THE PONTRJAGIN-THOM CONSTRUCTION AND COBORDISM

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Introduction

Thom’s 1954 paper [2] is probably most remembered for its pioneering results about cobordism classes of smooth manifolds. However, this material only fills the last of the four sections of the paper. The preceding material discusses and proves results in differential topology, then goes on to study the spaces $MSO(k)$ and $MO(k)$. These spaces, and their topological properties, are really the heart of the paper. In addition to relating them to cobordism theory, Thom also uses these spaces to answer questions about representability of homology classes by smooth submanifolds. In this talk, we will review the content of [2] which is now known as the Pontrjagin-Thom construction and how it is related to the oriented cobordism ring $\Omega^{SO}_{*}$.

1. Transversality

Thom’s results rely on the fundamental concept of transversality in differential topology. Transversality can be motivated by the following question: when does the diagram

\[
\begin{array}{ccc}
Y & \rightarrow & N \\
\downarrow^{i} & \downarrow & \\
M & \xrightarrow{f} & N
\end{array}
\]

have a pullback in smooth manifolds? We will address this question in the case where $i : Y \rightarrow N$ is a smooth embedding.

Definition 1.1. A map $f : M \rightarrow N$ of smooth manifolds is transverse to an embedded submanifold $i : Y \rightarrow N$ if, for all points $y \in Y$ and $m = f^{-1}(i(y)) \in M$, the induced map

\[df_m + di_y : T_m M \oplus T_y Y \rightarrow T_{i(y)} N\]

is surjective. In this case, we write $f \pitchfork i$.

Proposition 1.2. If $f : M \rightarrow N$ is a smooth map, $i : Y \rightarrow N$ is a smooth embedding, and $f \pitchfork i$, then the subspace $f^{-1}(i(Y)) \subset M$ can be endowed with the structure of a smooth submanifold.

The proof of the proposition makes use of the inverse function theorem to pull back local coordinate functions defining $i(Y) \subset N$ to $M$. The subset defined by the zero locus of such functions then locally defines a submanifold of $M$. Transversality implies a stronger conclusion about the pre-image $f^{-1}(i(Y))$, which we shall need in the next section.

Proposition 1.3. Let $X \subset M$ be a smooth submanifold. Then, there exists an open submanifold $N(X) \subset M$ containing $X$ for which

\[N(X) \cong \nu(X)\]

where $\nu(X)$ is the normal bundle of $X$ in $M$. We call $N(X)$ a tubular neighborhood of $X$. 

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In the case when $X$ is compact, this proposition is easily proved using the exponential map. Furthermore, in the case where $i(Y) \subset N$ is compact and $f$ is proper, the proof of Proposition 1.2 also shows that there is a tubular neighborhood $N(Y) \subset N$ of $Y$ such that $f^{-1}(N(Y)) \subset M$ is a tubular neighborhood of $f^{-1}(i(Y))$ in $M$.

If we take $Y$ to be a point, then Proposition 1.2 reproduces the usual sufficient condition for the pre-image of a point to be a smooth submanifold: we require that $i(Y) \in N$ is a regular value of the map $f$. Sard’s theorem tells us that for a generic point of $N$ this condition is satisfied. Equivalently, we may say that for any $i(Y) \in N$ and $f : M \to N$, and for any open neighborhood $U$ of $i(Y)$, $i(Y)$ is a regular value of the composition of $f$ with a generic diffeomorphism of $U$ which is the identity near the boundary of $U$. Thom proves a generalization of this result for when $Y$ is an arbitrary embedded submanifold of $N$. We will only need the following version of this result.

**Theorem 1.4.** Let $f : M \to N$ be a smooth, proper map of smooth manifolds and $Y \subset N$ a compact smooth submanifold. Let $U = N(Y)$ be an open tubular neighborhood of $Y$ in $N$. Then, the subspace $\text{Diff}^{\text{reg}}(U, \partial U) \subset \text{Diff}(U, \partial U)$ of diffeomorphisms $\varphi$ of $U$ which restrict to the identity near the boundary of $U$ for which the composition $\varphi \circ f$ is transverse to $Y$ is dense.

This is usually stated in the following equivalent form (which is more analogous to the statement of Sard’s theorem).

**Corollary 1.5.** Let $M$ and $N$ be smooth manifolds with $M$ compact and let $Y \subset N$ be a compact submanifold. Then the subspace of smooth maps $C^\infty_{Y}(M, N) \subset C^\infty(M, N)$ which are transverse to $Y$ is dense.

Using the theorem on smooth approximation, we see that this result can be extended to include all continuous maps.

**Corollary 1.6.** Let $M$ and $N$ be smooth manifolds with $M$ compact and let $Y \subset N$ be a compact submanifold. Then the subspace of smooth maps $C^\infty_{\partial Y}(M, N) \subset C^0(M, N)$ which are transverse to $Y$ is dense.

2. **The Pontrjagin-Thom construction**

We will now see how Thom applied the results stated in the previous section to relate various questions about submanifolds of a smooth manifold to results from algebraic topology. In the following, $M$ will be a smooth, compact, oriented manifold of dimension $n + k$.

Let $X \subset M$ be a smooth submanifold of dimension $n$. The normal bundle $\nu(X)$ of $X$ in $M$ is a rank $k$ oriented vector bundle over $X$. We see that this vector bundle is classified by a map

$$X \xrightarrow{\varphi} BSO(k)$$

Moreover, since $M$, hence $X$, is compact, we see that this map factors through the oriented Grassmannian

$$X \longrightarrow \text{Gr}^SO_k(\mathbb{R}^m) \longrightarrow BSO(k)$$

for sufficiently large $m$ (see [1]). That $\nu(X)$ is classified by this map means that each square in

$$\begin{array}{ccc}
\nu(X) & \longrightarrow & \tau^m_k \\
\downarrow & & \downarrow \\
X & \longrightarrow & \text{Gr}^SO_k(\mathbb{R}^m)
\end{array}$$

is a pullback along $\varphi$. That is, $\nu(X)$ is transverse to $Y$ in $N$. This is usually stated in the following equivalent form (which is more analogous to the statement of Sard’s theorem).
is a pullback of vector bundles, where \( \tau^m_k \) is the tautological oriented bundle over \( \text{Gr}_k^{SO}(\mathbb{R}^m) \) and \( E_k := ESO(k) \times_{SO(k)} \mathbb{R}^k \) is the universal vector bundle over \( BSO(k) \).

We now introduce the Thom space of a vector bundle. The motivation for this is that, in our current situation, we would like to extend the top row of the above diagram to all of \( M \supset N(X) \equiv \nu(X) \). Let \( E \to B \) be a vector bundle equipped with a metric. We can define the following sub-fiber bundles

\[
D(E) := \{ (x, v) \in E : ||v|| < 1 \}
\]

\[
\overline{D}(E) := \{ (x, v) \in E : ||v|| \leq 1 \}
\]

\[
S(E) := \{ (x, v) \in E : ||v|| = 1 \}.
\]

Note that \( \overline{D}(E) = D(E) \cup S(E) \). We define the Thom space of \( E \) to be

\[
\text{Th}(E) := \overline{D}(E)/S(E).
\]

This is naturally a pointed space. In the case when \( B \) is compact, \( \text{Th}(E) \) is the one point compactification of \( E \).

Applying this construction to \( \nu(X) \), we see that

\[
\text{Th}(\nu(X)) \equiv \overline{N(X)}/\partial N(X) \equiv M/(M - N(X)).
\]

By precomposing with the quotient map, this gives the desired extension

\[
M \longrightarrow M/(M - N(X)) \equiv \text{Th}(\nu(X)) \longrightarrow \text{Th}(\tau^m_k) \longrightarrow \text{Th}(E_k).
\]

The Thom space of the universal vector bundle \( E_k \) is usually denoted by \( MSO(k) \).

2.1. \textit{L-equivalent submanifolds}. In building towards defining the cobordism ring, Thom first defines a notion of “embedded cobordism” which he calls \textit{L-equivalence}.

**Definition 2.2.** Let \( M \) be an \( n+k \) dimensional compact, oriented smooth manifold and let \( X_0 \) and \( X_1 \) be \( n \) dimensional oriented smooth submanifolds. We say that \( X_0 \) and \( X_1 \) are \textit{L-equivalent} if there exists an oriented smooth submanifold with boundary \( W \) of \( M \times I \) such that \( \partial W = W \cap (M \times \partial I) \) and

\[
W \cap M \times \{0\} = X_0
\]

\[
W \cap M \times \{1\} = X_1.
\]

It is not difficult to check that this relation is an equivalence relation on submanifolds of \( M \) of a fixed dimension. If the codimension \( k \) is large enough (say, \( k > n+1 \)), the set of \textit{L}-equivalence classes of submanifolds can be given the structure of an abelian group. First observe that in this case, the \textit{L}-equivalence class of a submanifold does not depend on the embedding of the submanifold, since all embeddings will be isotopic. Then, we can define the addition operation on \textit{L}-equivalence classes by disjoint union of representing submanifolds \( [X]_L + [Y]_L := [X \cup Y]_L \) (here, we also need large enough codimension to ensure we can embed the two submanifolds so they do not intersect). The identity class is given by \( [\emptyset]_L \) and the inverse to \( [X]_L \) is provided by \( [-X]_L \) where \( -X \) is the submanifold \( X \) with reversed orientation. Denote the group of \textit{L}-equivalence classes of \( n \) dimensional oriented submanifolds of \( M \) by \( L^m_O(M) \).

We now state Thom’s main theorem relating \( L^m_O(M) \) to the universal Thom space \( MSO(k) \).

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\(^1\)The definition can also be made in the unoriented case.
Theorem 2.3. Let $M$ be a compact, smooth, oriented manifold of dimension $n + k$. There is a well-defined bijection of sets $L_k^SO(M) \to [M, MSO(K)]$. For sufficiently large $k$, this map is a group homomorphism.

We will only prove the first statement here.

Proof. The map $L_k^SO(M) \to [M, MSO(K)]$ will be given by

$$[X] \mapsto [f : M \to MSO(k)]$$

the extension of the classifying map $\varphi : X \to BSO(k)$.

First, we check that this is well-defined. Let $X_0$ and $X_1$ be $L$-equivalent submanifolds of $M$ and let $W \subset M \times I$ be the submanifold witnessing their equivalence. We will use the following fact from differential topology.

Lemma 2.4. The submanifold $W$ above can be chosen in such a way so that it has a tubular neighborhood with boundary $N(W)$ where $N(W) \cap (M \times \{i\})$ is a tubular neighborhood of $X_i$ for $i = 0, 1$.

The reason this can be done is that we can always arrange $W$ so that it intersects the boundary $M \times \partial I$ transversely. Using the lemma, we consider the map $F : M \times I \to MSO(k)$ which is the extension of the classifying map of the normal bundle $\nu(W) \to BSO(k)$. By construction, this map restricts to the maps $f_i$ extended from the classifying maps $X_i \to BSO(k)$ on $M \times \{i\}$. This provides the desired homotopy between $f_0$ and $f_1$.

Next, we prove this map is surjective. Given a homotopy class of maps, we may choose a representative $f : M \to MSO(k)$. Since $M$ is compact, if we consider the sequence of inclusions

$$\cdots \to \text{Th}(\tau^m_k) \to \text{Th}(\tau^{m+1}_k) \to \cdots \to MSO(k)$$

we see that the map $f$ factors through $\text{Th}(\tau^m_k)$ for some sufficiently large $m$. Let $b$ be the basepoint of $\text{Th}(\tau^m_k)$. We see that $M - f^{-1}(b)$ and $\text{Th}(\tau^m_k) - \{b\} \cong D(\tau^m_k)$ are both smooth manifolds. Then, using Corollary [1.6], we can perturb the restricted map $f$ slightly to obtain a smooth map $g$ transverse to the compact submanifold $Gr^SO_k(\mathbb{R}^m)$ of $D(\tau^m_k)$. In particular, this can be done in such a way that the extension of $g$ to all of $M$ is homotopic to $f$. With the map $g$ in hand, we see that the pre-image $X := g^{-1}(Gr^SO_k(\mathbb{R}^m))$ is a smooth submanifold of $M$. Moreover, we see that a tubular neighborhood $N^m_k$ of $Gr^SO_k(\mathbb{R}^m)$ in $D(\tau^m_k)$ is pulled back to a tubular neighborhood of $X$ in $M - f^{-1}(b)$

$$\nu(X) \cong N(X) = (g|_X)^* N^m_k.$$ 

Then, using the fact that, as vector bundles, $N^m_k \cong D(\tau^m_k) \cong \tau^m_k$, we see that $\nu(X) \cong (g|_X)^* \tau^m_k$. Hence, the classifying map of $\nu(X)$ is homotopic to the restriction of $g$ to $X$. It follows that the extension of the classifying map to $M - f^{-1}(b)$ is homotopic to $g$ itself, so the extension of the classifying map to all of $M$ is homotopic to $f$.

Finally, we must show that if submanifolds $X_0$ and $X_1$ correspond to homotopic maps $f_0, f_1 : M \to MSO(k)$, then $X_0$ and $X_1$ are $L$-equivalent. This just comes down to applying the first part of the previous argument to a map of the form $F : M \times I \to MSO(k)$. Doing so, we can obtain a submanifold $W \subset M \times I$ which witnesses the $L$-equivalence of $X_0$ and $X_1$. \qed
3. Cobordism

We first recall the definition of cobordant manifolds and Thom’s definition of the cobordism ring.

**Definition 3.1.** Two compact, oriented, smooth manifolds of the same dimension, \(X_0\) and \(X_1\), are cobordant if there exists a compact, oriented, smooth manifold \(M\) such that \(\partial M = X_0 \cup (-X_1)\).

We easily see that being cobordant is an equivalence relation on manifolds of a fixed dimension. We can give the set of compact, oriented, smooth manifolds of arbitrary dimension the structure of an abelian group as follows. Addition is given by disjoint union of representatives \([X] + [Y] := [X \coprod Y]\). The identity is given by the class of the empty manifold, and inverses are provided by reversing the orientation of a representative: \([-X] = |X|\). We see that this abelian group is graded by the dimension of representing manifolds. We can additionally endow it with a multiplicative structure which respects this grading, given by cartesian product of representing manifolds: \([X] \cdot [Y] = [X \times Y]\). It is again easy to check that this gives the set of cobordism classes the structure of a graded ring, which we shall denote \(\Omega^\text{SO}_n\). Thom also introduces the unoriented cobordism ring \(\Omega^\text{O}_n\) in an analogously, which is usually denoted by \(\Omega^\text{O}_n\).

The definition of the group structure of the cobordism ring is similar the definition of \(L\)-equivalence classes above. Indeed, this was Thom’s motivation for introducing \(L\)-equivalence.

**Lemma 3.2.** For sufficiently large \(k\), the group \(L_n^\text{SO}(S^{n+k})\) can be identified with the \(n\)-th component of the cobordism ring \(\Omega_n^\text{SO}\).

*Proof.* We define a map of sets \(L_n^\text{SO}(S^{n+k}) \to \Omega^\text{SO}_n\) by sending an \(L\)-equivalence class \([X]_L\) to the corresponding cobordism class \([X]\) of any representing \(X \subset S^{n+k}\). This map is clearly well-defined and surjective, since any compact smooth manifold can be embedded in \(\mathbb{R}^{n+k}\) for sufficiently large \(k\), hence in \(S^{n+k}\). To show it is injective, we must only observe that a manifold with boundary witnessing a cobordism can be embedded in sufficiently large dimensional Euclidean space as well, and thus in its compactification \(S^{n+k}\). This result is a slight generalization of Whitney’s embedding theorem, and is proved in [2]. This map is clearly a homomorphism. \(\Box\)

Before stating Thom’s theorem on the cobordism ring, we will need a fact, also proved in [2], about the homotopy groups of \(\text{MSO}(k)\).

**Proposition 3.3.** The homotopy groups \(\pi_{n+k}(\text{MSO}(k))\) are independent of \(k\) as long as \(n < k\).

That is, for sufficiently large \(k\), the homotopy groups \(\pi_{n+k}(\text{MSO}(k))\) stabilize\(^3\) We now come to the most well remembered theorem in [2].

**Theorem 3.4.** The groups \(\Omega^\text{SO}_n\) are isomorphic to the stable homotopy groups \(\pi_{n+k}(\text{MSO}(k))\) for each \(n\).

*Proof.* This is a consequence of Theorem 2.3 and Lemma 3.2. \(\Box\)

Thom goes on to compute pieces of the cobordism ring. The full stable homotopy groups of \(\text{MSO}(k)\) are difficult to compute, but ignoring torsion makes the problem considerably easier. The reason is that one can apply Serre’s \(\mathcal{C}\)-theory to reduce the computation of the rational

\(^2\)Between the two objects, the unoriented cobordism ring is easier to calculate. This is due to the simpler structure of the Thom space \(\text{MO}(k) := \text{Th}(\text{EO}(k) \times \text{O}(k) \mathbb{R}^k)\).

\(^3\)The proper way to discuss this is in terms of spectra. Roughly, we would like to construct a spectrum whose \(k\)-th space if \(\text{MSO}(k)\), which is usually denoted \(\text{MSO}\). The proof of Proposition 3.3 constructs the maps \(\Sigma \text{MSO}(k) \to \text{MSO}(k + 1)\) needed to define this spectrum.
homotopy groups $\pi_{n+k}(MSO(k)) \otimes \mathbb{Q}$ to computing the rational cohomology of $BSO(k)$ (for details, see chapter 18 of [1]). We will only state the result here.

**Theorem 3.5.** The rational oriented cobordism ring $\Omega^*_{SO} \otimes \mathbb{Q}$ is a polynomial algebra generated by unique elements in degrees $4n$ for $n$ any natural number.

These generators correspond to the rational Pontrjagin classes of $BSO(k)$. In terms of compact oriented smooth manifolds, these classes are represented by complex projective spaces $\mathbb{CP}^{2n}$ of even dimension.

**References**
