathrm{El}(\mathcal{R})$. From the preceding we have $\mathcal{A}^\square$ SSE over $\mathcal{R}$ to $(A_j)^\square$, with $I - A_j = (I - B)(F_1F_2 \cdots F_k)G^{-1}$ and therefore

$$(I - A_j)G = (I - B)(F_1F_2 \cdots F_k) \ .$$

Let H_i denote the evaluation of F_i at $t = 0$. Repeating the previous procedure, with the role of left and right interchanged, we find B_k with $(B_k)^\square$ and $B^\square$ SSE over $\mathcal{R}$, and

$$(H_k)^{-1} \cdots (H_2)^{-1}(H_1)^{-1}(I - A_j)G = (I - B_k) \ .$$

Define $H = H_1H_2 \cdots H_k$. Then $H^{-1}(I - A_j)G = I - B_k$. Evaluating at $t = 0$, we see $H = G$. Then $B_k = G^{-1}A_jG$; as in Case 2, $(A_j)^\square$ is SSE over $\mathcal{R}$ to $(B_k)^\square$. This finishes the proof (given Lemma 7.4).

□

Lemma 7.4. *Let $\mathcal{R}$ be a ring. Suppose A and B are matrices over $t\mathcal{R}[t]$; ℓ is a positive integer; E is a basic elementary matrix whose nonzero offdiagonal entry is $E(i_0, j_0) = rt^\ell$, with $r \in \mathcal{R}$; and $E(I - A) = I - B$ or $(I - A)E = I - B$.*

Then the matrices $\mathcal{A}^\square$ and $\mathcal{B}^\square$ are SSE over $\mathcal{R}$.

Proof. Without loss of generality, suppose for notational simplicity that $(i_0, j_0) = (1, 2)$.

We first give a proof assuming that $E(I - A) = I - B$. Let $A = tA_1 + \cdots + t^kA_k$, with the A_i over $\mathcal{R}$, and for later notational convenience set $A_i = 0$ if $i > k$. Since $E(A - I) = B - I$, we have $B = EA - E + I = EA - (E - I)I$. Therefore $B = tB_1 + \cdots + t^{k+l}B_{k+\ell}$, with $B_\ell(1, 2) = A_\ell(1, 2) - r$, and

$$B_{i+\ell}(1, j) = A_{i+\ell}(1, j) + rA_i(2, j) \ , \quad 1 \leq i \leq k \ ,$$

and in all other entries $B = A$.

We first consider the case $\ell = 1$. Let X be the $n \times n$ matrix such that $X(1, 2) = 1$ and other entries of X are zero. Let u_i be the row vector which is the second row of A_i. Let U_i be the $n \times n$ matrix whose first row is u_i and whose other rows are zero.

Then the matrix $\mathcal{B}^\square$, in block form with $n \times n$ blocks, is

$$\mathcal{B}^\square = \begin{pmatrix} A_1 - rX & A_2 + rU_1 & A_3 + rU_2 & \dots & A_{k-1} + rU_{k-2} & A_k + rU_{k-1} & rU_k \\ I & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & I & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & I & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & I & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & I & 0 \end{pmatrix} .$$

We we will perform a string of elementary SSEs over $\mathcal{R}$ which will transform $\mathcal{B}^\square$ into $\mathcal{A}^\square$. We use lines within matrices to emphasize block patterns, especially for blocking compatible with a multiplication.

First we perform the column splitting which splits off columns which isolate all entries with coefficient r. Letting e_1 denote the size n column vector with first entry 1 and other entries zero, we define the $n(k+1) \times n(2k+1)+1$ matrix

$$W = \left(\begin{array}{cccccc|ccc|c} A_1 & A_2 & \dots & A_{k-1} & A_k & 0 & rU_1 & \cdots & rU_k & -re_1 \\ I & 0 & \dots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & I & \dots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & I & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \dots & 0 & I & 0 & 0 & \cdots & 0 & 0 \end{array}\right) .$$

and the $(n(2k+1)+1) \times n(k+1)$ matrix

$$M = \left(\begin{array}{cc} I_n & 0 \\ 0 & I_{nk} \\ \hline 0 & I_{nk} \\ \hline e_2 & 0 \end{array}\right)$$

in which I_j as usual means a $j \times j$ identity matrix and e_2 is the row vector $\left(\begin{smallmatrix} 0 & 1 & 0 & \cdots & 0 \end{smallmatrix}\right)$. Then $\mathcal{B}^{\square} = WM$ and we define $B^{(1)} = MW$, SSE over $\mathcal{R}$ to $\mathcal{B}^{\square}$. In block form,

$$B^{(1)} = \left(\begin{array}{cccccc|ccc|c}
A_1 & A_2 & \dots & A_{k-1} & A_k & 0 & rU_1 & \cdots & rU_k & -re_1 \\
I & 0 & \dots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & I & \dots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & \dots & I & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \dots & 0 & I & 0 & 0 & \cdots & 0 & 0 \\
\hline
I & 0 & \dots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & I & \dots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & \dots & I & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \dots & 0 & I & 0 & 0 & \cdots & 0 & 0 \\
\hline
u_1 & u_2 & \dots & u_{k-1} & u_k & 0 & 0 & \cdots & 0 & 0
\end{array}\right) .$$

Next we perform a diagonal refactorization of $B^{(1)}$. Define the diagonal matrix D by setting

$$\begin{aligned}
D(t,t) &= 1 && \text{if } 1 \le t \le (k+1)n \\
D\big((k+i)n+t,\, (k+i)n+t)\big) &= u_i(t) && \text{if } 1 \le i \le k \ \text{ and } \ 1 \le t \le n \\
&= 1 && \text{if } t = (2k+1)n+1 \ .
\end{aligned}$$

Define a matrix X which is equal to $B^{(1)}$ except that $X(1,t) = r$ if $(k+1)n + 1 \le t \le (2k+1)n$. Then $B^{(1)} = XD$. Define $B^{(2)} = DX$. In block form,

$$B^{(2)} = \left(\begin{array}{cccccc|ccc|c}
A_1 & A_2 & \dots & A_{k-1} & A_k & 0 & R & \cdots & R & -re_1 \\
I & 0 & \dots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & I & \dots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & \dots & I & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \dots & 0 & I & 0 & 0 & \cdots & 0 & 0 \\
\hline
U_1' & 0 & \dots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & U_2' & \dots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & \dots & U_{k-1}' & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \dots & 0 & U_k' & 0 & 0 & \cdots & 0 & 0 \\
\hline
u_1 & u_2 & \dots & u_{k-1} & u_k & 0 & 0 & \cdots & 0 & 0
\end{array}\right)$$

in which every entry of the top row of R is r and the other entries of R are zero, and U_i' denotes the diagonal matrix with $U_i'(t,t) = u_i(t)$, for $1 \le t \le n$.

Next, amalgamate the columns $(k+1)n+1, \dots, (2k+1)n$ (the columns through the R blocks) to a single column to form $B^{(3)}$. For this define

$$Y = \left(\begin{array}{ccccccc|c|c}
A_1 & A_2 & A_3 & \cdots & A_{k-1} & A_k & 0 & re_1 & -re_1 \\
I & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & I & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I & \cdots & 0 & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & I & 0 & 0 & 0 \\
\hline
U_1' & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & U_2' & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & U_3' & \cdots & 0 & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & 0 & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & U_{k-1}' & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & U_k' & 0 & 0 & 0 \\
\hline
u_1 & u_2 & u_3 & \cdots & u_{k-1} & u_k & 0 & 0 & 0
\end{array}\right) \quad \text{and}$$

$$Z = \left(\begin{array}{c|c|c}
I_{(k+1)n} & 0 & 0 \\
\hline
0 & 1 \cdots 1 & 0 \\
\hline
0 & 0 & 1
\end{array}\right)$$

in which the central block of Z is a row vector of size kn with every entry 1. Then $B^{(2)} = YZ$ and we define $B^{(3)} = ZY$. In block form,

$$B^{(3)} = \begin{pmatrix} A_1 & A_2 & A_3 & \cdots & A_{k-1} & A_k & 0 & re_1 & -re_1 \\ I & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & \cdots & 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & I & 0 & 0 & 0 \\ u_1 & u_2 & u_3 & \cdots & u_{k-1} & u_k & 0 & 0 & 0 \\ u_1 & u_2 & u_3 & \cdots & u_{k-1} & u_k & 0 & 0 & 0 \end{pmatrix} .$$

Next we similarly amalgamate the last two rows, to obtain the matrix

$$B^{(4)} = \left(\begin{array}{cccccc|c|c} A_1 & A_2 & A_3 & \cdots & A_{k-1} & A_k & 0 & 0 \\ I & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & I & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & I & \cdots & 0 & 0 & 0 & 0 \\ I & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & I & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & I & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & I & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \cdots & 0 & I & 0 & 0 \\ \hline u_1 & u_2 & u_3 & \cdots & u_{k-1} & u_k & 0 & 0 \end{array}\right) .$$

This matrix is a zero extension of $\mathcal{A}^{\square}$ and therefore is SSE over $\mathcal{R}$ to $\mathcal{A}^{\square}$ (see Proposition 6.5). This finishes the proof in the case $\ell = 1$ that the matrices $\mathcal{A}^{\square}$ and $\mathcal{B}^{\square}$ are SSE over $\mathcal{R}$.

The proof for the case $\ell > 1$ is very similar. We will discuss it for the case $\ell = 3$, from which the general argument should be clear. For $\ell = 3$, with the same notation as in the case $\ell = 1$, and recalling $A_i = 0$ if $i > k$, we have

$$\mathcal{B}^{\square} = \begin{pmatrix} A_1 & A_2 & A_3 - rX & A_4 + rU_1 & \cdots & A_{k+1} + rU_{k-2} & A_{k+2} + rU_{k-1} & A_{k+3} + rU_k \\ I & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & I & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & I & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & I & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & I & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & I & 0 \end{pmatrix} .$$

As in the case $\ell = 1$, we split columns to isolate the terms involving r. The resulting matrix $B^{(1)}$ here has a form involving a shift of the $\ell = 1$ form in the new rows:

$$B^{(1)} = \left(\begin{array}{cccccccc|ccc|c}
A_1 & A_2 & A_3 & A_4 & \cdots & A_{k+1} & A_{k+2} & 0 & rU_1 & \cdots & rU_k & -re_1 \\
I & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & I & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & I & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & I & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & I & 0 & 0 & \cdots & 0 & 0 \\
\hline
0 & 0 & I & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & I & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & I & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & I & 0 & 0 & \cdots & 0 & 0 \\
\hline
0 & 0 & u_1 & u_2 & \cdots & u_{k-1} & u_k & 0 & 0 & \cdots & 0 & 0
\end{array}\right).$$

From here the argument proceeds as in the case $\ell = 1$, through slightly different matrices,

$$B^{(2)} = \left(\begin{array}{cccccccc|ccc|c}
A_1 & A_2 & A_3 & A_4 & \cdots & A_{k+1} & A_{k+2} & 0 & R & \cdots & R & -re_1 \\
I & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & I & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & I & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & I & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & I & 0 & 0 & \cdots & 0 & 0 \\
\hline
0 & 0 & U_1' & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & U_2' & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & U_{k-1}' & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & U_k' & 0 & 0 & \cdots & 0 & 0 \\
\hline
0 & 0 & u_1 & u_2 & \cdots & u_{k-1} & u_k & 0 & 0 & \cdots & 0 & 0
\end{array}\right)$$

and

$$B^{(3)} = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 & \cdots & A_{k+1} & A_{k+2} & 0 & re_1 & -re_1 \\ I & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & I & 0 & 0 & 0 \\ 0 & 0 & u_1 & u_2 & \cdots & u_{k-1} & u_k & 0 & 0 & 0 \\ 0 & 0 & u_1 & u_2 & \cdots & u_{k-1} & u_k & 0 & 0 & 0 \end{pmatrix} .$$

This completes our proof that $\mathcal{A}^{\square}$ and $\mathcal{B}^{\square}$ are SSE over $\mathcal{R}$ in the case $E(I-A) = I-B$.

Now suppose $(I-A)E = I-B$. In place of $\mathcal{A}^{\square}$, we consider a matrix form corresponding to a role reversal for rows and columns:

$$A^{\mathrm{col}} = \begin{pmatrix} A_1 & I & 0 & \cdots & 0 & 0 & 0 \\ A_2 & 0 & I & \cdots & 0 & 0 & 0 \\ A_3 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{k-2} & 0 & 0 & \cdots & 0 & I & 0 \\ A_{k-1} & 0 & 0 & \cdots & 0 & 0 & I \\ A_k & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix} .$$

With the roles of row and column reversed, the arguments we've given show that A^{col} and B^{col} are SSE over $\mathcal{R}$. What remains is to see that A^{col} and $\mathcal{A}^{\square}$ are SSE over $\mathcal{R}$. For this we define a matrix A' with the block form

$$A' = \left(\begin{array}{cc|cc|cc|c|cc|cc} A_1 & A_2 & A_3 & 0 & A_4 & 0 & \cdots & A_{k-1} & 0 & A_k & 0 \\ I & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & I_n & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & I_{2n} & \cdots & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \hline \cdots & \cdots & \multicolumn{2}{c|}{\cdots} & \multicolumn{2}{c|}{\cdots} & \cdots & \multicolumn{2}{c|}{\cdots} & \multicolumn{2}{c}{\cdots} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & I_{(k-3)n} & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & I_{(k-2)n} \\ I & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{array} \right) .$$

In the display of A' above and next, a block I without subscript is I_n.

For example, if $k = 4$ then

$$A' = \left(\begin{array}{cc|cc|ccc} A_1 & A_2 & A_3 & 0 & A_4 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & I & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I \\ I & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) .$$

Three steps remain.

First, the matrix A' is SSE over $\mathcal{R}$ to $\mathcal{A}^{\square}$ by a string of $k-2$ block row amalgamations. Beginning with $A' = A'_0$: amalgamate to block row 2 the block rows with I in block column 1 to form A'_1. From the resulting matrix, amalgamate to block row 3 the rows with I in column 2, to form A'_2. Etc. The last block row amalgamation produces $\mathcal{A}^{\square}$. For example, with A' above for $k = 4$ and

$$X = \begin{pmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \\ 0 & I & 0 & 0 & 0 \end{pmatrix} , \quad Y = \begin{pmatrix} A_1 & A_2 & A_3 & 0 & A_4 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I \end{pmatrix}$$

we have $A' = A'_0 = XY$ and

$$A'_1 = YX = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 & 0 \\ I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I \\ 0 & I & 0 & 0 & 0 \end{pmatrix} .$$

The next step produces $A'_2 = \mathcal{A}^{\square}$.

Second, the matrix A' is conjugate to the matrix A^* obtained from A' by (i) replacing in block row 1 the blocks A_j, $2 \le j \le k$, with the identity block I_n and (ii) replacing the I blocks in block column 1 with $A_2, \dots, A_k$ (with A_j appearing above A_{j+1}, $1 \le i < k$). An SSE from A' to A^* is achieved by a string of diagonal refactorizations of the blocks A_j. For example, in the display for $k = 4$, let X be the matrix obtained from A' by replacing the A_2 block with I. Let D be the block diagonal matrix with block indices matching those of A', and with $D = A_2$ in the second diagonal block and $D = I$

otherwise. Then $XD = A'$ and DX has A_2 occupying the $2,1$ block as desired. To move A_j to its target position in the first block column takes $j-1$ moves of this type.

Third and last, the matrix A^* is SSE over $\mathcal{R}$ to the matrix A^{col} by a string of block column amalgamations, just as A' is SSE over $\mathcal{R}$ to $\mathcal{A}^{\Box}$ by a string of block row amalgamations.

This finishes the proof of the lemma. $\Box$

We record a corollary of Theorem 7.2.

Corollary 7.5. *Suppose $\mathcal{R}$ is a ring, and suppose P and Q are square matrices over $\mathcal{R}[t]$. Suppose A' and B' are matrices over $\mathcal{R}$ such that P and Q are $\mathrm{El}(\mathcal{R}[t])$ equivalent (respectively) to $I - tA'$ and $I - tB'$. Then the following are equivalent:*

1. *A' and B' are SSE over $\mathcal{R}$.*
2. *P and Q are $\mathrm{El}(\mathcal{R})[t]$ equivalent.*

8. SSE and $\mathrm{Nil}_0(\mathcal{R})$

Nilpotent matrices N, N' over $\mathcal{R}$ represent the same element of $\mathrm{Nil}_0(\mathcal{R})$ if and only if $I - tN$ and $I - tN'$ represent the same element of $NK_1(\mathcal{R})$. It therefore follows from Theorem 7.2 that there is another characterization of when nilpotent matrices N, N' represent the same element of $\mathrm{Nil}_0(\mathcal{R})$:

Theorem 8.1. *Suppose N and N' are nilpotent matrices over a ring $\mathcal{R}$. Then the following are equivalent.*

1. *$[N] = [N']$ in $\mathrm{Nil}_0(\mathcal{R})$.*
2. *N and N' are SSE over $\mathcal{R}$.*

(There is also a shorter proof of Theorem 8.1, avoiding Theorem 7.2, which we forego.) Consequently, we can think of Theorem 7.2 as a generalization of the correspondence $\mathrm{Nil}_0(\mathcal{R}) \to NK_1(\mathcal{R})$, from nilpotent matrices to arbitrary matrices.

Remark 8.2. Theorem 7.2 is an alternate ingredient for a proof that $\mathrm{Nil}_0(\mathcal{R})$ and $NK_1(\mathcal{R})$ are isomorphic. If the matrix A in $\mathcal{M}(\mathcal{R})$ is nilpotent, then the map $\beta\colon A \mapsto I - tA$ is the standard map inducing the group isomorphism $\mathrm{Nil}_0(\mathcal{R}) \to \mathrm{NK}_1(\mathcal{R})$. It is straightforward to check that β induces a well defined homomorphism $\mathrm{Nil}_0(\mathcal{R}) \to \mathrm{NK}_1(\mathcal{R})$, which is surjective on account of the Higman trick (see [41] Proposition 3.5.3, or [30] Theorem 3.2.22). The more difficult part of the proof is to show that this epimorphism is injective. For example, Weibel proves this with a sophisticated composition of maps (see [41], Section III.3.5). Rosenberg approaches this by defining a map inducing the inverse, but (he agrees that) the proof [30, p.150] that the map is well defined is incomplete. The map of Theorem 7.2 restricts to define an inverse to the standard epimorphism $\mathrm{Nil}_0(\mathcal{R}) \to \mathrm{NK}_1(\mathcal{R})$, and therefore gives an alternate proof for this step, in the spirit of Rosenberg's approach. It also identifies the elements of $\mathrm{Nil}_0(\mathcal{R})$ as SSE-$\mathcal{R}$ classes.

References

[1] Hyman Bass. *Algebraic K-theory.* W. A. Benjamin, Inc., New York-Amsterdam, 1968.

[2] Louis Block, John Guckenheimer, Michał Misiurewicz, and Lai Sang Young. Periodic points and topological entropy of one-dimensional maps. In *Global theory of dynamical systems (Proc. Internat. Conf., Northwestern Univ., Evanston, Ill., 1979)*, volume 819 of *Lecture Notes in Math.*, pages 18–34. Springer, Berlin, 1980.

[3] M. Boyle and J. B. Wagoner. Positive algebraic K-theory and shifts of finite type. In *Modern dynamical systems and applications*, pages 45–66. Cambridge Univ. Press, Cambridge, 2004.

[4] Mike Boyle. Positive K-theory and symbolic dynamics. In *Dynamics and randomness (Santiago, 2000)*, volume 7 of *Nonlinear Phenom. Complex Systems*, pages 31–52. Kluwer Acad. Publ., Dordrecht, 2002.

[5] Mike Boyle and David Handelman. Algebraic shift equivalence and primitive matrices. *Trans. Amer. Math. Soc.*, 336(1):121–149, 1993.

[6] Mike Boyle, K. H. Kim, and F. W. Roush. Path methods for strong shift equivalence of positive matrices. *Acta Appl. Math.*, 126:65–115, 2013.

[7] Mike Boyle and Scott Schmieding. The generalized spectral conjecture for nonnegative matrices. arXiv:1501.04697; to appear, Linear Algebra and its Applications.

[8] Mike Boyle and Scott Schmieding. Finite group extensions of shifts of finite type: K-theory, Parry and Livšic. arXiv:1503.02050, 2015.

[9] Mike Boyle and Scott Schmieding. Strong shift equivalence and algebraic K-theory. arXiv:1501.04695, 2015.

[10] Mike Boyle and Michael C. Sullivan. Equivariant flow equivalence for shifts of finite type, by matrix equivalence over group rings. *Proc. London Math. Soc. (3)*, 91(1):184–214, 2005.

[11] P. M. Cohn. *Free rings and their relations*, volume 19 of *London Mathematical Society Monographs*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London, second edition, 1985.

[12] Edward G. Effros. *Dimensions and C^*-algebras*, volume 46 of *CBMS Regional Conference Series in Mathematics*. Conference Board of the Mathematical Sciences, Washington, D.C., 1981.

[13] F. T. Farrell. The nonfiniteness of Nil. *Proc. Amer. Math. Soc.*, 65(2):215–216, 1977.

[14] Hans Fitting. Über den Zusammenhang zwischen dem Begriff der Gleichartigkeit zweier Ideale und dem Äquivalenzbegriff der Elementarteilertheorie. *Math. Ann.*, 112(1):572–582, 1936.

[15] S. M. Gersten. Higher K-theory of rings. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf. Seattle Res. Center, Battelle Memorial Inst., 1972)*, pages 3–42. Lecture Notes in Math., Vol. 341. Springer, Berlin, 1973.

[16] Daniel Grayson. Higher algebraic K-theory. II (after Daniel Quillen). In *Algebraic K-theory (Proc. Conf., Northwestern Univ., Evanston, Ill., 1976)*, pages 217–240. Lecture Notes in Math., Vol. 551. Springer, Berlin, 1976.

[17] Daniel R. Grayson. The K-theory of endomorphisms. *J. Algebra*, 48(2):439–446, 1977.

[18] Dennis R. Harmon. NK_1 of finite groups. *Proc. Amer. Math. Soc.*, 100(2):229–232, 1987.

[19] Nigel Higson and John Roe. *Analytic K-homology.* Oxford Mathematical Monographs. Oxford University Press, Oxford, 2000. Oxford Science Publications.

[20] K. H. Kim, F. W. Roush, and J. B. Wagoner. Characterization of inert actions on periodic points. I. *Forum Math.*, 12(5):565–602, 2000.
[21] K. H. Kim, F. W. Roush, and J. B. Wagoner. Characterization of inert actions on periodic points. II. *Forum Math.*, 12(6):671–712, 2000.
[22] Wolfgang Krieger. On dimension functions and topological Markov chains. *Invent. Math.*, 56(3):239–250, 1980.
[23] Douglas Lind and Brian Marcus. *An Introduction to Symbolic Dynamics and Coding*. Cambridge University Press, Cambridge, 1995.
[24] M. Maller and M. Shub. The integral homology of Smale diffeomorphisms. *Topology*, 24(2):153–164, 1985.
[25] Amnon Neeman. Noncommutative localisation in algebraic K-theory. II. *Adv. Math.*, 213(2):785–819, 2007.
[26] Amnon Neeman and Andrew Ranicki. Noncommutative localisation in algebraic K-theory. I. *Geom. Topol.*, 8:1385–1425, 2004.
[27] A. Nenashev. K_1 by generators and relations. *J. Pure Appl. Algebra*, 131(2):195–212, 1998.
[28] M. Pollicott, R. Sharp, S. Tuncel, and P. Walters. The mathematical research of William Parry FRS. *Ergodic Theory Dynam. Systems*, 28(2):321–337, 2008.
[29] Andrew Ranicki. *High-dimensional knot theory*. Springer Monographs in Mathematics. Springer-Verlag, New York, 1998. Algebraic surgery in codimension 2, With an appendix by Elmar Winkelnkemper.
[30] Jonathan Rosenberg. *Algebraic K-theory and its applications*, volume 147 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994.
[31] A. H. Schofield. *Representation of rings over skew fields*, volume 92 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1985.
[32] Clayton Sherman. Group representations and algebraic K-theory. In *Algebraic K-theory, Part I (Oberwolfach, 1980)*, volume 966 of *Lecture Notes in Math.*, pages 208–243. Springer, Berlin, 1982.
[33] Ton Vorst. Localization of the K-theory of polynomial extensions. *Math. Ann.*, 244(1):33–53, 1979. With an appendix by Wilberd van der Kallen.
[34] J. B. Wagoner. Strong shift equivalence theory and the shift equivalence problem. *Bull. Amer. Math. Soc. (N.S.)*, 36(3):271–296, 1999.
[35] J. B. Wagoner. Strong shift equivalence and K_2 of the dual numbers. *J. Reine Angew. Math.*, 521:119–160, 2000. With an appendix by K. H. Kim and F. W. Roush.
[36] J. B. Wagoner. Strong shift equivalence theory. In *Symbolic dynamics and its applications*, volume 60 of *Proc. Sympos. Appl. Math.*, pages 121–154. Amer. Math. Soc., Providence, RI, 2004.
[37] R. B. Warfield, Jr. Stable equivalence of matrices and resolutions. *Comm. Algebra*, 6(17):1811–1828, 1978.
[38] C. A. Weibel. Mayer-Vietoris sequences and module structures on NK_*. In *Algebraic K-theory, Evanston 1980 (Proc. Conf., Northwestern Univ., Evanston, Ill., 1980)*, volume 854 of *Lecture Notes in Math.*, pages 466–493. Springer, Berlin, 1981.
[39] Charles Weibel. NK_0 and NK_1 of the groups C_4 and D_4. Addendum to "Lower algebraic K-theory of hyperbolic 3-simplex reflection groups" by J.-F. Lafont and I. J. Ortiz [mr2495796]. *Comment. Math. Helv.*, 84(2):339–349, 2009.
[40] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.

[41] Charles A. Weibel. *The K-book*, volume 145 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2013. An introduction to algebraic K-theory.

[42] R. F. Williams. Classification of subshifts of finite type. *Ann. of Math. (2)*, 98:120–153, 1973; erratum, ibid. 99:380–381, 1974.

[43] R. F. Williams. Strong shift equivalence of matrices in $\mathrm{GL}(2, \mathbf{Z})$. In *Symbolic dynamics and its applications (New Haven, CT, 1991)*, volume 135 of *Contemp. Math.*, pages 445–451. Amer. Math. Soc., Providence, RI, 1992.