Lecture 7: Improper Integrals

Today: Improper Integrals of Type I and Type II

We recall the statement of the Evaluation Theorem.

**Theorem** (Evaluation Theorem). *If* $f$ *is continuous on the interval* $[a, b]$, *then*

$$
\int_{a}^{b} f(x) \, dx = F(b) - F(a),
$$

*where* $F$ *is an antiderivative of* $f$.

In today’s class, we relax the condition on the Evaluation Theorem in two different ways, and introduce the *improper integrals*.

**Improper Integrals of Type I: Infinite Intervals**

First, we relax the condition on the finite interval by looking at the following example

**Example.** Find the area of the region that lies under the curve $y = x^{-2}$, above the $x$-axis, and to the right of the line $x = 1$.

It may seem that the region would have infinite area because the region itself is infinite. But let’s take a closer look. If we want to know the area between $x = 1$ and $x = 2$, we can set up the definite integral

$$
\int_{1}^{2} x^{-2} \, dx = \left[ -x^{-1} \right]_{1}^{2} = 1 - \frac{1}{2} = \frac{1}{2}.
$$

Again, if we want to know the area between $x = 1$ and $x = 3$, we set up the definite integral

$$
\int_{1}^{3} x^{-2} \, dx = \left[ -x^{-1} \right]_{1}^{3} = 1 - \frac{1}{3} = \frac{2}{3}.
$$
Similarly, if we just draw an arbitrary vertical line \( x = t \) with \( t > 1 \), and want to know the area between \( x = 1 \) and \( x = t \), the definite integral would tell us that the area is

\[
\int_1^t x^{-2} \, dx = \left[ -x^{-1} \right]_1^t = 1 - \frac{1}{t} = \frac{t - 1}{t}.
\]

The following figures shows our computation results.

Notice that if we take \( t \to \infty \), the area of the shaded region would approach

\[
\lim_{t \to \infty} \frac{t - 1}{t} = 1.
\]

So we say that the area of the infinite region is equal to 1, and write that

\[
\int_1^\infty x^{-2} \, dx = \lim_{t \to \infty} \int_1^t x^{-2} \, dx = \lim_{t \to \infty} \frac{t - 1}{t} = 1.
\]

With this example in mind, we can define the integral of a function over an infinite interval in the following way.

**Definition** (Improper Integral, Type I). (a) If \( \int_a^t f(x) \, dx \) exists for every number \( t \geq a \), then

\[
\int_a^\infty f(x) \, dx = \lim_{t \to \infty} \int_a^t f(x) \, dx,
\]

provided this limit exists (as a finite number).

(b) If \( \int_t^b f(x) \, dx \) exists for every number \( t \leq b \), then

\[
\int_{-\infty}^b f(x) \, dx = \lim_{t \to -\infty} \int_t^b f(x) \, dx,
\]

provided this limit exists (as a finite number).

We say an improper integral is **convergent** if the corresponding limit exists, and **divergent** if the limit does not exist.

(c) If both \( \int_a^\infty f(x) \, dx \) and \( \int_{-\infty}^a f(x) \, dx \) are convergent for a number \( a \), then we define

\[
\int_{-\infty}^\infty f(x) \, dx = \int_a^\infty f(x) \, dx + \int_{-\infty}^a f(x) \, dx.
\]
Example. (a) Determine whether the improper integral $\int_1^{\infty} \frac{1}{x} \, dx$ is convergent or divergent.

(b) For what values of $p$ is the improper integral $\int_1^{\infty} \frac{1}{x^p} \, dx$ convergent?

(a) Use the definition, we have

$$\int_1^{\infty} \frac{1}{x} \, dx = \lim_{t \to \infty} \int_1^t \frac{1}{x} \, dx = \lim_{t \to \infty} \left[ \ln |x| \right]_1^t = \lim_{t \to \infty} \ln(t) = \infty.$$  

So the improper integral is divergent.

(b) We know from part (a) that when $p = 1$, the integral is divergent. Now let’s assume that $p \neq 1$, then

$$\int_1^{\infty} \frac{1}{x^p} \, dx = \lim_{t \to \infty} \int_1^t \frac{1}{x^p} \, dx = \lim_{t \to \infty} \left[ \frac{x^{1-p}}{1-p} \right]_1^t = \lim_{t \to \infty} \frac{t^{1-p} - 1}{1-p}.$$  

Here we have two cases:

Case 1. If $p < 1$, then $1 - p > 0$, and

$$\lim_{t \to \infty} \frac{t^{1-p} - 1}{1-p} = \infty.$$  

Case 2. If $p > 1$, then $1 - p < 0$, and

$$\lim_{t \to \infty} \frac{t^{1-p} - 1}{1-p} = \frac{-1}{1-p} = \frac{1}{p-1}.$$  

Thus the improper integral $\int_1^{\infty} \frac{1}{x^p} \, dx$ is convergent if $p > 1$, and is divergent if $p \leq 1$.

Example. Evaluate the improper integral

$$\int_{-\infty}^{\infty} \frac{1}{1 + x^2} \, dx$$  

Let’s choose $a = 0$ to evaluate this improper integral.

$$\int_{-\infty}^{\infty} \frac{1}{1 + x^2} \, dx = \int_{-\infty}^{0} \frac{1}{1 + x^2} \, dx + \int_{0}^{\infty} \frac{1}{1 + x^2} \, dx$$

$$= \lim_{s \to -\infty} \int_s^{0} \frac{1}{1 + x^2} \, dx + \lim_{t \to \infty} \int_0^t \frac{1}{1 + x^2} \, dx$$

$$= \lim_{s \to -\infty} \left[ \arctan(x) \right]_s^0 + \lim_{t \to \infty} \left[ \arctan(x) \right]_0^t$$

$$= \arctan(0) - \lim_{s \to -\infty} \arctan(s) + \lim_{t \to \infty} \arctan(t) - \arctan(0).$$
We look at the graph of \( y = \arctan(x) \), and notice that there are two horizontal asymptotes at \( y = \pm \pi/2 \).

Thus
\[
\int_{-\infty}^{\infty} \frac{1}{1 + x^2} \, dx = \lim_{t \to \infty} \arctan(t) - \lim_{s \to -\infty} \arctan(s) = \frac{\pi}{2} - \left( -\frac{\pi}{2} \right) = \pi.
\]

**Exercise.** Is the improper integral \( \int_{0}^{\infty} e^{-x} \, dx \) convergent or divergent?

By definition of improper integral,
\[
\int_{0}^{\infty} e^{-x} \, dx = \lim_{t \to \infty} \int_{0}^{t} e^{-x} \, dx = \lim_{t \to \infty} \left[ -e^{-x} \right]_{0}^{t} = \lim_{t \to \infty} (-e^{-t}) - (-e^{0}) = 1.
\]
Thus this improper integral is convergent.

**Improper Integrals of Type II: Discontinuous Integrand**

The first type of improper integrals concerns the area of a region that extends infinitely on the horizontal direction. We now introduce the second type of improper integral on functions that have vertical asymptotes.

**Definition** (Improper Integral, Type II). (a) If \( f \) is continuous on \([a, b)\) and is discontinuous at \( b \), then
\[
\int_{a}^{b} f(x) \, dx = \lim_{t \to b^{-}} \int_{a}^{t} f(x) \, dx,
\]
provided that this limit exists (as a finite number).

(b) If \( f \) is continuous on \((a, b]\) and is discontinuous at \( a \), then
\[
\int_{a}^{b} f(x) \, dx = \lim_{t \to a^{+}} \int_{t}^{b} f(x) \, dx,
\]
provided that this limit exists (as a finite number).

(c) If \( f \) is continuous on \([a, b]\) except at \( c \in (a, b) \), and both \( \int_{a}^{c} f(x) \, dx \) and \( \int_{c}^{b} f(x) \, dx \) are convergent, then we define
\[
\int_{a}^{b} f(x) \, dx = \int_{a}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx.
\]
Example. (a) For what values of $p$ is the integral $\int_0^1 \frac{1}{x^p} \, dx$ improper?

(b) For what values of $p$ is the integral $\int_0^1 \frac{1}{x^p} \, dx$ divergent?

(a) The only type of discontinuity on $[0, 1]$ that could arise in the integrand is when $x^p = 0$. However, we notice that when $p \leq 0$, $-p \geq 0$, and

$$\frac{1}{x^p} = x^{-p}$$

is continuous throughout $[0, 1]$. When $p > 0$, the integrand is not continuous at 0. Thus the integral is improper if and only if $p > 0$.

(b) We assume that $p > 0$ first. Then

$$\int \frac{1}{x^p} \, dx = \int x^{-p} \, dx = \frac{x^{1-p}}{1-p} + C.$$ 

Now we take the limit

$$\lim_{{t \to 0^+}} \int_t^1 \frac{1}{x^p} \, dx = \lim_{{t \to 0^+}} \left[ \frac{1}{1-p} - \frac{t^{1-p}}{1-p} \right].$$ 

We shall notice that

$$\lim_{{t \to 0^+}} t^{1-p} = \begin{cases} 
0, & \text{if } p < 1, \\
1, & \text{if } p = 1, \\
\infty, & \text{if } p > 1.
\end{cases}$$

However, when $p = 1$, the denominator $1 - p$ would be zero. Thus the improper integral is divergent if and only if $p \geq 1$.

Exercise. Find the mistake in the following evaluation:

$$\int_0^2 \frac{1}{x-1} \, dx = \left[ \ln |x-1| \right]_0^2 = \ln(1) - \ln(1) = 0.$$ 

The integrand is discontinuous at $x = 1$, which is inside the range of integration. Thus this should be an improper integral

$$\int_0^2 \frac{1}{x-1} \, dx = \int_0^1 \frac{1}{x-1} \, dx + \int_1^2 \frac{1}{x-1} \, dx.$$ 

We know the second integral is divergent from the previous example, thus this integral should also be divergent.
Comparison Test

The following theorem sometimes helps us determine if an improper integral is convergent or divergent.

**Theorem 19** (Comparison Test for Improper Integrals). Suppose that $f$ and $g$ are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

1. If $\int_a^\infty f(x) \, dx$ is convergent, then $\int_a^\infty g(x) \, dx$ is convergent.
2. If $\int_a^\infty g(x) \, dx$ is divergent, then $\int_a^\infty f(x) \, dx$ is divergent.

The idea of this theorem could be presented in the following figure.

If the graph of $f(x)$ stays above the graph of $g(x)$ to the right of $x = a$, then we should expect to have

$$\int_a^\infty f(x) \, dx \geq \int_a^\infty g(x) \, dx$$

Surely, if $\int_a^\infty g(x) \, dx$ is divergent, being larger than this quantity, $\int_a^\infty f(x) \, dx$ must also be divergent, and if $\int_a^\infty f(x) \, dx$ is convergent, being smaller than this quantity, $\int_a^\infty g(x) \, dx$ must also be convergent.

**Remark.** The converse of these two statements may not hold true. If $\int_a^\infty g(x) \, dx$ is convergent, it does not tell use whether $\int_a^\infty f(x) \, dx$ is convergent or divergent. Similarly, if $\int_a^\infty f(x) \, dx$ is divergent, we do not know whether $\int_a^\infty g(x) \, dx$ is convergent or divergent.

**Example.** Show that $\int_{-\infty}^\infty e^{-x^2} \, dx$ is convergent.

**Proof.** We first write

$$\int_{-\infty}^\infty e^{-x^2} \, dx = \int_{-\infty}^{-1} e^{-x^2} \, dx + \int_{-1}^{1} e^{-x^2} \, dx + \int_{1}^\infty e^{-x^2} \, dx.$$
Notice that the second integral is an ordinary definite integral, which should always evaluate to a finite number. For the other two integrals, we use the fact that $x^2 \geq |x|$ always holds true whenever $|x| \geq 1$, and therefore $e^{-x^2} \leq e^{-|x|}$ on these two infinite intervals (see figure below).

We know from a previous exercise that

$$\int_{1}^{\infty} e^{-|x|} dx = \int_{1}^{\infty} e^{-x} dx$$

is convergent. Thus

$$\int_{1}^{\infty} e^{-x^2} dx \leq \int_{1}^{\infty} e^{-|x|} dx$$

must also be convergent. By a similar argument, the other improper integral is also convergent. Thus the original improper integral over the whole real line is convergent. \qed