Phase Transitions and Renormalization:
Using quantum techniques to understand critical phenomena.

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Outline

1. Phase Transitions and Universality
   - Universality
   - The Ising Model
   - Critical Behavior

2. Quantum Field Theory and Renormalization
   - A Different Approach
   - The Need to Renormalize
   - Wilsonian Renormalization

3. Solving for Critical Behavior
The Main Idea

Landau’s classification of phase transitions:
- A parameter that breaks symmetry of the order parameter.
- A parameter that controls the amount of ordering in the system ($T$).

Critical behavior will be dependent on these parameters, plus the symmetry of the system. Using this, we can describe a wide variety of systems with the same tools (ferromagnets, liquid-crystals, cosmology, BEC, etc.).
The Ising Model

In formulas

The spin Ising model describes the behavior of a ferromagnetic material.

- There is a lattice of spin-$\frac{1}{2}$ “particles” $\Lambda$. Spins are projected to a preferred axis (can also consider general preferred subspaces).
- The spin ($\uparrow$ or $\downarrow$) of the particle at position $x \in \Lambda$ is described by a spin field $s(x)$.
- There is a magnetic field $h(x)$ parallel to the preferred axis.
- Energetically favorable for spins on neighboring sites to be aligned with each other.
- Energetically favorable for spins to be aligned with the magnetic field.

$$\mathcal{H} \sim - \sum_{x \in \Lambda} h(x)s(x) - \sum_{\langle x, x' \rangle} s(x)s(x').$$
Each arrow represents the value of the spin field at that point. The order parameter of the system is the total of all the spins and is called the magnetization $M$,

$$M = \sum_{x \in \Lambda} s(x).$$
If we tweak the temperature, eventually we reach a critical temperature $T_c$ where the discontinuity in $M$ disappears. The point where $T = T_c$ and $h = 0$ is called the critical point.
Studying physics near the critical point shows that a number of physical quantities scale exponentially. Moreover, the exponential scaling factors are the same for any statistical system in the universality class.

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Correlation length \(\xi \sim \left( \frac{1}{T-T_c} \right)^\nu\).
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Correlation length \(\xi \sim \left(\frac{1}{T-T_c}\right)^\nu\).
What went wrong?

The problem is that near the critical point, the correlation length diverges: $\xi \to \infty$.

- Mean field theory assumes that statistical fluctuations are small compared to the average behavior.

- Large correlation length results in different parts of the system coupling to each other and large scale fluctuations.
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Connections to Quantum Field Theory

We last saw that large statistical fluctuations occurring near the critical point violate the assumptions of mean field theory. Luckily, there is another way to approach this problem. Start with a comparison:

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source
Connections to Quantum Field Theory (contd.)

Takeaway: we can view the continuum limit of our statistical theory as a quantum field theory. If we want to study the Ising model near a critical point (large $\xi$), we need to study the corresponding quantum field theory at small mass scales:

$$m^{-1} \sim \xi \to \infty.$$ 

We will focus on the scaling factor $\nu$ of the correlation length. This comes from the correlation function, so we need to look at the quantum propagator.
Problems with a naive approach

Ideally, then, we would just define our theory with a small mass and perform the relevant calculations to determine the propagator. Sadly, this does not work.

- Some interacting quantum field theories lead to divergent predictions of physical results (this is due to the ill-defined nature of the path integral).
- To solve this issue: “renormalize” the theory by subtracting infinite counter-terms from the mass, interaction strength, and other parameters.
- This means that the physical mass of the theory is not actually defined by the parameter $m$ in the action!
We need another way to define our theory with small mass term. To do so, we turn to Wilson’s approach to renormalization.

- Original problem with the path integral was defining it over fields of arbitrarily high energy.
- Instead, consider a finite energy cutoff on our theory, $\Lambda$.
- The parameters of the theory will depend on $\Lambda$. As we slowly change $\Lambda$, these parameters will change continuously.
- Thinking of the defining parameters of the theory as coordinates, varying $\Lambda$ produces a flow in the “space of theories” called the RG flow.
Making the mass small

This gives us a way to specify that the mass is small which is consistent with renormalization.

- Start off at some finite energy scale and gradually lower $\Lambda$.
- Eventually, $\Lambda$ will become comparable in size to the physical mass.
- The longer this takes, the smaller the physical mass is.
- Requiring the mass to be small then requires the theory to pass close to a sink in the RG flow field.
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3 Solving for Critical Behavior
With these tools, we can approach the original problem. The key is that the RG flow of our theory comes close to a sink, or fixed point.

- Near an RG fixed point, most parameters of the theory become less important.
- Usually, there is only one relevant parameter remaining (the mass in our case).
- Using this reduction, we can compute the propagator using some routine techniques from QFT.
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- To first order in perturbation theory, we already get $\nu = 0.60$.
- Continuing to higher orders reproduces the experimental result of 0.63.
Why does this work?

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- Near the fixed point, most parameters become “irrelevant.”
- Translation: near a critical point, our system is described by the scaling laws of a few parameters $\implies$ universality.
For Further Reading I

Michael E. Peskin and Daniel V. Schroeder
*An Introduction to Quantum Field Theory.*

Leo P. Kadanoff
*Statistical Physics*
World Scientific, 2000