Lecture 6: Partial Fractions

Today: Rational Functions, Partial Fractions

We continue our exploration of integration techniques today.

Rational Functions

Another class of elementary functions that we often encounter is the rational functions.

Definition. A rational function is a function of the form

\[ f(x) = \frac{P(x)}{Q(x)}, \]

where both \( P(x) \) and \( Q(x) \) are polynomials. If the degree of \( P \) is less than the degree of \( Q \), we call \( f \) a proper rational function.

A theorem in algebra tells us that if a rational function \( \frac{P(x)}{Q(x)} \) is not proper, that is, \( \deg(P) \geq \deg(Q) \), we can always find polynomials \( S(x) \) and \( R(x) \) such that \( \deg(R) < \deg(Q) \), and

\[ P(x) = S(x)Q(x) + R(x), \]

or equivalently,

\[ \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}. \]

Here, the rational function \( \frac{R(x)}{Q(x)} \) is guaranteed to be proper.

Example. Write each rational function as the sum of a polynomial and a proper rational function.

\begin{align*}
(a) \quad & \frac{x^2 + 1}{x^3 + 3x^2 + 3x + 1}, & (b) \quad & \frac{x^2 - 2x - 3}{x + 1}, & (c) \quad & \frac{x^3 - 1}{x + 1}, & (d) \quad & \frac{x^4 + 1}{x^2 + 1}.
\end{align*}

The technique for dividing polynomials to find \( S(x) \) and \( R(x) \) is long division. It is very similar to the long division for integers that we learned in elementary school.

(a) This is already a proper rational function.
(b) Perform long division

\[
\begin{array}{c|ccccc}
\multicolumn{2}{r}{} & x & -3 \\
\hline
x + 1) & x^2 - 2x - 3 \\
 & -x^2 & -x \\
\hline
 & 3x & -3 \\
 & 3x & +3 \\
\hline
 & 0 & \end{array}
\]
Thus \( S(x) = x - 3 \) and \( R(x) = 0 \), and we write
\[
\frac{x^2 - 2x - 3}{x + 1} = x - 3.
\]

(c) Perform long division
\[
\begin{array}{c|ccccc}
 x^2 - x + 1 \\
\hline
 x + 1 & x^3 & - 1 \\
 & - x^3 - x^2 & \\
 & - x^2 & x - 1 \\
 & x^2 + x & - x - 1 \\
 & & & & - 2
\end{array}
\]
Thus \( S(x) = x^2 - x + 1 \) and \( R(x) = -2 \), and we write
\[
\frac{x^3 - 1}{x + 1} = x^2 - x + 1 - \frac{2}{x + 1}.
\]

(d) Perform long division
\[
\begin{array}{c|cccc}
 x^2 - 1 \\
\hline
 x^2 + 1 & x^4 & + 1 \\
 & - x^4 - x^2 & - x^2 + 1 \\
 & x^2 + 1 & 2
\end{array}
\]
Thus \( S(x) = x^2 - 1 \) and \( R(x) = 2 \), and we write
\[
\frac{x^4 + 1}{x^2 + 1} = x^2 - 1 + \frac{2}{x^2 + 1}.
\]

Exercise. Evaluate the indefinite integrals
\[
\begin{align*}
(a) & \quad \int \frac{x^2 + 1}{x^3 + 3x^2 + 3x + 1} \, dx, \\
(b) & \quad \int \frac{x^2 - 2x - 3}{x + 1} \, dx, \\
(c) & \quad \int \frac{x^3 - 1}{x + 1} \, dx, \\
(d) & \quad \int \frac{x^4 + 1}{x^2 + 1} \, dx.
\end{align*}
\]

(a) Notice that the denominator \( x^3 + 3x^2 + 3x + 1 = (x + 1)^3 \). Substitute \( u = x + 1 \), we get
\[
\begin{align*}
\int \frac{x^2 + 1}{x^3 + 3x^2 + 3x + 1} \, dx &= \int \frac{x^2 + 1}{(x + 1)^3} \, dx \\
&= \int \frac{(u - 1)^2 + 1}{u^3} \, du \\
&= \int \frac{u^2 - 2u + 2}{u^3} \, du \\
&= \int \left( \frac{1}{u} - \frac{2}{u^2} + \frac{2}{u^3} \right) \, du \\
&= \ln |u| + \frac{2}{u} - \frac{1}{u^2} + C \\
&= \ln |x + 1| + \frac{2}{x + 1} - \frac{1}{(x + 1)^2} + C.
\end{align*}
\]
(b) The long division result tells us that
\[
\int \frac{x^2 - 2x - 3}{x + 1} \, dx = \int (x - 3) \, dx = \frac{x^2}{2} - 3x + C.
\]

(c) The long division result tells us that
\[
\int \frac{x^3 - 1}{x + 1} \, dx = \int \left( x^2 - x + 1 - \frac{2}{x + 1} \right) \, dx = \frac{x^3}{3} - \frac{x^2}{2} + x - 2 \ln |x + 1| + C.
\]

(d) The long division result tells us that
\[
\int \frac{x^4 + 1}{x^2 + 1} \, dx = \int \left( x^2 - 1 + \frac{2}{x^2 + 1} \right) \, dx = \frac{x^3}{3} - x + 2 \arctan(x) + C.
\]

Partial Fractions

If we want to find the indefinite integral of a rational function, our first step would be to write it as the sum of a polynomial and a proper rational function. We know how to integrate a polynomial, so the remaining work is to find the integral of a proper rational function.

A corollary of the Fundamental Theorem of Algebra tells us that any polynomial \( Q(x) \) can be factored as a product of linear factors of the form \( ax + b \) and irreducible quadratic factors of the form \( ax^2 + bx + c \) (where the discriminant \( b^2 - 4ac < 0 \)).

**Example.** Factor the following polynomials

(a) \( x^4 - 16 \)
(b) \( 2x^3 + 3x^2 - 2x \)
(c) \( x^3 - x^2 - x + 1 \)

(a) Use the difference of squares formula twice:
\[
x^4 - 16 = (x^2 + 4)(x^2 - 4) = (x^2 + 4)(x + 2)(x - 2).
\]

(b) Factor an \( x \) first, then factor the remaining quadratic:
\[
2x^3 + 3x^2 - 2x = x(2x^2 + 3x - 2) = x(x + 2)(2x - 1).
\]

(c) Factor an \( x - 1 \) first, then factor the remaining quadratic with difference of squares:
\[
x^3 - x^2 - x + 1 = x^2(x - 1) - (x - 1) = (x^2 - 1)(x - 1) = (x + 1)(x - 1)^2.
\]

The next step is to express the proper rational function \( R(x)/Q(x) \) as a sum of **partial fractions** by the following algorithm.
Algorithm (Partial Fraction Decomposition). To find a partial fraction decomposition of a proper rational function \( R(x)/Q(x) \), follow these steps:

1. Factor the denominator \( Q(x) \) into linear and irreducible quadratic factors
   \[ Q(x) = (a_1x^2 + b_1x + c_1)^{k_1} \cdots (a_mx^2 + b_mx + c_m)^{k_m}(d_1x + e_1)^{\ell_1} \cdots (d_nx + e_n)^{\ell_n}. \]

2. If none of the factors above are repeated, \( i.e., \) if
   \( k_1 = \cdots = k_m = \ell_1 = \cdots = \ell_n = 1 \),
   then there exist constants \( A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_m, \) and \( C_1, C_2, \ldots, C_n \) such that
   \[ \frac{R(x)}{Q(x)} = \sum_{i=1}^{m} \frac{A_i x + B_i}{a_i x^2 + b_i x + c_i} + \sum_{j=1}^{n} \frac{C_j}{d_j x + e_j}. \]

3. If one of the irreducible quadratic factors is repeated, for example, if \( Q(x) \) has the factor \( (ax^2 + bx + c)^k \) where \( k \geq 2 \), then instead of the single partial fraction
   \[ \frac{Ax + B}{ax^2 + bx + c}, \]
   we replace it by the sum
   \[ \frac{A_1 x + B_1}{ax^2 + bx + c} + \frac{A_2 x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_k x + B_k}{(ax^2 + bx + c)^k}. \]

4. If one of the linear factors is repeated, for example, if \( Q(x) \) has the factor \( (dx + e)^\ell \) where \( \ell \geq 2 \), then instead of the single partial fraction
   \[ \frac{C}{dx + e}, \]
   we replace it by the sum
   \[ \frac{C_1}{dx + e} + \frac{C_2}{(dx + e)^2} + \cdots + \frac{C_\ell}{(dx + e)^\ell}. \]

5. Solve for all the undetermined constant coefficients.

Example. Perform partial fraction decomposition on the following proper rational functions.

(a) \( \frac{16x - 64}{x^4 - 16} \)
(b) \( \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x} \)
(c) \( \frac{4x}{x^3 - x^2 - x + 1} \)

(a) We know \( x^4 - 16 = (x^2 + 4)(x + 2)(x - 2) \) from the previous exercise. Thus the partial fraction decomposition has the form
   \[ \frac{16x - 64}{x^4 - 16} = \frac{Ax + B}{x^2 + 4} + \frac{C}{x + 2} + \frac{D}{x - 2}. \]
Multiply both sides by \((x^2 + 4)(x+2)(x-2)\), we get

\[16x - 64 = (Ax + B)(x+2)(x-2) + C(x^2 + 4)(x-2) + D(x^2 + 4)(x+2)\]

Expand the right-hand side and collect terms of the same power, we get

\[16x - 32 = (A + C + D)x^3 + (B - 2C + 2D)x^2 + (-4A + 4C + 4D)x + (-4B - 8C + 8D)\]

The coefficients on the left and right must match up, so we get the following system of equations:

\[
\begin{align*}
A + C + D &= 0 \\
B - 2C + 2D &= 0 \\
-4A + 4C + 4D &= 16 \\
-4B - 8C + 8D &= -64
\end{align*}
\]

Solve for the undetermined coefficients, we get

\[A = -2, \quad B = 8, \quad C = 3, \quad D = -1.\]

So the partial fraction decomposition is

\[
\frac{16x - 32}{x^4 - 16} = \frac{-2x + 8}{x^2 + 4} + \frac{3}{x + 2} - \frac{1}{x - 2}.
\]

(b) We know \(2x^3 + 3x^2 - 2x = x(x+2)(2x-1)\) from the previous exercise. Thus the partial fraction decomposition has the form

\[
\frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x} = \frac{A}{x} + \frac{B}{x + 2} + \frac{C}{2x - 1}.
\]

Multiply both sides by \(x(x+2)(2x-1)\) to get

\[x^2 + 2x - 1 = A(x+2)(2x-1) + Bx(2x-1) + Cx(x+2)\]

Expand the right-hand side and collect terms of the same power, we get

\[x^2 + 2x - 1 = (2A + 2B + C)x^2 + (3A - B + 2C)x - 2A\]

The coefficients on the left and right must match up, so we get the following system of equations:

\[
\begin{align*}
2A + 2B + C &= 1 \\
3A - B + 2C &= 2 \\
-2A &= -1
\end{align*}
\]
Solve for the undetermined coefficients, we get
\[ A = \frac{1}{2}, \quad B = -\frac{1}{10}, \quad C = \frac{1}{5}. \]

So the partial fraction decomposition is
\[ \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x} = \frac{1}{2} \cdot \frac{1}{x} - \frac{1}{10} \cdot \frac{1}{x + 2} + \frac{1}{5} \cdot \frac{1}{2x - 1}. \]

(c) We know \( x^3 - x^2 - x + 1 = (x - 1)^2(x + 1) \) from the previous exercise. Thus the partial fraction decomposition has the form
\[ \frac{4x}{x^3 - x^2 - x + 1} = \frac{A}{x - 1} + \frac{B}{(x - 1)^2} + \frac{C}{x + 1}. \]

Multiply both sides by \((x - 1)^2(x + 1)\), we get
\[ 4x = A(x - 1)(x + 1) + B(x + 1) + C(x - 1)^2. \]

Expand the right-hand side and collect terms of the same power, we get
\[ 4x = (A + C)x^2 + (B - 2C)x + (-A + B + C). \]

The coefficients on the left and right must match up, so we get the following system of equations:
\[
\begin{cases}
A + C & = 0 \\
B - 2C & = 4 \\
-A + B + C & = 0
\end{cases}
\]

Solve for the undetermined coefficients, we get
\[ A = 1, \quad B = 2, \quad C = -1. \]

So the partial fraction decomposition is
\[ \frac{4x}{x^3 - x^2 - x + 1} = \frac{1}{x - 1} + \frac{2}{(x - 1)^2} - \frac{1}{x + 1}. \]

**Integrals of Rational Functions**

Now we have all the tools that enable us to find the indefinite integral of any rational function. We will start with some integrals of proper rational functions whose denominators are irreducible.

**Example.** Evaluate the indefinite integrals
(a) \[ \int \frac{2}{x+1} \, dx \]  
(b) \[ \int \frac{1}{x^2 + 4} \, dx \]  
(c) \[ \int \frac{x+3}{x^2 + 2x + 2} \, dx \]

(a) Use a substitution \( u = x + 1 \), we can get

\[ \int \frac{2}{x+1} \, dx = \ln |x+1| + C. \]

(b) Use an inverse substitution \( x = 2u \), we can get

\[ \int \frac{1}{x^2 + 4} \, dx = \frac{1}{4} \int \frac{1}{u^2 + 1} \cdot 2 \, du = \frac{1}{2} \arctan \left( \frac{x}{2} \right) + C. \]

(c) This one requires a little bit more algebraic manipulation.

\[ \int \frac{x+3}{x^2 + 2x + 2} \, dx = \int \frac{x+3}{(x+1)^2 + 1} \, dx = \int \frac{x+1}{(x+1)^2 + 1} \, dx + \int \frac{1}{(x+1)^2 + 1} \, dx. \]

Use the substitution \( u = x + 1 \) for both integrals, while use the substitution \( v = u^2 + 1 \) again for the first integral, we get

\[ \int \frac{x+3}{x^2 + 2x + 2} \, dx = \frac{1}{2} \int \frac{dv}{v} + 2 \int \frac{du}{u^2 + 1} = \frac{1}{2} \ln |x^2 + 2x + 2| + 2 \arctan(x + 1) + C. \]

Example. Evaluate the indefinite integrals

(a) \[ \int \frac{16x - 64}{x^4 - 16} \, dx \]  
(b) \[ \int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x} \, dx \]  
(c) \[ \int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} \, dx \]

(a) By partial fraction decomposition, we know that

\[ \int \frac{16x - 64}{x^4 - 16} \, dx = - \int \frac{2x}{x^2 + 4} \, dx + 8 \int \frac{1}{x^2 + 4} \, dx + 3 \int \frac{1}{x + 2} \, dx - \int \frac{1}{x - 2} \, dx \]

\[ = - \ln |x^2 + 4| + 4 \arctan \left( \frac{x}{2} \right) + 3 \ln |x + 2| - \ln |x - 2| + C. \]

(b) By partial fraction decomposition, we know that

\[ \int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x} \, dx = \frac{1}{2} \int \frac{1}{x} \, dx - \frac{1}{10} \int \frac{1}{x + 2} \, dx + \frac{1}{5} \int \frac{1}{2x - 1} \, dx \]

\[ = \frac{1}{2} \ln |x| - \frac{1}{10} \ln |x + 2| + \frac{1}{10} \ln |2x - 1| + C. \]

(c) The integrand in this example is not yet a proper rational function. We perform long
division to write this as the sum of a polynomial and a proper rational function.

\[
\begin{align*}
\frac{x + 1}{x^3 - x^2 - x + 1} &= \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} \\
& \quad - \frac{x^4 + x^3 + x^2 - x}{x^3 - x^2 + 3x + 1} \\
& \quad - \frac{x^3 + x^2 + x - 1}{4x} \\
\end{align*}
\]

Thus

\[
\frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} = x + 1 + \frac{4x}{x^3 - x^2 - x + 1}.
\]

By the partial fraction decomposition, we know that

\[
\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} \, dx = \int (x + 1) \, dx + \int \frac{1}{x - 1} \, dx + \int \frac{2}{(x - 1)^2} \, dx - \int \frac{1}{x + 1} \, dx
\]

\[
= \frac{x^2}{2} + x + \ln |x - 1| - \frac{2}{x - 1} - \ln |x + 1| + C
\]

\[
= \frac{x^2}{2} + x - \frac{2}{x - 1} + \ln \left| \frac{x - 1}{x + 1} \right| + C.
\]

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