Lecture 12: Power Series

Today: Power Series, Radius of Convergence, Term-by-Term Differentiation and Integration, Power Series of Specific Functions

We are now ready to sum up infinitely many functions.

Power Series and Radius of Convergence

Definition. A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n + \cdots$$

where $x$ is a variable and the $c_n$'s are constants. We call the constants $c_n$ the coefficients of the power series. More generally, a power series centered at $a$, or simply, a power series at $a$, is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + c_3 (x-a)^3 + \cdots + c_n (x-a)^n + \cdots$$

Remark. We adopt the special treatment of $(x-a)^0 \equiv 1$ even when $x = a$.

Example. Consider the power series centered at $x = 2$

$$\sum_{n=0}^{\infty} (x-2)^n.$$ 

There is no reason to expect this power series to converge for all values of $x$. For example, when $x = 0$, this is a geometric series with common ratio $-2$, which is divergent. We also notice that this is simply a geometric series with common ratio $x - 2$, so it is convergent whenever $|x - 2| < 1$, or equivalently, $1 < x < 3$.

Example. Now consider the power series centered at $x = 2$

$$\sum_{n=0}^{\infty} \frac{(x-2)^n}{n}.$$ 

We use the Ratio Test to determine the values of $x$ such that it converges.

$$\lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{n+1} \cdot \frac{n}{(x-2)^n} \right| = \lim_{n\to\infty} \frac{|x-2| \cdot n}{n+1} = |x-2|$$

Thus the power series is convergent when $|x - 2| < 1$, or equivalently, $1 < x < 3$; it is divergent when $|x - 2| > 1$, or equivalently, $x < 1$ or $x > 3$. The Ratio Test is inconclusive.
when \( x = 1 \) or \( x = 3 \), so we need to discuss these two cases separately. When \( x = 3 \), the power series is the harmonic series, which diverges. When \( x = 1 \), the power series is the alternating harmonic series, which converges. In conclusion, the power series converges when \( 1 \leq x < 3 \).

**Exercise.** For what values of \( x \) is the series \( \sum_{n=0}^{\infty} e^{n^2}x^n \) convergent?

We use the Ratio Test to determine the values of \( x \) such that it converges.

\[
\lim_{n \to \infty} \left| \frac{e^{(n+1)^2}x^{n+1}}{e^{n^2}x^n} \right| = \lim_{n \to \infty} \frac{e^{n^2+2n+1}}{e^{n^2}} \cdot |x| = \lim_{n \to \infty} e^{2n+1}|x|
\]

This limit diverges to \( \infty \) when \( x \neq 0 \). Thus the given series converges only when \( x = 0 \).

**Exercise.** Recall the Bessel’s differential equation

\[
x^2 \frac{d^2 f}{dx^2} + x \frac{df}{dx} + (x^2 - \alpha^2) f = 0.
\]

The Bessel functions of the first kind, denoted by \( J_\alpha(x) \), are solutions to Bessel’s differential equation. For this exercise, we will only take \( \alpha \) to be non-negative integers. In this case, the Bessel function \( J_\alpha(x) \) is given by

\[
J_\alpha(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+\alpha} \cdot n!(n+\alpha)!} \cdot x^{2n+\alpha}.
\]

Find the domain of the Bessel function of the first kind \( J_\alpha(x) \).

Let \( a_n \) be the coefficients. Then

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}x^{2n+\alpha+2}}{a_n x^{2n+\alpha}} \right| = \lim_{n \to \infty} \left| \frac{(-1)^{n+1}}{2^{2n+\alpha+2} \cdot (n+1)!(n+\alpha+1)!} \cdot \frac{2^{2n+\alpha} \cdot n!(n+\alpha)! \cdot x^{2n+\alpha+2}}{(-1)^n x^{2n+\alpha}} \right|
\]

\[
= \lim_{n \to \infty} \frac{2^{2n+\alpha} \cdot n!(n+\alpha)!}{2^{2n+\alpha+2} \cdot (n+1)!(n+\alpha+1)!} \cdot x^2
\]

\[
= \lim_{n \to \infty} \frac{x^2}{4(n+1)(n+\alpha+1)} = 0
\]

for all \( x \). Thus the Ratio Test tells us that the series converges for all values of \( x \in \mathbb{R} \). In another word, the domain of the Bessel function of the first kind \( J_\alpha(x) \) is \( (-\infty, \infty) \).

From these examples above, we expect that given a power series centered at \( x = a \), there is an interval (possibly degenerate) around the point \( a \) on which the power series is convergent. The following theorem summarizes all possible scenarios for this to happen, and therefore leads to the definition of radius of convergence.
Theorem 37. Given a power series \( \sum_{n=0}^{\infty} c_n(x-a)^n \). Exactly one of the following three cases holds:

1. The series converges only if \( x = 1 \);
2. The series converges for all \( x \in \mathbb{R} \);
3. There exists a positive real number \( R \) such that the series converges if \( |x-a| < R \) and diverges if \( |x-a| > R \).

Definition. The number \( R \) in case (3) of the previous theorem is called the radius of convergence of the power series. By convention, the radius of convergence is \( R = 0 \) for case (1) and \( R = \infty \) for case (2). The interval of convergence of a power series is the interval that consists of all values of \( x \) for which the series is convergent.

Remark. For a power series centered at \( a \) with radius of convergence \( R \), the interval of convergence is not necessarily \((a-R,a+R)\). Indeed, the theorem tells us that the power series is convergent on \((a-R,a+R)\) and is divergent on \((-\infty,a-R)\) and \((a+R,\infty)\), but it does not tell us the convergence at the endpoints \( a \pm R \). The endpoints must be checked individually with other convergence tests.

Example. Find the radius of convergence and interval of convergence of the series \( \sum_{n=0}^{\infty} x^n e^n \).

Compute the ratio of consecutive terms

\[
\lim_{n \to \infty} \left| \frac{x^{n+1}e^{n+1}}{x^n e^n} \right| = \lim_{n \to \infty} \left| \frac{x^{n+1}e^{n+1}}{x^n e^n} \right| = \lim_{n \to \infty} \frac{|x|}{e} = \frac{|x|}{e}.
\]

By the Ratio Test, the series is convergent if \( |x|/e < 1 \), or equivalently, \( |x| < e \). It is divergent if \( |x|/e > 1 \), or equivalently, \( |x| > e \). Thus the radius of convergence is \( R = e \).

To find the interval of convergence, we need to look at the endpoints \( \pm e \). When \( x = e \), the series becomes \( \sum 1 \), thus diverges. When \( x = -e \), the series becomes \( \sum (-1)^n \), thus also diverges. Therefore, we conclude that the interval of convergence is \((-e,e)\).

Example. Find the radius of convergence and interval of convergence of the series \( \sum_{n=1}^{\infty} \frac{x^n}{ne^n} \).

Compute the ratio of consecutive terms

\[
\lim_{n \to \infty} \left| \frac{x^{n+1}}{(n+1)e^{n+1}} \right| = \lim_{n \to \infty} \left| \frac{x^{n+1}e^{n+1}}{(n+1)xe^n} \right| = \lim_{n \to \infty} \frac{n}{n+1} \cdot \frac{|x|}{e} = \frac{|x|}{e}.
\]

By the Ratio Test, the series is convergent if \( |x|/e < 1 \), or equivalently, \( |x| < e \). It is divergent if \( |x|/e > 1 \), or equivalently, \( |x| > e \). Thus the radius of convergence is \( R = e \).

To find the interval of convergence, we need to look at the endpoints \( \pm e \). When \( x = e \), the series becomes the harmonic series, thus diverges. When \( x = -e \), the series becomes the alternating harmonic series, thus converges. Therefore, the interval of convergence is \([-e,e)\).
Term-by-Term Differentiation and Integration

The following theorem provide us with the powerful tool of term-by-term differentiation and integration within the radius of convergence of a power series.

**Theorem 38** (Term-by-Term Differentiation and Integration). Let

\[\sum_{n=0}^{\infty} c_n (x - a)^n\]

be a power series centered at \(a\) with radius of convergence \(R > 0\). Then the function \(f(x)\) defined by

\[f(x) = \sum_{n=0}^{\infty} c_n(x - a)^n\]

on the interval \((a - R, a + R)\) is differentiable (hence continuous). Furthermore,

\[f'(x) = \sum_{n=1}^{\infty} n c_n (x - a)^{n-1}, \quad \int f(x) \, dx = C + \sum_{n=0}^{\infty} \frac{c_n(x - a)^{n+1}}{n + 1},\]

and the radii of convergence of the power series in these two equations are both \(R\).

We will not prove this theorem. But we shall use this theorem to find power series expressions for specific functions.

**Example.** Take the Bessel function of the first kind of order 0

\[J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n \cdot n! \cdot n} \cdot x^{2n}.\]

The power series is convergent on all of \(\mathbb{R}\). Thus its derivative can be computed term-by-term

\[J_0'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{2^{2n} \cdot n! \cdot n!} \cdot x^{2n-1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2^{2n-1} \cdot (n-1)! n} \cdot x^{2n-1}\]

\[= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2^{2n+1} \cdot (n+1)!} \cdot x^{2n+1} = -J_1(x).\]

**Representing Functions as Power Series**

We have seen the sum of convergent geometric series.

\[\frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n \quad (1)\]

for \(|x| < 1\). We will use this formula to find the power series expressions for a variety of functions.
Example. We first try to find the power series representation of \( \frac{1}{1 + x^2} \) and its interval of convergence. Notice that
\[
\frac{1}{1 + x^2} = \frac{1}{1 - (-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}.
\]
This is a geometric series with common ratio \(-x^2\). Thus it converges if \( | -x^2 | < 1 \), or equivalently, \( |x| < 1 \). Therefore, its interval of convergence is \((-1, 1)\).

Example. Now we try to find the power series representation of \( \frac{1}{x + 2} \) and its interval of convergence. We factor a 2 from the denominator to write it in the form of the left-hand side of equation (1).
\[
\frac{1}{x + 2} = \frac{1}{2(1 + \frac{x}{2})} = \frac{1}{2} \cdot \frac{1}{1 - (-\frac{x}{2})} = \frac{1}{2} \sum_{n=0}^{\infty} \left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^{n+1}}
\]
This is a geometric series with common ratio \(-x/2\). Thus it converges if \( | -x/2 | < 1 \), or equivalently, \( |x| < 2 \). Therefore, its interval of convergence is \((-2, 2)\).

Exercise. Find a power series representation of \( \frac{x^2}{1 + x^2} \) and determine its interval of convergence.

We may notice that
\[
\frac{x^2}{1 + x^2} = 1 - \frac{1}{1 + x^2} = 1 - \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 + \sum_{n=0}^{\infty} (-1)^{n+1} x^{2n}
\]
\[
= 1 - 1 + x^2 - x^4 + x^6 - x^8 + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} x^{2n}.
\]

Another way is to notice that
\[
\frac{x^2}{1 + x^2} = x^2 \cdot \frac{1}{1 + x^2} = x^2 \sum_{n=0}^{\infty} (-1)^n x^{2n} = \sum_{n=0}^{\infty} (-1)^n x^{2n+2}.
\]
Both series are geometric with common ratio \(-x^2\). Thus they converge if \( | -x^2 | < 1 \), or equivalently, \( |x| < 1 \). Therefore, the interval of convergence is \((-1, 1)\).

In these three cases, we transformed the given functions into the form
\[
\frac{1}{1 - u},
\]
then use the sum of convergent geometric series to write them into power series. The next several examples uses term-by-term differentiation and integration on the equation
\[
\frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n
\] (1)
to find power series.
Example. Differentiating each side of the equation

\[
\frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n,
\]

we get

\[
\frac{1}{(1 - x)^2} = \sum_{n=1}^{\infty} nx^{n-1}.
\]

By Theorem 38, the radius of convergence \( R = 1 \) does not change.

Example. Integrating both sides of the equation

\[
\frac{1}{1 + x} = \sum_{n=0}^{\infty} (-1)^n x^n,
\]

we get

\[
\ln(1 + x) = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} = C + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n.
\]

To determine what the value of \( C \) is, let \( x = 0 \), then \( \ln(1 + 0) = C \). Thus \( C = 0 \) and

\[
\ln(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n.
\]

By Theorem 38, the radius of convergence \( R = 1 \) does not change.

Exercise. Find a power series representation for \( \arctan(x) \), and determine its interval of convergence.

We know that

\[
\int \frac{1}{1 + x^2} \, dx = \arctan(x) + C.
\]

Thus

\[
\arctan(x) = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}.
\]

To find \( C \), let \( x = 0 \), then we get \( \arctan(0) = C \). Thus \( C = 0 \), and therefore

\[
\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}.
\]

By Theorem 38, the radius of convergence \( R = 1 \) does not change. To find the interval of convergence, we need to inspect the endpoints \( x = \pm 1 \).

Notice that \( (\pm 1)^{2n+1} = (\pm 1)^{2n} \cdot (\pm 1) = \pm 1 \). Thus the convergence at the endpoints are the same. Let’s look at \( x = 1 \). Then the power series becomes

\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1},
\]

which is convergent by the Alternating Series Test. The interval of convergence is \( [-1, 1] \).
Example. Besides giving power series representations of known functions, we may also use term-by-term differentiation and integration to determine the function corresponding to a given power series. For example, we know from a homework problem that the power series
\[ \sum_{n=0}^{\infty} \frac{x^n}{n!} \]
is convergent for all \( x \in \mathbb{R} \). Now define \( f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \). Then its derivative
\[ f'(x) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = f(x). \]
However, we know all functions that satisfies \( f'(x) = f(x) \) are of the form
\[ f(x) = Ce^x. \]
To find the value of \( C \), let \( x = 0 \), then we have
\[ f(0) = Ce^0 = C, \text{ and } f(0) = 1 + 0 + 0 + \cdots = 1. \]
Thus \( C = 1 \) and \( f(x) = e^x \). Therefore, we get the power series representation of the exponential function
\[ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \]
The radius of convergence of this power series is \( \infty \). Thus the equality above holds for all \( x \in \mathbb{R} \).
In particular, if we take \( x = 1 \), we get the following expression for the number \( e \) as a sum of an infinite series:
\[ e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots \]
In some context, this expression is adopted as the definition of the number \( e \).