Lecture 9: Sequences and Limits

Today: Sequences, Limits, Convergence, Subsequences, Boundedness, Monotonicity

When physicists study complex functions, they often employ Newton’s idea of representing functions as infinite sums of elementary functions. In the next few lectures, we will study the exact meaning of taking infinite sums, as well as writing a complex function as an infinite sum of elementary functions. We start our discussion with the basic concepts of sequences and limits.

Sequences and Limits

Definition. A sequence is an infinite list of numbers written in a definite order:

\[ \{a_1, a_2, a_3, \ldots, a_n, \ldots \} \]

The number \( a_1 \) is called the first term; the number \( a_2 \) is the second term. In general, we call the number \( a_n \) the \( n \)-th term. The sequence \( \{a_1, a_2, a_3, \ldots, a_n, \ldots \} \) can also be denoted by \( \{a_n\}_{n=1}^{\infty} \), or simply \( \{a_n\} \) when there is no confusion in the context.

Example. Some sequences can be defined by giving a formula for the \( n \)-th term. For example,

(a) The simplest sequence \( \{1, 1, 1, 1, \ldots \} \) of repeating 1’s, can be described by

\[ \{a_n\}, \text{ where } a_n = 1. \]

(b) The sequence \( \{1, 2, 3, 4, \ldots \} \) of consecutive integers, can be described by

\[ \{a_n\}, \text{ where } a_n = n. \]

Remark. In the examples above, we have a corresponding number \( a_n \) for every positive integer \( n \). It is sometimes helpful to define a sequence as a function \( f \) whose domain is the set of positive integers. In this way, a sequence of numbers can be written as \( \{f(n)\}_{n=1}^{\infty} \).

Example. A sequence does not need to be all integers. In fact, we can use complicated functions to define a sequence of numbers. Additionally, the index of a list does not need to start at 1. For example,

(a) \( \{2/n\}_{n=1}^{\infty} \) is a list of rational numbers. If we write out the terms of this sequence, we get

\[ \left\{2, \frac{2}{3}, \frac{2}{5}, \frac{2}{7}, \frac{2}{9}, \ldots \right\}. \]
(b) \( \{\sqrt{n}\}_{n=0}^{\infty} \) is a list of real numbers that are square roots of integers. If we write out the terms of this sequence we get
\[
\{0, 1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5}, \sqrt{6}, \ldots \}
\]

(c) \( \{\sin(n\pi/4)\}_{n=2}^{\infty} \) is a list of numbers that follows the value of sine. If we write out the terms of this sequence, we get
\[
\left\{1, \frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}, -1, -\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}, 1, \ldots \right\}
\]

**Example.** Some sequences are very difficult to be described by a simple defining function.

(a) The sequence \( \{b_n\}_{n=0}^{\infty} \), where \( b_n \) is the number of babies born in the United States on the \( n \)-th day after January 1, 1970.

(b) The sequence \( \{p_n\}_{n=1}^{\infty} \), where \( p_n \) is the \( n \)-th digit of \( \pi \) in decimal after the decimal point. If we write out the terms of this sequence, we get
\[
\{1, 4, 1, 5, 9, 2, 6, \ldots \}
\]

(c) The Fibonacci sequence \( \{f_n\}_{n=1}^{\infty} \) can be defined recursively by
\[
f_1 = 1, \quad f_2 = 1, \quad f_n = f_{n-1} + f_{n-2} \text{ for } n \geq 3.
\]

The first few terms of the Fibonacci sequence are
\[
\{1, 1, 2, 3, 5, 8, 13, 21, \ldots \}.
\]

A sequence such as \( \{2/n\}_{n=1}^{\infty} \) can be plotted on a number line.

It can also be plotted as the graph of a function.

From both figures, we can see that the terms of the sequence \( a_n = 2/n \) are approaching 0 as \( n \) gets large. In fact, the difference between each term and 0,
\[
|a_n - 0| = \left| \frac{2}{n} - 0 \right| = \frac{2}{n},
\]
can be made as small as possible by taking \( n \) sufficiently large. We hereby introduce the \textit{limit} of a sequence.

**Definition.** A sequence \( \{ a_n \} \) has the \textbf{limit} \( L \) if we can make the terms \( a_n \) as close to \( L \) as we like by taking \( n \) sufficiently large. In this case, we write

\[
\lim_{n \to \infty} a_n = L, \quad \text{or} \quad a_n \to L \text{ as } n \to \infty.
\]

If a sequence has a limit, we say that the sequence \textbf{converges} (or is \textbf{convergent}). Otherwise, we say that the sequence \textbf{diverges} (or is \textbf{divergent}).

A more precise definition of limit is given by

**Definition.** A sequence \( \{ a_n \} \) has the \textbf{limit} \( L \) if for every \( \epsilon > 0 \), there exists an integer \( N \) such that if \( n \geq N \), then \( |a_n - L| < \epsilon \).

This definition can be illustrated by the following figure.

No matter how small we choose the value of \( \epsilon > 0 \), the graph of the sequence must eventually be enclosed between the horizontal lines \( y = L + \epsilon \) and \( y = L - \epsilon \), although usually a small choice of \( \epsilon \) would require a large \( N \) for this to happen.

In the case of a divergent sequence, we may have the special case of \textbf{diverging to infinity}. Here is the precise definition.

**Definition.** We say that a sequence \( \{ a_n \} \) \textbf{diverges to} \( \infty \) if for every number \( M \geq 0 \), there exists an integer \( N \) such that if \( n \geq N \), then \( a_n \geq M \). We denote this by

\[
\lim_{n \to \infty} a_n = \infty.
\]

Similarly, we say that a sequence \( \{ a_n \} \) \textbf{diverges to} \( -\infty \) if for every number \( M \leq 0 \), there exists an integer \( N \) such that if \( n \geq N \), then \( a_n \leq M \). We denote this by

\[
\lim_{n \to \infty} a_n = -\infty.
\]
Properties of Limits

The usual limit laws for functions hold for sequences.

**Theorem 20.** If \( \{a_n\} \) and \( \{b_n\} \) are convergent sequences, and \( c \) is a constant, then

(a) \( \lim_{n \to \infty} (a_n \pm b_n) = \lim_{n \to \infty} a_n \pm \lim_{n \to \infty} b_n \).

(b) \( \lim_{n \to \infty} ca_n = c \lim_{n \to \infty} a_n \).

(c) \( \lim_{n \to \infty} a_n b_n = \lim_{n \to \infty} a_n \cdot \lim_{n \to \infty} b_n \).

(d) If \( a_n > 0 \) and \( p > 0 \), then \( \lim_{n \to \infty} a_n^p = \left( \lim_{n \to \infty} a_n \right)^p \).

(e) If \( \lim_{n \to \infty} b_n \neq 0 \), then \( \lim_{n \to \infty} \frac{a_n}{b_n} = \frac{\lim_{n \to \infty} a_n}{\lim_{n \to \infty} b_n} \).

**Theorem 21** (Squeeze Theorem). If \( a_n \leq b_n \leq c_n \) for \( n \geq N \), and \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} c_n = L \), then

\( \lim_{n \to \infty} b_n = L \).

**Theorem 22** (Absolute Value). \( \lim_{n \to \infty} |a_n| = 0 \) if and only if \( \lim_{n \to \infty} a_n = 0 \).

The concept of sequences is closely related to functions, as well as their limits.

**Theorem 23.** If \( \{a_n\} \) is a sequence defined by a function \( a_n = f(n) \), and if \( \lim_{x \to \infty} f(x) = L \), then

\( \lim_{n \to \infty} a_n = L \).

**Theorem 24** (Continuity and Convergence). If \( \lim_{n \to \infty} a_n = L \) and the function \( f \) is continuous at \( L \), then

\( \lim_{n \to \infty} f(a_n) = f(L) \).

**Example.** Find the following limits, if exist.

(a) \( \lim_{n \to \infty} \frac{n}{n + 1} \), (b) \( \lim_{n \to \infty} \frac{\ln(n)}{n} \), (c) \( \lim_{n \to \infty} \frac{(-1)^n}{n} \), (d) \( \lim_{n \to \infty} \cos(n\pi) \).

(a) We have multiple ways to evaluate this limit now. For example,

\[
\lim_{n \to \infty} \frac{n}{n + 1} = \lim_{n \to \infty} \left(1 - \frac{1}{n + 1}\right) = \lim_{n \to \infty} 1 - \lim_{n \to \infty} \frac{1}{n + 1} = 1 - 0 = 1,
\]

or,

\[
\lim_{n \to \infty} \frac{n}{n + 1} = \lim_{n \to \infty} \frac{1}{1 + \frac{1}{n}} = \frac{\lim_{n \to \infty} 1}{\lim_{n \to \infty} 1 + \lim_{n \to \infty} \frac{1}{n}} = \frac{1}{1 + 0} = 1.
\]
(b) We cannot apply Theorem 20 because both the numerator and the denominator diverge to \(\infty\). However, if we consider the function \(f(x) = \frac{\ln(n)}{n}\), we can apply L'Hôpital's Rule to obtain the limit

\[
\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{1/x}{1} = 0.
\]

By Theorem 23, we conclude that

\[
\lim_{n \to \infty} \frac{\ln(n)}{n} = 0.
\]

(c) We know that

\[
\lim_{n \to \infty} \left|\frac{(-1)^n}{n}\right| = \lim_{n \to \infty} \frac{1}{n} = 0.
\]

Thus by Theorem 22, we have

\[
\lim_{n \to \infty} \frac{(-1)^n}{n} = 0.
\]

(d) Notice that this is the limit of the sequence \(\{-1, 1, -1, 1, -1, 1, \ldots\}\). The terms oscillates between 1 and \(-1\) infinitely without approaching any value. Thus this limit does not exist.

**Example.** Find the following limits, if exist.

(a) \[\lim_{n \to \infty} \cos\left(\frac{\pi}{n}\right)\]

(b) \[\lim_{n \to \infty} \frac{n!}{n^n}\]

(a) We notice that \(\lim_{n \to \infty} \frac{\pi}{n} = 0\) and \(\cos(x)\) is continuous at \(x = 0\). By Theorem 24, we have

\[
\lim_{n \to \infty} \cos\left(\frac{\pi}{n}\right) = \cos\left(\lim_{n \to \infty} \frac{\pi}{n}\right) = \cos(0) = 1.
\]

(b) Both the numerator and the denominator diverge to \(\infty\) as \(n\) approaches \(\infty\). On top of that, we do not have a good continuous function to describe the numerator \(n!\), so we cannot apply L'Hôpital's Rule. We have to proceed by the definition of \(n!\), which is the product of positive integers up to \(n\). Thus one term of this sequence can be written as

\[
\frac{n!}{n^n} = \frac{1 \times 2 \times 3 \times \cdots \times n}{n \times n \times n \times \cdots \times n} = \frac{1}{n} \cdot \frac{2 \times 3 \times \cdots \times n}{n \times n \times \cdots \times n}.
\]

Notice that the second quantity is always at most 1. Thus

\[
0 \leq \frac{n!}{n^n} \leq \frac{1}{n}.
\]

But we also know that \(\lim_{n \to \infty} \frac{1}{n} = 0\). By the Squeeze Theorem, we conclude that

\[
\lim_{n \to \infty} \frac{n!}{n^n} = 0.
\]
Convergence of Sequences

When we handle a sequence of numbers, the first question to consider is whether the sequence converges or diverges. It is sometimes helpful to pass the same question to the level of subsequences.

Definition. Let \( \{a_n\}_{n=1}^{\infty} \) be a sequence of numbers, and let \( \{n_k\}_{k=1}^{\infty} \) be a sequence of integers such that
\[
n_1 < n_2 < n_3 < \cdots < n_{k-1} < n_k < n_{k+1} < \cdots
\]
Then the sequence \( \{a_{n_k}\}_{k=1}^{\infty} \) is called a subsequence of \( \{a_n\}_{n=1}^{\infty} \).

Example. (a) The sequence of fractions \( \{1/2n\} \) is a subsequence of \( \{1/n\} \).
(b) The constant sequence \( \{1, 1, 1, \ldots\} \) is a subsequence of the alternating sequence \( \{(-1)^n\} \).
(c) The sequence of odd integers \( \{1, 3, 5, 7, 9, \ldots\} \) is a subsequence of the sequence of integers \( \{1, 2, 3, 4, 5, \ldots\} \).

Theorem 25. If a sequence \( \{a_n\} \) converges to \( L \), then every subsequence of \( \{a_n\} \) converges to the same limit \( L \). Conversely, if a subsequence \( \{a_{n_k}\}_{k=1}^{\infty} \) of a sequence \( \{a_n\}_{n=1}^{\infty} \) diverges, then we can conclude that the sequence \( \{a_n\}_{n=1}^{\infty} \) also diverges.

Example. For what values of \( r \) is the sequence \( \{r^n\} \) convergent?

Consider the function \( f(x) = r^x \) defined on the interval \([1, \infty)\). We know that when \( r > 1 \), this is an increasing exponential function, so \( f(x) \to \infty \) as \( x \to \infty \); when \( 0 < r < 1 \), this is a decreasing exponential function, so \( f(x) \to 0 \) as \( x \to \infty \). When \( r = 0 \), the function \( f(x) \) is the constant zero function, and when \( r = 1 \), the function \( f(x) \) is the constant 1 function. Thus
\[
\lim_{x \to \infty} r^x = \begin{cases} 
0, & \text{if } 0 \leq r < 1, \\
1, & \text{if } r = 1, \\
\infty, & \text{if } r > 1.
\end{cases}
\]

Using Theorem 23, we have
\[
\lim_{n \to \infty} r^n = \begin{cases} 
0, & \text{if } 0 \leq r < 1, \\
1, & \text{if } r = 1, \\
\infty, & \text{if } r > 1.
\end{cases}
\]

Now if \( -1 < r < 0 \), we have
\[
\lim_{n \to \infty} |r^n| = \lim_{n \to \infty} |r|^n = 0
\]
from the previous result. If \( r = -1 \), the sequence alternates between \( \pm 1 \), thus diverges. Finally, if \( r < -1 \), then \( r^2 > 1 \), and we know that

\[
\lim_{n \to \infty} |r^{2n}| = \lim_{n \to \infty} (r^2)^n = \infty
\]

from the previous result. By Theorem 25, we can conclude that \( \{r^n\} \) diverges. Therefore, the sequence \( \{r^n\} \) is convergent if and only if \( -1 < r \leq 1 \).

**Monotonicity of Sequences**

**Definition.** A sequence \( \{a_n\} \) is called

- **increasing** if \( a_n \leq a_{n+1} \) for all \( n \);
- **strictly increasing** if \( a_n < a_{n+1} \) for all \( n \);
- **decreasing** if \( a_n \geq a_{n+1} \) for all \( n \);
- **strictly decreasing** if \( a_n > a_{n+1} \) for all \( n \).

A sequence is **monotone** if it is either increasing or decreasing.

**Example.** We take a look at some of the previous examples in this topic.

- The constant sequence \( \{1, 1, 1, 1, \ldots\} \) is both increasing and decreasing.
- The sequence of positive integers \( \{1, 2, 3, 4, \ldots\} \) is strictly increasing.
- The sequence \( \{1/n\} \) is strictly decreasing.
- The alternating sequence \( \{(-1)^n\} \) is neither increasing nor decreasing.
- The sequence \( \{r^n\} \) is
  - increasing if \( r = 0 \) or \( r \geq 1 \);
  - strictly increasing if \( r > 1 \);
  - decreasing if \( 0 \leq r \leq 1 \);
  - strictly decreasing if \( 0 < r < 1 \).

**Exercise.** Show that the sequence

\[
\left\{ \frac{n + 1}{n^2 + 1} \right\}_{n=1}^\infty
\]

is decreasing.

We notice that all the terms in this sequence are positive. In general, there are two ways to show a sequence of positive numbers is decreasing (resp. increasing). If the difference \( a_{n+1} - a_n \) of consecutive terms is non-positive, or if the ratio \( a_{n+1}/a_n \) of consecutive terms is less than or equal to 1, then we can say that the sequence \( \{a_n\} \) is decreasing.
Difference. The difference of consecutive terms is
\[
\frac{(n+1) + 1}{(n+1)^2 + 1} - \frac{n + 1}{n^2 + 1} = \frac{-n^2 - 3n}{(n^2 + 1)(n+1)^2 + 1} \leq 0.
\]
Thus the sequence is decreasing.

Ratio. The ratio of consecutive terms is
\[
\frac{(n+1) + 1}{(n+1)^2 + 1} \div \frac{n + 1}{n^2 + 1} = \frac{(n+1) + 1}{(n+1)^2 + 1} \cdot \frac{n^2 + 1}{n + 1} = \frac{n^3 + 2n^2 + n + 2}{n^3 + 3n^2 + 4n + 2} \leq 1.
\]
Thus the sequence is decreasing.

Definition. A sequence \( \{a_n\} \) is
- **bounded above** if there exists a number \( M \) such that \( a_n \leq M \) for all \( n \); we say \( M \) is an upper bound of \( \{a_n\} \).
- **bounded above** if there exists a number \( m \) such that \( a_n \geq m \) for all \( n \); we say \( m \) is a lower bound of \( \{a_n\} \).

A sequence is bounded if it is both bounded above and bounded below.

Example. Once again, we look at some of our previous examples.
- The constant sequence \( \{1,1,1,1,\ldots\} \) is bounded.
- The sequence of positive integers \( \{1,2,3,4,\ldots\} \) is bounded below but not above.
- The sequence \( \{1/n\} \) is bounded.
- The alternating sequence \( \{(-1)^n\} \) is bounded.
- The sequence \( \{r^n\} \) is
  - bounded if \(-1 \leq r \leq 1\);
  - bounded below but not above if \( r > 1 \);
  - neither bounded above nor bounded below if \( r < 1 \);
  - never bounded above but not below.

We state one form of the Completeness Axiom of the set of real numbers \( \mathbb{R} \) here.

**Axiom 1** (Least Upper Bound Property of \( \mathbb{R} \)). If \( S \) is a non-empty set of real numbers, and if it has an upper bound \( M \in \mathbb{R} \), then it has a least upper bound \( \ell \in \mathbb{R} \), which means
- \( \ell \) is an upper bound of \( S \);
- If \( m \in \mathbb{R} \) is an upper bound of \( S \), then \( \ell \leq m \).

The statement of the next theorem is equivalent to the least upper bound property.

**Theorem 26** (Monotone Convergence Theorem). Every bounded monotone sequence of real numbers is convergent.
This theorem will be very important in the next topic, when we discuss the convergence criteria of series.

**Remark.** The converse of this theorem only holds partially. A convergent sequence must be bounded, but it does not need to be monotone (*e.g.* \((-1)^n/n\)). On the other hand, if we remove any assumptions of this theorem, the theorem does not hold any more. A bounded sequence does not necessarily converge (*e.g.* \((-1)^n\)); a monotone sequence does not necessarily converge, either (*e.g.* \(n\)).