1 Introduction

We have already discussed the concept of periods for holomorphic 1-forms and the analogy between divisors and holomorphic line bundles on a compact complex manifold. Having these objects in mind, we now try to address the following questions concerning compact Riemann surfaces:

- **Question 1.** When a degree zero divisor on a compact Riemann surface is a principal divisor?
- **Question 2.** Let \( X \) be a compact Riemann surface of genus 1 so that the space \( \Omega(X) \) of holomorphic 1-forms is 1-dimensional. As mentioned before, \( \Lambda := \{ \int_\alpha \omega \mid \omega \in \Omega(X), \alpha \in H_1(X, \mathbb{Z}) \} \) is a lattice in \( \mathbb{C} \) and we can associate with \( X \) the complex torus \( \mathbb{C} / \Lambda \). Is there any analogous construction for the higher genera?

**Example 1.** For divisors on the Riemann sphere vanishing of the degree is also a sufficient condition for being principal: a degree zero divisor \( D \) can be written as \( D = \sum_{i=1}^{k} p_i - \sum_{i=1}^{k} q_i \) for points \( p_i, q_i \) of \( \mathbb{CP}^1 \). This is the divisor of the meromorphic function \( f(z) = \prod_{i=1}^{k} (z - p_i) \prod_{i=1}^{k} (z - q_i) \). In higher genera it is very easy to construct examples of degree zero divisors which are not principal: assuming \( X \) is of positive genus, for two distinct points \( p, q \in X \) the degree zero divisor \( p - q \) is not principal. For if \( (f) = p - q \), the meromorphic function \( f \) defines a holomorphic degree one map \( f : X \to \mathbb{CP}^1 \). Such a map is an isomorphism and this contradicts \( g(X) \geq 1 \).

It turns out that these two questions are closely related. *Abel’s theorem* answers Question 1. We first start with the motivation behind it.

2 Motivation

Fix a compact Riemann surface \( X \) of genus \( g \). The long exact sequence in cohomology for the short exact sequence \( 0 \to \mathcal{O}^* \to \mathcal{M}^* \to \mathcal{M}^*/\mathcal{O}^* \to 0 \) of sheaves on \( X \) allows us to identify the quotient of the additive group of divisors \( \text{Div}(X) = H^0(X, \mathcal{M}^*/\mathcal{O}^*) \) to the subgroup \( \text{Div}_P(X) = H^0(X, \mathcal{M}^*) / H^0(X, \mathcal{O}^*) = \mathcal{M}^*(X)/\mathcal{O}^*(X) \) of principal divisors with the Picard group.
Pic\( (X) = H^1 (X, \mathcal{O}^*) \) of isomorphism classes of holomorphic line bundles over \( X \). The smaller group \( \text{Pic}_0 (X) = \text{Div}_0 (X) / \text{Div}_P (X) \) of linear equivalence classes of degree zero divisors is actually the kernel of the first Chern class map \( c_1 : H^1 (X, \mathcal{O}^*) \to H^2 (X, \mathbb{Z}) \) due to the fact that the first Chern class of a line bundle is just the Poincaré dual of a generic section and under the identification \( H^2 (X, \mathbb{Z}) \cong H_0 (X, \mathbb{Z}) \cong \mathbb{Z} \) a divisor, considered as a 0-cycle, is mapped to its degree. On the other hand, looking at the long exact sequence of cohomology groups associated with \( 0 \to \mathbb{Z} \hookrightarrow \mathcal{O} \xrightarrow{\exp (2\pi i)} \mathcal{O}^* \to 0 \), the kernel of \( c_1 : H^1 (X, \mathcal{O}^*) \to H^2 (X, \mathbb{Z}) \) may be identified with \( H^1 (X, \mathcal{O}) / H^1 (X, \mathbb{Z}) \). Up to now we have an isomorphism between \( \text{Div}_0 (X) / \text{Div}_P (X) \) and \( H^1 (X, \mathcal{O}) / H^1 (X, \mathbb{Z}) \). But the Serre duality establishes an isomorphism \( H^1 (X, \mathcal{O}) \cong H^0 (X, \Omega^*) = \Omega (X)^* \) under which the lattice \( H^1 (X, \mathcal{O}) \) in \( H^1 (X, \mathcal{O}) \) corresponds to the lattice \( H_1 (X, \mathbb{Z}) \) in \( \Omega (X)^* \) where each 1-cycle \( \gamma \) is considered as the functional \( \omega \mapsto \int_\gamma \omega \) on the space of global holomorphic 1-forms. In summary:

\[
\text{Div} (X) / \text{Div}_P (X) \supset \text{Div}_0 (X) / \text{Div}_P (X) \cong H^1 (X, \mathcal{O}) / H^1 (X, \mathbb{Z}) \cong \Omega (X)^* / H_1 (X, \mathbb{Z})
\]

Abel’s theorem carefully defines a homomorphism \( \text{Div}_0 (X) \to \Omega (X)^* / H_1 (X, \mathbb{Z}) \) and identifies its kernel with the subgroup of principal divisors. This homomorphism is in fact surjective and this is the content of a theorem due to Jacobi that will be proved in §4. It might be very illuminating to explicitly write down the isomorphism \( \text{Div}_0 (X) / \text{Div}_P (X) \cong \Omega (X)^* / H_1 (X, \mathbb{Z}) \) appeared in (1) by keeping track of various identifications we made and then compare it with the homomorphism (2) in the statement of Abel’s theorem.

It should be mentioned that we get an answer to Question 2 as well: \( \Omega (X)^* / H_1 (X, \mathbb{Z}) \) is the generalization of the construction for elliptic curves we explained before. Picking an ordered basis \( \{ \omega_1, \ldots, \omega_g \} \) for \( \Omega (X) \), \( H_1 (X, \mathbb{Z}) \) embeds in \( \mathbb{C}^g \) as the following subgroup:

\[
\text{Per} (\omega_1, \ldots, \omega_g) := \left\{ \left( \int_\gamma \omega_1, \ldots, \int_\gamma \omega_g \right) \mid \gamma \in H_1 (X, \mathbb{Z}) \right\}
\]

We claim that this is a lattice of full rank in \( \mathbb{C}^g \) or equivalently, fixing a basis \( \{ \gamma_1, \ldots, \gamma_{2g} \} \) for the free abelian group \( H_1 (X, \mathbb{Z}) \), the columns of the \( g \times 2g \) matrix \( \Pi := \left[ \int_{\gamma_j} \omega_i \right]_{1 \leq i \leq g, 1 \leq j \leq 2g} \) are linearly independent over \( \mathbb{R} \). It suffices to show that \( \left[ \begin{array}{c} 1 \\ 1 \end{array} \right] \) is non-singular. If the row vector \( [c_1 \ldots c_{2g}] \) is in its kernel, then the integral of \( \sum_{i=1}^g c_i \omega_i + \sum_{i=g+1}^{2g} c_i \bar{\omega}_i \) over any \( \gamma_j \) and thus over any arbitrary 1-cycle is zero. The Poincaré duality implies that such a closed form is exact. But the closed form \( \sum_{i=1}^g c_i \omega_i + \sum_{i=g+1}^{2g} c_i \bar{\omega}_i \) is not exact unless all \( c_i \)'s vanish. This is due to the fact

\[1\text{On an arbitrary complex manifold we only have an embedding of }\frac{H^0 (X, \mathcal{M}^*) / H^0 (X, \mathcal{O}^*)}{H^0 (X, \mathcal{M}^*) / H^0 (X, \mathcal{O}^*)} \text{ in } H^1 (X, \mathcal{O}^*). \text{ In the projective case this is an isomorphism or equivalently any holomorphic line bundle has a meromorphic section, cf. [Griffiths-Harris, p. 161]. See [Forster, p. 225] for another proof of the fact that vector bundles on compact Riemann surfaces have non-zero meromorphic sections. The statement holds in the non-compact case too since a non-compact Riemann surface is a Stein manifold. In the case of line bundles over Riemann surfaces (not necessarily compact) an alternative approach is to prove that } H^1 (X, \mathcal{M}^*) \text{ vanishes, cf. [Forster, p. 230].} \]

\[2\text{It should be mentioned that holomorphic line bundles over a compact Riemann surface whose first Chern class vanishes are precisely those line bundles that can be equipped with a flat connection or equivalently line bundles with locally constant transition functions. See the fourth exercise here for a much more general statement.} \]
that $H_{dR}^1(X) = \Omega(X) \oplus \overline{\Omega(X)}$. We deduce that the subgroup $\text{Per}(\omega_1, \ldots, \omega_g)$ spanned by the columns of $\Pi$ is a lattice in $\mathbb{C}^g$. Hence $\mathbb{C}^g / \text{Per}(\omega_1, \ldots, \omega_g)$ is a complex torus (and in fact has a much richer structure, see §5). This is called the Jacobian variety of $X$ and is denoted by $\text{Jac}(X)$ and arises as the answer to Question 2 in §1.

3 Abel’s Theorem

Theorem 1. Let $X$ be a compact Riemann surface of genus $g$. Fix a basis $\{\omega_1, \ldots, \omega_g\}$ for $\Omega(X)$. Define the Abel-Jacobi map by

\[
\begin{aligned}
\left\{ u : \text{Div}_0(X) \to \text{Jac}(X) = \mathbb{C}^g / \text{Per}(\omega_1, \ldots, \omega_g) \right\},
\sum_{i=1}^k p_i - \sum_{i=1}^k q_i \mapsto \left( \sum_{i=1}^k f_{p_i} \omega_1, \ldots, \sum_{i=1}^k f_{p_i} \omega_g \right) \mod \text{Per}(\omega_1, \ldots, \omega_g) \\
\end{aligned}
\]

A simpler description of this map is $D \mapsto (\int_c \omega_1, \ldots, \int_c \omega_g)$ where $c$ is a 1-chain with $\partial c = D$. The kernel of this map is precisely the subgroup $\text{Div}_P(X)$ of principal divisors.

Note that in this definition it does not matter which path from $q_i$ to $p_i$ we pick in computing components of $\left( \sum_{i=1}^k f_{p_i} \omega_1, \ldots, \sum_{i=1}^k f_{p_i} \omega_g \right)$ because changing integration path by a 1-cycle changes this vector only by an element of the period lattice $\text{Per}(\omega_1, \ldots, \omega_g)$. In other words a degree 0 divisor $D$, i.e. a 0-cycle, is principal if and only if for any 1-chain $c$ satisfying $\partial c = D$ the vector $(\int_c \omega_1, \ldots, \int_c \omega_g)$ lies in the period lattice $\text{Per}(\omega_1, \ldots, \omega_g)$.

We only sketch a short proof for the if part and the reader can find the whole proof in [Forster, §20]. We have to show that the above map vanishes on the difference of two linearly equivalent divisors. There are maximal subsets of linearly equivalent effective divisors:

Definition 1. Let $D$ be an effective divisor on the compact Riemann surface $X$ and consider the space of global sections of the corresponding sheaf:

$$\mathcal{L}(D) = H^0(X, \mathcal{O}_D) = \{ f \in \mathcal{M}(X) \mid (f) \geq -D \} \neq 0$$

A linear system determined by a positive dimensional subspace $V$ of $\mathcal{L}(D)$ is defined as the subset $\{(f) + D \mid f \in V\}$ of effective divisors. When $V = \mathcal{L}(D)$ we have the complete linear system $|D|$ determined by $D$ which consists of all effective divisors linearly equivalent to $D$. These objects have natural structures of projective spaces: two meromorphic functions $f, g \neq 0$ have the same divisors iff $f = \lambda g$ for a $\lambda \in \mathbb{C}^*$. This indicates that the complete linear system $|D|$ can be identified with the projectivization of the vector space $\mathcal{L}(D)$ of global sections and other systems determined by subspaces of $\mathcal{L}(D)$ are just linear-projective subspaces.

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3This is the Hodge decomposition for compact Riemann surfaces. In fact, it is easy to prove existence of an embedding $\Omega(X) \oplus \overline{\Omega(X)} \hookrightarrow H_{dR}^1(X)$ that takes $(\omega, \eta)$ to the deRham cohomology class of the (complex valued) closed form $\omega + \eta$. Note that $i f_X(\omega + \eta) \wedge \omega = i f_X(\omega + \eta) \wedge \omega \geq 0$ and $i f_X(\omega + \eta) \wedge \eta = i f_X(\eta) \wedge \eta \leq 0$ with equality iff $\omega$ and $\eta$ are zero. This is due to the fact that for a holomorphic 1-form $\alpha$, $i \alpha \wedge \bar{\alpha}$ is a non-negative multiple of any volume form compatible with the orientation induced from the complex structure. But by Stokes’ theorem these integrals vanish when $\omega + \eta$ is exact. Hence this map is injective and now comparing dimensions establishes the decomposition $H_{dR}^1(X) = \Omega(X) \oplus \overline{\Omega(X)}$. 

3
It is easy to modify the definition of $u$ in order to obtain a map $u': \text{Div}(X) \to \mathbb{C}^g/\text{Per}(\omega_1, \ldots, \omega_g)$. Pick a base point $q \in X$ and set:

$$
\begin{align*}
\left\{ \begin{array}{l}
u': \text{Div}(X) \to \text{Jac}(X) = \mathbb{C}^g/\text{Per}(\omega_1, \ldots, \omega_g) \\
\sum_{i=1}^k n_ip_i \mapsto \left(\sum_{i=1}^k n_ip_i, \sum_{i=1}^k n_i \int_q^{p_i} \omega_1, \ldots, \sum_{i=1}^k n_i \int_q^{p_i} \omega_g\right) \mod \text{Per}(\omega_1, \ldots, \omega_g)
\end{array} \right.
\end{align*}
$$

(3)

Again this map is well-defined, $u'(D) = u(D - \text{deg}(D), q)$ and for two divisors $D_1, D_2$ of the same degree $u'(D_1) - u'(D_2) = u(D_1 - D_2)$. It suffices to show that $u'$ is constant on any complete linear system. If we restrict $u'$ to effective divisors of degree $m \in \mathbb{N}$, in fact we get a holomorphic map from $X^m$ (or the $m$-fold symmetric power $S^m(X)$) to the complex torus. To see this, just note that around any $m$-tuple $\tilde{p} = (\tilde{p}_1, \ldots, \tilde{p}_m) \in X^m$ there is an open neighborhood $U_1 \times \cdots \times U_m$ with $U_i$ a simply connected neighborhood of $\tilde{p}_i$ in $X$ and the map $u'|_{X^m} : X^m \to \text{Jac}(X)$ locally lifts to

$$
\begin{align*}
\left\{ \begin{array}{l}
U_1 \times \cdots \times U_m \to \mathbb{C}^g \\
(p_1, \ldots, p_m) \mapsto \left(\sum_{i=1}^m \int_{\tilde{p}_i}^{p_i} \omega_1, \ldots, \sum_{i=1}^m \int_{\tilde{p}_i}^{p_i} \omega_g\right) + \left(\sum_{i=1}^m \int_{c_i} \omega_1, \ldots, \sum_{i=1}^m \int_{c_i} \omega_g\right)
\end{array} \right.
\end{align*}
$$

(4)

where the integration path from $\tilde{p}_i$ to $p_i$ in the simply connected domain $U_i$ can be chosen arbitrarily and $c_i$ is an arbitrary path from $q$ to $\tilde{p}_i$. Obviously the above map is holomorphic and hence the same is true for $u'|_{X^m} : X^m \to \text{Jac}(X)$. Now the argument is very simple: any holomorphic map from a complex projective space (more generally from any simply connected compact complex manifold) to a complex torus must be constant since it lifts to a holomorphic map to the universal cover of torus which is a complex vector space and such a map must be constant as its components are global holomorphic functions. Thus any $u'|_{\text{Div}}$ is constant.

**Example 2.** Let $\Lambda$ be a lattice in $\mathbb{C}$ and consider the genus 1 Riemann surface $X = \mathbb{C}/\Lambda$. It is equipped with a very natural group structure, the induced from that of $\mathbb{C}$ (that is, $\mathbb{C}/\Lambda$ is an elliptic curve over $\mathbb{C}$). Hence a divisor $D$ on $X$ can be evaluated as a sum of elements of this group. We claim that a degree zero divisor $D = \sum_{i=1}^k p_i - \sum_{i=1}^k q_i$ on $X$ is principal iff we have the identity $\sum_{i=1}^k p_i = \sum_{i=1}^k q_i$ in the group $\mathbb{C}/\Lambda$, i.e. $D$ is zero when is computed as a sum of elements of the group. The holomorphic 1-form $dz$ is invariant under translations and therefore induces a nowhere vanishing 1-form $\omega$ on $X$. Lifting $p_1, \ldots, p_k, q_1, \ldots, q_k$ to points $\tilde{p}_1, \ldots, \tilde{p}_k, \tilde{q}_1, \ldots, \tilde{q}_k$ of $\mathbb{C}$ and denoting the image of the line segment from $\tilde{q}_i$ to $\tilde{p}_i$ under the quotient map $\mathbb{C} \to \mathbb{C}/\Lambda$ by the path $c_i$ which starts from $\tilde{q}_i$ and ends at $\tilde{p}_i$, we have:

$$
\sum_{i=1}^k \int_{c_i} \omega = k \sum_{i=1}^k \int_{\tilde{q}_i}^{\tilde{p}_i} dz = \sum_{i=1}^k (\tilde{p}_i - \tilde{q}_i)
$$

According to Abel’s theorem, $D$ is principal iff the above sum belongs to the period lattice $\Lambda$ or equivalently $\sum_{i=1}^k p_i - \sum_{i=1}^k q_i = 0 + \Lambda \in \mathbb{C}/\Lambda$.

**Exercise 1.** The previous example indicates that given a lattice $\Lambda$ in $\mathbb{C}$ and two disjoint sets of complex numbers $\{a_1, \ldots, a_k\}$ and $\{b_1, \ldots, b_k\}$ ($a_i$’s are not necessarily distinct and the same is
true for \( b_i \)'s), a necessary and sufficient condition for existence of a meromorphic function on \( \mathbb{C} \) doubly periodic with respect to \( \Lambda \) whose zeros and poles (written with appropriate multiplicities) are \( a_1, \ldots, a_k \) and \( b_1, \ldots, b_k \) respectively is \( a_1 + \cdots + a_k \equiv b_1 + \cdots + b_k \mod \Lambda \). Give an elementary proof of the necessity part by only using basic tools from the theory of functions of one complex variable\(^4\).

4 Jacobi Inversion

Abel’s theorem provides us with an injective homomorphism \( j : \text{Pic}_0(X) = \text{Div}_0(X)/\text{Div}_F(X) \to \text{Jac}(X) \) which is induced by the map \( u \) in (2). The Jacobi inversion problem states that this is in fact an isomorphism. We will prove this in several steps:

**Step 1.** Since \( u \) is a homomorphism, it suffices to show its image contains an open subset.

**Step 2.** Fixing arbitrary points \( a_1, \ldots, a_g \in X \), one can only concentrate on the following map defined on \( X^g \) whose formula is similar to that of (3):

\[
\begin{aligned}
F : X^g &\to \text{Jac}(X) = \mathbb{C}^g / \text{Per} (\omega_1, \ldots, \omega_g) \\
(p_1, \ldots, p_g) &\mapsto \left( \sum_{i=1}^g \int_{a_i}^{p_i} \omega_1, \ldots, \sum_{i=1}^g \int_{a_i}^{p_i} \omega_g \right) \mod \text{Per} (\omega_1, \ldots, \omega_g)
\end{aligned}
\]  

(5)

We just need to show that \( F \)'s image contains an open subset of \( \text{Jac}(X) \). As explained in §3 this map is holomorphic and so one can invoke the inverse function theorem. Therefore it just suffices to show that the Jacobian of \( F \) is non-zero at a point. Just like (4) we can construct a lifting of \( F \) to the universal cover \( \mathbb{C}^g \) around any given point \( \tilde{p} = (\tilde{p}_1, \ldots, \tilde{p}_g) \):

\[
U_1 \times \cdots \times U_g \rightarrow \mathbb{C}^g : (p_1, \ldots, p_g) \mapsto \left( \sum_{i=1}^g \int_{\tilde{p}_i}^{p_i} \omega_1, \ldots, \sum_{i=1}^g \int_{\tilde{p}_i}^{p_i} \omega_g \right) + \text{const}
\]

(6)

where \( U_j \) is a simply connected open neighborhood of \( \tilde{p}_j \) equipped with holomorphic local coordinate \( z_j \). Suppose in the chart \((U_j, z_j)\) the 1-form \( \omega_i \) is given by \( f_i dz_j \). Equipping \( U_1 \times \cdots \times U_g \) with the coordinates system \( (z_1, \ldots, z_g) \) the Jacobian of (6) at \( \tilde{p} \) is the matrix \( [f_i(\tilde{p}_j)]_{1 \leq i, j \leq g} \). This is invertible iff there is not any non-zero global holomorphic 1-form which vanishes at \( \tilde{p}_1, \ldots, \tilde{p}_g \) because a holomorphic 1-form \( \sum_{i=1}^g c_i \omega_i \) vanishes at \( \tilde{p}_1, \ldots, \tilde{p}_g \) iff the row vector \([c_1, \ldots, c_g]\) lies in the kernel of this matrix. Such a situation is generic: for a generic effective degree \( g \) divisor \( D \) one has \( \{ \omega \in \Omega(X) \mid (\omega) \geq D \} = 0 \). This is due to the fact that intuitively an equation \( \omega(a) = 0 \) for a generic \( a \in X \) cuts off a codimension one subspace from the \( g \)-dimensional space \( \Omega(X) \). Note that by the Serre duality the dimension of \( \{ \omega \in \Omega(X) \mid (\omega) \geq D \} \) is the same as that of \( H^1(X, \mathcal{O}_D) \) and we arrive at the following definition:

**Definition 2.** A divisor \( D \) on the compact Riemann surface \( X \) is called **special** if both \( H^0(X, \mathcal{O}_D) \) and \( H^1(X, \mathcal{O}_D) \) are non-zero. Equivalently, denoting a canonical divisor by \( K \), \( D \) is called special

\(^4\)There is an elementary proof for the sufficiency part too. Having \( a_1, \ldots, a_k, b_1, \ldots, b_k \) satisfying the congruence property modulo the lattice, one has to construct some appropriate doubly periodic function and this can be done with the help of the **theta function** associated with the lattice \( \Lambda \).
when $|D|, |K - D| \neq 0^5$.

Hence $F : X^g \to \text{Jac}(X)$ is full rank over some Zariski open subset and this yields the \textit{Jacobi inversion theorem}:

\textbf{Theorem 2.} \textit{For any compact Riemann surface $X$ the map $j : \text{Pic}_0(X) \to \text{Jac}(X)$ is a group isomorphism.}

We want to involve a third step to investigate the map $F$ from (5) more carefully.

\textbf{Step 3.} For any degree zero divisor $D$ and any degree $g$ divisor $D'$, there is an effective degree $g$ divisor $D''$ with $D'' \sim D + D'$.

\textbf{Exercise 2.} Prove this claim with help of the Riemann-Roch theorem.

The preceding theorem implies that any point of the Jacobian is in the form of $u(D)$ for some suitable $D \in \text{Div}_0(X)$. The exercise indicates that there is an effective degree $g$ divisor $D''$ with $D'' - \sum_{i=1}^g a_i \sim D$. So $u(D)$ is the same as $u(D'' - \sum_{i=1}^g a_i)$ while this coincides with $F(D'')$ by definition. We summarize what has been proved about $F$:

\textbf{Corollary 1.} Fix a basis $\{\omega_1, \ldots, \omega_g\}$ for $\Omega(X)$ and base points $a_1, \ldots, a_g \in X$, and identify the set of effective degree $g$ divisors with the $g$-fold symmetric power $S^g(X)$. Then the map

\[ A : S^g(X) \to \text{Jac}(X) = \mathbb{C}^g / \text{Per} (\omega_1, \ldots, \omega_g) \]

\[ \sum_{i=1}^g p_i \mapsto \left( \sum_{i=1}^g f_{a_i}^{p_i} \omega_1, \ldots, \sum_{i=1}^g f_{a_i}^{p_i} \omega_g \right) \mod \text{Per} (\omega_1, \ldots, \omega_g) \]

is surjective and generically injective and fails to be locally biholomorphic only at special divisors.

Note that any fiber is a complete linear system. In fact, it can be proved that the rank of $A$ at the divisor $D \in S^g(X)$ is $g - \dim |D|$ and denoting the closed subvariety of special divisors by $Y$, $A$ restricts to a biholomorphic map $S^g(X) - Y \to \text{Jac}(X) - A(Y)$, cf. [Narasimhan, §15].

\textbf{Exercise 3.} Let $X$ be a compact Riemann surface of genus 2. Using the fact that a canonical divisor $K$ must be of degree 2, verify that $|K|$ is the unique special complete linear system of degree 2 and is 1-dimensional. Conclude that the map $A : S^2(X) \to \text{Jac}(X)$ is the blow-up of the 2-dimensional complex torus $\text{Jac}(X)$ at a point.

The corollary below summarizes both Abel’s and Jacobi’s theorems.

\textbf{Corollary 2.} \textit{For any compact Riemann surface $X$ there is an exact sequence of abelian groups:}

\[ 0 \to \mathbb{C}^* \to \mathcal{M}^*(X) \xrightarrow{f^{(\gamma)}} \text{Div}_0(X) \xrightarrow{\nu} \text{Jac}(X) \to 0 \]

\footnote{The dimensions of $\mathcal{L}(D) = H^0(X, \mathcal{O}_D)$ and $H^1(X, \mathcal{O}_D)$ are denoted by $l(D)$ and $i(D)$ respectively. The latter is called the \textit{index of speciality} and by the Serre duality is the same as $l(K - D)$.}
Finally note that any curve of positive genus can be embedded in its Jacobian. In fact, fixing a \( q \in X \) the degree zero divisor \( p - q \) is principal only when \( p = q \). Hence embedding \( X \) in \( \text{Div}_0(X) \) via \( p \in X \mapsto p - q \in \text{Div}_0(X) \) and then applying \( u \) yields an injective holomorphic map \( X \to \text{Jac}(X) : p \mapsto \left( \int_0^p \omega_1, \ldots, \int_0^p \omega_q \right) \mod \langle \omega_1, \ldots, \omega_k \rangle \). When \( g = 1 \) both sides are Riemann surfaces and this map must be an isomorphism. We are again back to the construction we started with in §1:

**Corollary 3.** Any Riemann surface of genus 1 is in the form of \( \mathbb{C}/\Lambda \) for a suitable lattice \( \Lambda \) in \( \mathbb{C} \).

5 Further Results

5.1 Riemann Bilinear Relations and the Normalized Period Matrix

The period matrix \( \Pi := \left[ \int_{\gamma_i}^{} \omega_j \right]_{1 \leq i \leq g, 1 \leq j \leq 2g} \) of \( X \) with respect to bases \( \{ \omega_1, \ldots, \omega_g \} \) for \( \Omega(X) \) and \( \{ \gamma_1, \ldots, \gamma_{2g} \} \) for \( H_1(X, \mathbb{Z}) \) was introduced in §2 and can be studied more carefully.

**Theorem 3.** There is a \( 2g \times 2g \) skew-symmetric integral matrix \( Q \) such that

\[ \Pi Q^{-1} \Pi^T = 0 \quad \text{and} \quad -i\Pi Q^{-1} \Pi^T > 0 \]

In fact, \( Q \) can be taken to be the intersection matrix \( [\gamma_i, \gamma_j] \) for \( 1 \leq i, j \leq 2g \) of the selected homology basis.

We are going to present a proof of this theorem based on the structure of the singular cohomology ring \( H^*_{\text{sing}}(X, \mathbb{Z}) \) of the compact oriented genus \( g \) surface \( X \). See [Griffiths-Harris, chapter 2, §2] or [Narasimhan, §14] for more geometric treatments. It is a standard fact that there is an ordered basis \( \{ \alpha^*_1, \ldots, \alpha^*_g, \beta^*_1, \ldots, \beta^*_g \} \) for \( H^1_{\text{sing}}(X, \mathbb{Z}) \) with the property that \( \alpha^*_i \sim \alpha^*_j = \beta^*_i \sim \beta^*_j = 0 \) for any \( i, j \) while \( \alpha^*_i \sim \beta^*_j = \delta_{ij} \). Here \( \{ \alpha_i \} \) is the generator of \( H^2_{\text{sing}}(X, \mathbb{Z}) \) determined by the orientation, i.e. the fundamental class. The dual basis \( \{ \alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g \} \) for \( H_1(X, \mathbb{Z}) \) by the Poincaré duality satisfies \( \forall i, j : \alpha_i, \alpha_j = \beta_i, \beta_j = 0, \alpha_i, \beta_j = \delta_{ij} \). Consider the complex deRham cohomology group \( H^1_{dR}(X) \) as the complexification of \( H^1_{\text{sing}}(X, \mathbb{Z}) \). Thus one can write a deRham class \( [\omega_i] \) as a linear combination of integral classes \( \alpha^*_k, \beta^*_k \) with coefficients which are integrals of the form over the dual basis for the first homology: \( \left[ \omega_i \right] = \sum_{k=1}^g (\int_{\alpha_k} \omega_i) \alpha^*_k + \sum_{k=1}^g (\int_{\beta_k} \omega_i) \beta^*_k \).

By conjugating we get an analogous description for the classes of anti-holomorphic forms: \( \left[ \bar{\omega}_i \right] = \sum_{k=1}^g (\int_{\alpha_k} \bar{\omega}_i) \alpha^*_k + \sum_{k=1}^g (\int_{\beta_k} \bar{\omega}_i) \beta^*_k \). Next we can form the cup product of these classes:

\( [\omega_i] \cup [\omega_j] = \left( \int_X \omega_i \wedge \omega_j \right) \cdot [X] = 0 \Rightarrow \)

\( \left( \sum_{k=1}^g (\int_{\alpha_k} \omega_i) \alpha^*_k + \sum_{k=1}^g (\int_{\beta_k} \omega_i) \beta^*_k \right) \cup \left( \sum_{k=1}^g (\int_{\alpha_k} \omega_j) \alpha^*_k + \sum_{k=1}^g (\int_{\beta_k} \omega_j) \beta^*_k \right) = 0 \Rightarrow \)

\( \sum_{k=1}^g \left( \int_{\alpha_k} \omega_i, \int_{\beta_k} \omega_j - \int_{\bar{\alpha}_k} \omega_j, \int_{\bar{\alpha}_k} \omega_i \right) \cdot [X] = 0 \)
Therefore, $\sum_{i=1}^{g} \int_{\alpha_i} \omega_i \cdot \int_{\beta_i} \omega_j = \sum_{i=1}^{g} \int_{\beta_i} \omega_i \cdot \int_{\alpha_i} \omega_j$ for any $i, j$. For any $(c_1, \ldots, c_g) \in \mathbb{C}^g - \{0\}$ the holomorphic 1-form $\omega := \sum_{i=1}^{g} c_i \omega_i$ is non-zero and hence $\int_X \omega \wedge \bar{\omega} > 0$. One the other hand $[\omega] \sim [\bar{\omega}] = \left( \int_X \omega \wedge \bar{\omega} \right) [X]$ where $[\omega] \sim [\bar{\omega}]$ can be written as:

$$[\omega] \sim [\bar{\omega}] = \left( \sum_{i=1}^{g} c_i [\omega_i] \right) - \left( \sum_{j=1}^{g} \bar{c}_j [\bar{\omega}_j] \right) =$$

$$\left( \sum_{i=1}^{g} c_i \left( \sum_{k=1}^{g} \int_{\alpha_k} \omega_i \cdot \alpha_k^* + \sum_{k=1}^{g} \int_{\beta_k} \omega_i \cdot \beta_k^* \right) \right) - \left( \sum_{j=1}^{g} \bar{c}_j \left( \sum_{k=1}^{g} \int_{\alpha_k} \omega_j \cdot \bar{\alpha}_k^* + \sum_{k=1}^{g} \int_{\beta_k} \omega_j \cdot \bar{\beta}_k^* \right) \right) =$$

$$\sum_{i=1}^{g} \sum_{j=1}^{g} \sum_{k=1}^{g} c_i \bar{c}_j \left( \int_{\alpha_k} \omega_i \cdot \int_{\beta_k} \omega_j - \int_{\beta_k} \omega_i \cdot \int_{\alpha_k} \omega_j \right) [X]$$

Hence the coefficient $\sum_{i=1}^{g} \sum_{j=1}^{g} \sum_{k=1}^{g} c_i \bar{c}_j \left( \int_{\alpha_k} \omega_i \cdot \int_{\beta_k} \omega_j - \int_{\beta_k} \omega_i \cdot \int_{\alpha_k} \omega_j \right)$ appeared above is a positive multiple of $-1$ provided that at least one of $c_i$’s is non-zero. This indicates that the matrix $\left[ f_{ij} \right]_{1 \leq i, j \leq g}$ is non-singular since otherwise there exists a non-zero vector $(c_1, \ldots, c_g)$ with the property that $\sum_{i=1}^{g} c_i \int_{\alpha_i} \omega_i = 0$ for any $k$ which contradicts what just mentioned. Consequently, by changing the chosen basis $\{\omega_1, \ldots, \omega_g\}$ for $\Omega(X)$, WLOG we may assume $\int_{\alpha_i} \omega_i = \delta_{ij}$ for any $i, j$. Now the identity $\sum_{k=1}^{g} \int_{\alpha_k} \omega_i \cdot \int_{\beta_k} \omega_j = \sum_{k=1}^{g} \int_{\beta_k} \omega_i \cdot \int_{\alpha_k} \omega_j$ simplifies to $\int_{\beta_i} \omega_j = \int_{\beta_j} \omega_i$ and the positivity of $\int_{\beta_i} \omega_j - \int_{\beta_j} \omega_i$. Integrating $\omega_i$ for $\Omega(X)$, we obtain $\int_{\beta_i} \omega_j - \int_{\beta_j} \omega_i > 0$. We conclude that the matrix $\left[ f_{ij} \right]_{1 \leq i, j \leq g}$ is symmetric and its imaginary part is positive definite. Hence picking a symplectic basis for the first homology, there is a basis for the space of holomorphic 1-forms in which the period matrix of the genus $g$ Riemann surface $X$ takes the form of $\left[ f_{ij} \right] \cdot Z$ with $Z$ a $g \times g$ symmetric matrix satisfying $\text{Im}Z > 0$. This is called a normalized period matrix. Note that the conditions of the theorem are satisfied once we put $Q = \begin{bmatrix} 0 & I_g \\ -I_g & 0 \end{bmatrix}$.

The existence of an integral skew-symmetric matrix $Q$ with the properties outlined in the theorem is a necessary and sufficient condition for a $g$-dimensional complex torus $V/\Lambda$ with the period matrix $\Pi_{g \times 2g}$ (which corresponds to certain ordered bases of $V^*$ and $\Lambda$) to be an abelian variety, i.e. to admits an embedding in some projective space, check [Griffiths-Harris, p. 303] for a proof. Conjugating with a suitable element of $\text{SL}_{2g}(\mathbb{Z})$, the integral skew-symmetric matrix $E$ can be put in the form of

$$E = \begin{bmatrix} 0 & \Delta \\ -\Delta & 0 \end{bmatrix} \quad \Delta = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_g \end{bmatrix}, \quad d_1, \ldots, d_g \in \mathbb{N} \quad d_1, |d_2|, \ldots, |d_g$$

Changing the basis for $\Lambda$ accordingly, it is easy to check that there is a coordinate system on the $g$-dimensional vector space $V$ (a basis for $V^*$) in which the corresponding period matrix is in the form of $\left[ \Delta \right] \cdot Z$ with $Z$ satisfying the same properties as before. In such a situation we say that the
abelian variety $V/\Lambda$ carries a polarization of type $\Delta$. In the case of Jacobian varieties, we have a principally polarized abelian variety because, as we have seen, the matrix $Q$ can be conjugated over integers with $\begin{bmatrix} 0 & I_g \\ -I_g & 0 \end{bmatrix}$, i.e. $\Delta = I_g$.

5.2 Siegel Upper Half Plane and Torelli Theorem

In the previous subsection we proved that fixing a symplectic basis $\{\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g\}$ for $H_1(X, \mathbb{Z})$ (a basis satisfying $\alpha_i, \alpha_j = \beta_i, \beta_j = 0, \alpha_i, \beta_j = \delta_{ij}$), there is a basis for $\Omega(X)$ such that the period matrix of $X$ with respect to these bases is in the form of $\begin{bmatrix} I_g & \ast \\ \ast & Z \end{bmatrix}$ where $Z^T = Z, \text{Im} Z > 0$. The space of such $g \times g$ matrices is called the Siegel upper half plane of degree $g$ and is denoted by $\mathcal{S}_g$. A symplectic basis for homology is unique only up to the action of the symplectic group $\text{Sp}_{2g}(\mathbb{Z})$. In fact, there is an action of $\text{Sp}_{2g}(\mathbb{Z})$ on $\mathcal{S}_g$ where $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ takes $Z$ to $(C + ZD)^{-1}(A + ZB)$. When $g = 1$ this is precisely the familiar action of $\text{Sp}_{2g}(\mathbb{Z}) = \text{SL}_2(\mathbb{Z})$ on the upper half plane $\mathbb{H}$. So in fact we can assign to a genus $g$ Riemann surface a point of the quotient space $\mathcal{A}_g = \text{Sp}_{2g}(\mathbb{Z})/\mathcal{S}_g$. This space is the moduli space of principally polarized abelian varieties. Hence there is a map from the moduli space of genus $g$ Riemann surfaces to the moduli space $\mathcal{A}_g$. The Torelli theorem states that this map is an embedding! Rigorously speaking, one can associate to the matrix $Z \in \mathcal{S}_g$ a transcendental function on $C^g \to C$ called the theta function whose zero locus defines a divisor $\Theta$ of $\text{Jac}(X) = C^g / \text{Per} (\omega_1, \ldots, \omega_g)$ called the theta divisor. The Torelli theorem states that the knowledge of Jacobian along with its theta divisor determines the compact Riemann surface:

**Theorem 4.** Suppose $X, Y$ are compact Riemann surfaces and $\phi : \text{Jac}(X) \to \text{Jac}(Y)$ an isomorphism with $\phi^* \Theta_Y = \Theta_X$. Then $X$ and $Y$ are isomorphic.

The reader can find a proof of this theorem in [Narasimhan, §18]. For a more geometric proof consult [Griffiths-Harris, p. 359]. The theta divisor has been studied extensively by Riemann. It can be proved that if we embed effective divisors of degree $g - 1$ in the Jacobian via the map in (2), we get the theta divisor up to a translate. The Riemann singularity theorem determines the order of vanishing of the theta function at a point of $\Theta$. Consult [Narasimhan, §16, §17, §19] for a concise treatment of these topics.

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6 A polarization on the complex torus $V/\Lambda$ is defined as the first Chern class $c_1(L)$ of a positive definite holomorphic line bundle $L \to V/\Lambda$. A complex torus is an abelian variety iff it admits a polarization. The class $c_1(L) \in H^2(V/\Lambda, \mathbb{Z})$ can be thought of as a skew-symmetric bilinear form $\Lambda \times \Lambda \to \mathbb{Z}$. After picking a basis for $\Lambda$ such a form is represented by an integral skew-symmetric matrix $E$.

7 Similarly, the group $\text{Sp}(\Delta, \mathbb{Z})$ consisting of transformations $\mathbb{Z}^{2g} \to \mathbb{Z}^{2g}$ preserving the skew-symmetric form on $\mathbb{Z}^{2g}$ defined by $\begin{bmatrix} 0 & \Delta \\ -\Delta & 0 \end{bmatrix}$ acts on $\mathcal{S}_g$. The action is properly discontinuous and the corresponding quotient is the moduli space of $g$-dimensional polarized abelian varieties of type $\Delta$. 

9
5.3 Schottky Problem

Consider the embedding \( l : \mathcal{M}_g \to \mathcal{A}_g \) described in the previous subsection. The Schottky problem asks for a characterization of \( l(\mathcal{M}_g) \) in \( \mathcal{A}_g \). The Siegel upper half plane \( \mathcal{S}_g \) of degree \( g \) along with its quotient \( \mathcal{A}_g \) are of dimension \( \frac{g(g+1)}{2} \) while the moduli space \( \mathcal{M}_g \) of genus \( g \) Riemann surfaces is \( 3g - 3 \) dimensional. These dimensions are the same for \( g = 1, 2, 3 \) and the problem is thus interesting only for \( g \geq 4 \) when the dimension of \( \mathcal{A}_g \) is strictly larger than that of \( \mathcal{M}_g \).

5.4 Albanese Varieties

At the end of §4 we proved that any compact Riemann surface \( X \) can be embedded in its Jacobian. The embedding \( X \hookrightarrow \text{Jac}(X) \) has a universal property in the sense that any morphism from \( X \) to a complex torus factors uniquely through it. This is an example of an Albanese variety. For a compact Kähler manifold \( X \) its Albanese variety is defined to be \( \text{Alb}(X) = \Omega(X)^* / H_1(X, \mathbb{Z}) \) which clearly generalizes the definition of the Jacobian variety of a compact Riemann surface. If one accepts that this is a complex torus, then checking the universal property is straightforward. The torus \( \text{Alb}(X) \) is dual\(^{8}\) of the Picard variety \( \text{Pic}^0(X) = H^1(X, \mathcal{O}) / H^1(X, \mathbb{Z}) \) which parametrizes isomorphism classes of holomorphic line bundles on \( X \) whose first Chern class is zero. In dimension one, Abel’s theorem asserts that for compact Riemann surfaces the Picard variety and the Albanese variety are isomorphic.

\(^{8}\)The dual of the complex torus \( V/\Lambda \) is defined to be the space of conjugate-linear functionals \( l : V \to \mathbb{C} \) modulo the lattice of those \( l \)'s which furthermore satisfy \( \text{Im}(l(\Lambda)) \subseteq \mathbb{Z} \).