Lecture 13: Taylor and Maclaurin Series

Today: Taylor's Theorem, Taylor Series, Maclaurin Series

Let's start our discussion with a function that can be represented by a power series. Suppose that

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

We first notice that

$$f(a) = c_0.$$

Now by Theorem 38, we can find the derivative of $f(x)$ by differentiating the individual terms of the power series.

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$

Notice that the derivative is also a power series, so we can proceed to compute all of its higher derivatives.

$$\begin{aligned}
f''(x) &= \sum_{n=2}^{\infty} n(n-1) c_n (x-a)^{n-2} \\
f'''(x) &= \sum_{n=3}^{\infty} n(n-1)(n-2) c_n (x-a)^{n-3} \\
&\cdots\cdots \\
f^{(k)}(x) &= \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) c_n (x-a)^{n-k}
\end{aligned}$$

When we evaluate the derivatives at a, we get the constant term in each power series

$$f'(a) = 1 \cdot c_1, \quad f''(a) = 2 \cdot 1 \cdot c_2, \quad f'''(a) = 3 \cdot 2 \cdot 1 \cdot c_3, \quad \ldots \quad f^{(k)}(a) = k! \cdot c_k$$

Solving the equation for the k-th coefficient c_k, we get

$$c_k = \frac{f^{(k)}(a)}{k!}$$

We have proved the following theorem.

Theorem 39. *If f has a power series expansion at a with radius of convergence $R > 0$, that is,*

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \text{ for all } |x-a| < R,$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

Remark. Substituting thiss formula back into the series, we see that if f has a power series expansion at a, then it must be of the form

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \end{aligned}$$

Definition. The series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

is called the **Taylor series of the function f at a**. When $a = 0$, the series becomes

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n,$$

and is given the special name **Maclaurin series**.

Example. We have seen in the previous lecture that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

is a power series expansion of the exponential function $f(x) = e^x$. The power series is centered at 0. The derivatives $f^{(k)}(x) = e^x$, so $f^{(k)}(0) = e^0 = 1$. So the Taylor series of the function f at 0, or the Maclaurin series of f, is

$$\sum_{n=0}^{\infty} \frac{x^n}{n!},$$

which agrees with the power series definition of the exponential function.

Definition. If $f(x)$ is the sum of its Taylor series expansion, it is the limit of the sequence of partial sums

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

We call the n-th partial sum **the n-th-degree Taylor polynomial of f at a**.

One important application of Taylor series is to approximate a function by its Taylor polynomials. This is very useful in physics and engineering, where people only need a good approximation for most scenarios, and polynomials are usually much easier to deal with than a transcendental function. The following theorem justifies the use of Taylor polynomials for function approximation.

Theorem 40 (Taylor's Theorem)**.** *Let $n \geq 1$ be an integer, and let $a \in \mathbb{R}$ be a point. If $f(x)$ is a function that is n times differentiable at the point a, then there exists a function $h_n(x)$ such that*

$$f(x) = T_n(x) + h_n(x)(x-a)^n, \text{ where } \lim_{x \to a} h_n(x) = 0.$$

The term

$$R_n(x) = f(x) - T_n(x) = h_n(x)(x-a)^n$$

is called the **Peano form of the remainder**.

Sometimes we would like a better estimate on the remainder term, so that we could have a better understanding of how good the Taylor polynomials approximate the original functions. However, we can only do this under stronger regularity assumptions on $f(x)$.

Theorem 41 (Lagrange Form of the Remainder)**.** *Let $n \geq 1$ be an integer, and let $a \in \mathbb{R}$ be a point. If $f(x)$ is a function that is $n+1$ times differentiable on an open interval I containing a, then for all $x \in I$, there exists a number z strictly between a and x such that*

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}$$

This is the **Lagrange form of the remainder**.

Example. Find the Maclaurin series for $f(x) = \sin(x)$, and show that its sum equals $\sin(x)$.

First, we need to find the derivatives of $f(x)$ at 0:

$$\begin{aligned} f(x) &= \sin(x), & f(0) &= 0, \\ f'(x) &= \cos(x), & f'(0) &= 1, \\ f''(x) &= -\sin(x), & f''(0) &= 0, \\ f'''(x) &= -\cos(x), & f'''(0) &= -1, \\ f^{(4)}(x) &= \sin(x), & f^{(4)}(0) &= 0, \\ &\cdots\cdots & &\cdots\cdots \end{aligned}$$

The derivatives repeat in a 4-cycle, so we can write the Maclaurin series as

$$\begin{aligned} & f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots \\ = {} & x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}. \end{aligned}$$

Use Ratio Test to find its radius of convergence:

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2n+3}}{(2n+3)!}\Big/\frac{(-1)^n x^{2n+1}}{(2n+1)!}\right| = \lim_{n\to\infty}\frac{|x|^2}{(2n+3)(2n+2)} = 0$$

for all $x \in \mathbb{R}$. Thus the radius of convergence $R = \infty$.

To show that the sum of the Maclaurin series equals to the function $f(x) = \sin(x)$, we consider the n-th remainder term in Lagrange form

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}x^{n+1},$$

where z is a number strictly between 0 and x. Notice that $f^{(n+1)}(z)$ is a sine function or a cosine function, so $|f^{(n+1)}(z)| \le 1$. Then we have

$$-\frac{x^{n+1}}{(n+1)!} \le R_n(x) \le \frac{x^{n+1}}{(n+1)!}$$

However, we know that

$$\lim_{n\to\infty}\frac{x^{n+1}}{(n+1)!} = 0$$

for all $x \in \mathbb{R}$, so by Squeeze Theorem,

$$\lim_{n\to\infty} R_n(x) = 0$$

for all $x \in \mathbb{R}$. But since $R_n(x) = f(x) - T_n(x)$, this implies that the Taylor polynomials converges to $f(x)$ for all $x \in \mathbb{R}$, *i.e.*, the sum of the Maclaurin series equals $f(x) = \sin(x)$.

Example. Find the Taylor series for $f(x) = e^x$ at $a = 1$.

All derivatives of $f(x)$ are e^x, so $f^{(n)}(1) = e$ for all $n \ge 0$. Thus its Taylor series at 1 is

$$\sum_{n=0}^{\infty}\frac{e}{n!}(x-1)^n$$

with radius of convergence $R = \infty$. The following transformation verifies that we found the right expression for the Taylor series:

$$e^x = e\cdot e^{x-1} = e\sum_{n=0}^{\infty}\frac{(x-1)^n}{n!} = \sum_{n=0}^{\infty}\frac{e}{n!}(x-1)^n.$$

Exercise. Find the Maclaurin series of $f(x) = \cos(x)$.

First, we find the derivatives of $f(x)$ at 0:

$$\begin{aligned}
f(x) &= \cos(x), & f(0) &= 1,\\
f'(x) &= -\sin(x), & f'(0) &= 0,\\
f''(x) &= -\cos(x), & f''(0) &= -1,\\
f'''(x) &= \sin(x), & f'''(0) &= 0,\\
f^{(4)}(x) &= \cos(x), & f^{(4)}(0) &= 1,\\
&\cdots\cdots & &\cdots\cdots
\end{aligned}$$

The derivatives repeat in a 4-cycle, so we can write the Maclaurin series as

$$\begin{aligned}
&f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots\\
&= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}.
\end{aligned}$$

We can verify by Ratio Test to see that its radius of convergence is $R = \infty$.

We can use the Lagrange form of the remainder to prove that the Maclaurin series converges to the function $f(x) = \cos(x)$ for all $x \in \mathbb{R}$. The detail is left as an exercise.

Example. Find the Maclaurin series for $f(x) = x\cos(x)$.

We know that the Maclaurin series for $\cos(x)$ is

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}$$

Thus

$$f(x) = x\cos(x) = x\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n+1}.$$

Example. Find the Maclaurin series for $f(x) = e^{-x^2}$.

We know that the Maclaurin series for the exponential function e^u is

$$e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!}$$

Substitute $u = -x^2$ in the expression above, we get

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}x^{2n}.$$

Remark. The Taylor series / Maclaurin series of a infinitely differentiable function does not necessarily equal to the original function. A proof is required to show that they are equal (or not equal) for a function under consideration. We used the Lagrange form of the remainder to prove it for $\sin(x)$ and used the differential equation method to prove it for e^x.

We collect the following table of important Maclaurin series for reference.

Function	**Maclaurin Series**	
$\dfrac{1}{1-x}$	$\displaystyle\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$	$R = 1$
e^x	$\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	$R = \infty$
$\sin(x)$	$\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	$R = \infty$
$\cos(x)$	$\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	$R = \infty$
$\arctan(x)$	$\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$	$R = 1$
$\ln(1+x)$	$\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	$R = 1$

One application of Taylor series is to justify the use of L'Hôpital's Rule.

Example. Find the limit $\displaystyle\lim_{x\to 0} \frac{\sin(x)}{x}$.

We could evaluate the limit with L'Hôpital's Rule, but let's use the Maclaurin series instead.

$$\lim_{x\to 0} \frac{\sin(x)}{x} = \lim_{x\to 0} \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}{x} = \lim_{x\to 0} \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots\right) = 1.$$

The result agrees with the answer we get from L'Hôpital's Rule:

$$\lim_{x\to 0} \frac{\sin(x)}{x} = \lim_{x\to 0} \frac{\cos(x)}{1} = \frac{\cos(0)}{1} = 1.$$

Another important application of Taylor series is that they enable us to integrate functions that we previous could not handle.

Example. Evaluate $\int e^{-x^2}\,dx$ as an infinite series.

We know the Maclaurin series for the integrand is

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$$

with radius of convergence $R = \infty$. Using term-by-term integration, we get

$$\int e^{-x^2}\,dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} x^{2n+1}$$

with radius of convergence $R = \infty$.