Conditional Probability

Math 202, Winter 2021

Disclaimer: These lecture notes are written for class-planning purposes. It is likely that the notes contain typos and mistakes. You are encouraged to let me know when you see any of those.

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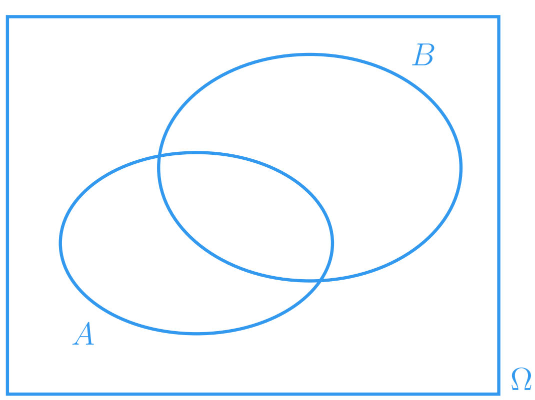
We have seen several ways to combine two given events. If we have events A and B, we could combine them to form events "A and B" and "A or B," which corresponds to the intersection $A \cap B$ and the union $A \cup B$, respectively. We could also consider the complements $\neg A$ and $\neg B$. In this lecture note, we will explore a new way of combining two given events – by conditioning one on the other.

1 Basic definitions

1.1 Consider two events A and B in a sample space Ω. There may be no obvious relation between these two events, so they may intersect at the most general position as two subsets of Ω, as shown in the following Venn diagram.



Instead of looking at the entire sample space Ω, we focus our attention to all outcomes in the event B, and consider the probability of A *given that* B has occurred. This has essentially created a new sample space $\Omega_B = B$, where the event we are interested in here is $A \cap B$. To compute the corresponding probability, we would take the ratio

$$\frac{|A \cap B|}{|\Omega_B|} = \frac{|A \cap B|}{|B|}.$$

Dividing both the numerator and the denominator by $|\Omega|$ will get us

$$\frac{|A \cap B|/|\Omega|}{|B|/|\Omega|} = \frac{P(A \cap B)}{P(B)},$$

which we are going to take as the definition of *conditional probability*.

1.2 Definition (*Conditional probability*)
Let A and B be two events of the sample space Ω, with $P(B) > 0$. The **conditional probability of A given B**, denoted by $P(A|B)$, is defined to be

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

1.3 Example
A fair dice is rolled twice. If the first roll is a 6, what is the probability that the second roll is also 6?

The sample space of this random scenario is represented by the following table.

Ω	1	2	3	4	5	6
1						
2						
3						
4						
5						
6						

Define the events

$$B = \{\text{outcomes with first roll 6}\}, \text{ and } A = \{\text{outcomes with second roll 6}\},$$

which are highlighted by blue and orange, respectively, in the table above. We see that

$$|A| = 6, \ |B| = 6, \text{ and } |A \cap B| = 1.$$

Therefore, the conditional probability of A given B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{|A \cap B|}{|B|} = \frac{1}{6}.$$

This result agrees with our prior understanding that past rolls of 6 does not affect the probability of rolling a 6 (or any number) in the future.

1.4 Example

A fair dice is rolled twice. If the first roll is a 6, what is the probability that the sum of the two rolls is 9?

We once again the following table to represent the sample space of this random scenario.

Ω	1	2	3	4	5	6
1						
2						
3						
4						
5						
6						

Define the events

$$B = \{\text{outcomes with first roll } 6\}, \text{ and } A = \{\text{outcomes with sum of } 9\},$$

which are highlighted by blue and orange, respectively, in the table above. In this case,

$$|A| = 4, \ |B| = 6, \ |A \cap B| = 1, \text{ and } |\Omega| = 36.$$

We can compute the conditional probability of A given B:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{|A \cap B|}{|B|} = \frac{1}{6}.$$

Comparing this to the unconditional probability of A

$$P(A) = \frac{|A|}{|\Omega|} = \frac{4}{36} = \frac{1}{9},$$

we got a higher conditional probability. This is a case where the probability of event A *depends on* the event B.

1.5 Definition (*Independent events*)

Two events A and B are **independent** if $P(A \cap B) = P(A)P(B)$.

1.6 Example

In Example 1.3, the events A and B are independent, as the probability of rolling two 6s is the product of the probability of rolling 6 at the first roll and the probability of rolling 6 at the second roll.

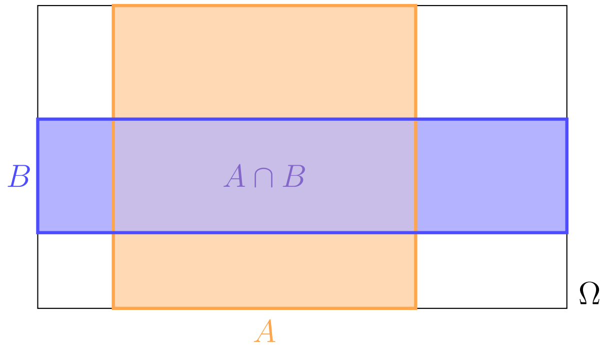
In Example 1.4, the events A and B are not independent, as $P(A \cap B) = 1/36$, but $P(A)P(B) = 1/54$.

1.7 Remark

An immediate consequence of probabilistic independence of A and B in terms of conditional probability is that imposing a condition of B on A would not change the probability of A, and *vice versa*. Algebraically, we have

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A),$$

and similarly, $P(B|A) = P(B)$. Geometrically speaking, the events A and B can be drawn as *cylinder sets* in two respective coordinates in a (rectangular) Venn diagram.



1.8 Question

A bag contains 2 blue marbles and 3 red marbles. We draw two marbles in a sequence without replacing the first marble into the bag before the second draw. Define the events

$$A = \{\text{outcomes with the first marble being blue}\},$$
$$B = \{\text{outcomes with the second marble being blue}\}.$$

Compute the probabilities $P(A)$, $P(B)$, $P(A \cap B)$, and $P(B|A)$. Are A and B independent events?

1.9 We could employ the same strategy as in Examples 1.3 and 1.4. If we label the marbles 1–5, where 1 and 2 are blue, and the rest are red, the following table represents the sample space of this random scenario.

Ω	**1**	**2**	**3**	**4**	**5**
1					
2					
3					
4					
5					

The diagonal entries are blacked out because of the no replacement rule, namely, the two marbles drawn must be distinct. This gives a sample space of size $|\Omega| = 20$. We highlight the events A and B with orange and green, respectively. We have

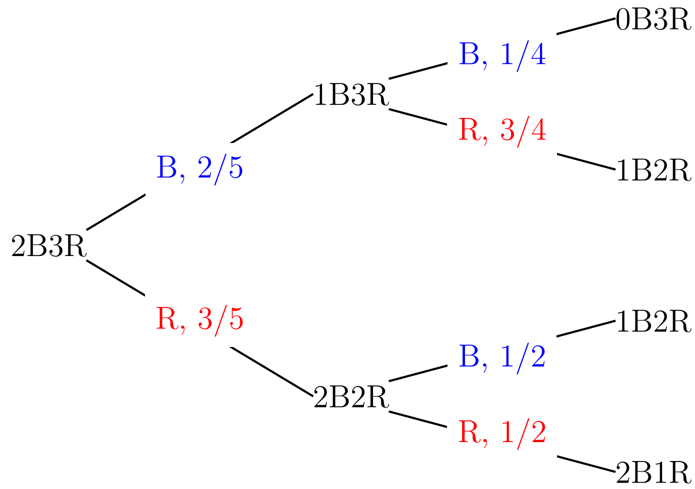
$$|A| = 8, \ |B| = 8, \ \text{and} \ |A \cap B| = 2.$$

We can compute the probabilities from here:

$$P(A) = \frac{|A|}{|\Omega|} = \frac{8}{20} = \frac{2}{5}; \quad P(B) = \frac{|B|}{|\Omega|} = \frac{8}{20} = \frac{2}{5};$$
$$P(A \cap B) = \frac{|A \cap B|}{|\Omega|} = \frac{2}{20} = \frac{1}{10};$$
$$P(B|A) = \frac{P(A \cap B)}{P(B)} = \frac{1/10}{2/5} = \frac{1}{4}.$$

We see from these probabilities that A and B are not independent events.

1.10 An alternative strategy that is very useful in probability is to use a *tree diagram* to represent all outcomes and their respective probabilities. Here is the tree diagram we constructed in class.



The probability of A can be read off the diagram at $P(A) = 2/5$. We can also read off the tree diagram that the conditional probability of event B given event A is $P(B|A) = 1/4$. The probability of $A \cap B$, which is equivalent to drawing two blue marbles with 0B3R left, is

$$P(A \cap B) = \frac{2}{5} \cdot \frac{1}{4} = \frac{1}{10}.$$

The probability of B would be the sum of

$$P(B) = \frac{2}{5} \cdot \frac{1}{4} + \frac{3}{5} \cdot \frac{1}{2} = \frac{2}{5}.$$

from the tree diagram