Naive Set Theory

Math 202, Winter 2021

Disclaimer: These lecture notes are written for class-planning purposes. It is likely that the notes contain typos and mistakes. You are encouraged to let me know when you see any of those.

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Set theory is the foundation of modern mathematics. It is the basis of the formal language which mathematicians use to communicate mathematical ideas. The *axiomatic approach* to set theory is the bedrock to rigorous mathematical reasoning in all fields of mathematics. In this course, we do not need the level of rigor in advance mathematics, and we will take an intuitive approach to set theory, often referred to as *naive set theory*.

1 Basic definitions

1.1 Definition (*Sets*)
The basic undefined term here is that of a **set**, which is a collection of *distinct* objects. We will often use an upper case letter (or some variants of it) to denote a set. The objects in a set do not need to have mathematical significance whatsoever, and they are called the **elements** or **members** of the set.

1.2 Example (*Some non-mathematical examples*)
Let's first see some examples of sets that has little to do with mathematics.

- Let A be the set of all *students* who registered for Math 202 this quarter. Then you, a student taking Math 202 right now, is an element of the set A; while Shuyi, the *instructor* of Math 202 this quarter, is not an element of the set A.
- Let B be the set of all letters in the word "finite". Then B contains 5 elements: the letters "f", "i", "n", "t", and "e". Note that a set does not contain repeated elements, thus the letter "i" is only counted once in the set. In another word, a letter is either in the set B, or not in the set B.

- Let C be the set of planets in the solar system. Then the Earth is an element of the set C; while the Sun is not an element of the set C, because it is not a planet.
- Pluto is of course a peculiar case for the set C. One might want to say that Pluto was an element of C before 2006, and was eliminated from this set by the International Astronomical Union (IAU) then. However, mathematicians usually do not allow a set to change over time (or contexts). Instead, we specify the different contexts, and define *different* sets for each context. For example, a more precise way to describe the set C is the set of planets in the solar system *as defined by IAU in 2006*. In this case, the set C would include the eight planets as we know today, but not Pluto. At the same time, we could specify that C_0 (note that we are not using the same letter) is the set of planets *accepted by the general public before 2006*. Then the set C_0 would include Pluto, together with the eight planets from Mercury to Neptune. This way, we completely eliminated the ambiguity of whether Pluto is considered a planet or not.

1.3 Example (*Some mathematical examples*)
The next few examples are more related to mathematical contexts.

- Let D be the set of single-digit positive integers. Then what D contains are the whole numbers 1 through 9.
- Let P be the set of prime numbers below 10. Then what P contains are the numbers 2, 3, 5, and 7.
- Let Q be the set of integers that are between 10 and 99 and also are perfect squares. The numbers satisfying these conditions are 16, 25, 36, 49, 64, and 81. Hence, they are all the elements of the set Q.

1.4 Notation
It is sometimes cumbersome to give a full-sentence word description every time we specify a set. We have a few ways to simplify our notations for a set.

- **Enumeration within curly brackets:**
When the elements of a set can be easily enumerated, we can write down all elements of the set (without omission or repetition), separate them with commas, and enclose everything within a pair of curly brackets. For example, we can write

$$C = \{\text{Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, Neptune}\}$$

for the set of planets; we can also write

$$P = \{2, 3, 5, 7\}$$

for the set of prime numbers below 10. We may also define sets as arbitrarily as we want through this method; for example,

$$S = \{★, ❄, ✈\}$$

is a set that contains three objects: a star ★, a snowflake ❄, and an airplane ✈.

- **Description within curly brackets:**
It is not always convenient to enumerate all elements of a set. For example, there is

no practical way to enumerate all Northwestern students, without using hundreds of pages. In these cases, we simply put the precise description of the set inside a pair of curly brackets. We can write

$$N = \{\text{Northwestern students}\}$$

for the set of all Northwestern students.

- **Defining criteria:** The overwhelmingly accepted notation for defining a set is to describe it as the collection of all elements satisfying some criteria. For example, consider the set Q in Example 1.3. It could be rewritten as

  The first criterion specifies that n is between 10 and 99.

  $$Q = \{n \mid 10 \leq n \leq 99, \text{ and } n = k^2 \text{ for some integer } k\}.$$

  The second criterion specifies that n is a perfect square.

  Elements of this set are temporarily called n.

  To decode this notation, we could read it as

  > "Q is the set of all n such that n is between 10 and 99, and n is the square of some integer k."

  The letters n and k are only meaningful in this particular definition of Q; they do not mean anything in any other contexts. These are called "dummy variables," and we cannot use them anywhere outside the "$\{\,|\,\}$" notation without redefining them.

It is worth mentioning that a particular set could have multiple ways to define, as long as all definitions agree upon whether any object is an element of the set or not. For example, the following three sets are *the same*, meaning that they contain the same elements.

$$\{16, 25, 36, 49, 64, 81\}$$
$$\{\text{perfect squares between 10 and 99}\}$$
$$\{n \mid 10 \leq n \leq 99, \text{ and } n = k^2 \text{ for some integer } k\}$$

1.5 Notation

If S is a set, and an object s is an element of S, then we write

$$s \in S,$$

which reads "s belongs to S," or "s is in S." If s is not in the set S, then we write

$$s \notin S,$$

which reads "s is not in S". For example, we have

$$2 \in P \text{ and } 16 \in Q.$$

We also have

$$\text{Shuyi} \notin A \text{ and Pluto} \notin C.$$

1.6 Remark (*Law of excluded middle*)
Given any set S and any object s, exactly one of the following holds: $s \in S$ or $s \notin S$.

1.7 Definition (*Size/cardinality*)
Given a set S, the **size**, or **cardinality**, of S is the number of distinct elements in S. We use $|S|$ to denote the size of S. The size of a finite set is always a non-negative integer.

1.8 Example
Let's consider the sets in Examples 1.2 and 1.3.

- As I checked on Sunday evening, there are 23 of you registered for this course. Thus the size of the set A is 23, and we write $|A| = 23$.
- The size of B is the number of *different* letters in the word "finite", and $|B| = 5$.
- There are 8 planets in the solar system, thus $|C| = 8$. For the Pluto lovers, $|C_0| = 9$.
- We have an enumeration for each of D, P, and Q. Thus it is not difficult to see that

$$|D| = 9, \ |P| = 4, \text{ and } |Q| = 6.$$

1.9 Remark
There are sets with an infinite cardinality, meaning that they contain infinitely many elements. For example, the set of integers (often denoted by the blackboard-bold letter $\mathbb{Z}$) is infinite, because there are infinitely many integers. However, as the title of this course suggests, we are not going to study these sets of infinite cardinality in detail.

1.10 Definition (*Empty set*)
There is a set of size 0. This is the set that does not contain any objects. This set is called the **empty set**. We use the symbol $\varnothing$ to represent the empty set. Note that the curly bracket notation for the empty set is $\varnothing = \{\}$.

2 Subsets

2.1 Definition (*Subsets*)
Given two sets A and B. We say that A is a **subset** of B if every element of A is also an element of B. This is denoted by $A \subseteq B$. If any one element of A is not in B, then A is not a subset of B, and we write $A \nsubseteq B$.

2.2 A schematic for $A \subseteq B$ and $A \nsubseteq B$ could be illustrated by the following *Venn diagrams*.

Note that in the $A \subseteq B$ example, the set A is entirely contained in the set B; while in the $A \nsubseteq B$ example, part of the set A is not inside of the set B.

2.3 Example

We are already familiar with examples of subsets in life. To list a few,

- Let A be the set of all students who registered for Math 202 this quarter, and let N be the set of all Northwestern students. Then $A \subseteq N$.
- Let D be the set of single-digit positive integers, and let P be the set of prime numbers below 10. To enumerate the elements of each set,

$$D = \{1, 2, 3, 4, 5, 6, 7, 8, 9\},$$
$$P = \{2, 3, 5, 7\}.$$

  It is clear from the enumeration that every element of P is also an element of D. Thus $P \subseteq D$.

2.4 Example (*Two special subsets*)

For any set S, there are at least two subsets of S: the set S itself, and the empty set $\varnothing$.

- Every element of S is an element of S (obviously). This explains why $S \subseteq S$.
- The case with the empty set is a bit more peculiar. There is no element in $\varnothing$. Hence, there is nothing wrong with saying

  > "Every element in $\varnothing$ is in S,"

  or

  > "Every element in $\varnothing$ is not in S."

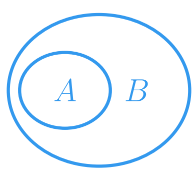
  In fact, it is even okay to say

  > "Every element in $\varnothing$ is made of mac-n-cheese balls,"

  because there is not a single element that can stand out to refute these statements.

2.5 Remark

If $A \subseteq B$, then $|A| \leq |B|$. In plain language, if A is a subset of B, then the size of A is at most the size of B. This is intuitive once we consider the Venn diagram for a subset relation,



where A is entirely contained in B. We could also verify this fact with the previous examples.

- The size of A (the set of all students who registered for Math 202 this quarter) is the number of students in this course, which is 23 at the moment of writing this note. There are more than $21,000$ students at Northwestern. Therefore, $|A| \leq |N|$.
- There are 9 single-digit positive integers, and 4 prime numbers below 10. Thus $|D| = 9$, $|P| = 4$, and $|P| \leq |D|$.
- For an arbitrary set S, we clearly have $|S| = |S|$. Thus it is also true that $|S| \leq |S|$.
- For an arbitrary set S, we know that $|S| \geq 0$, and $|\varnothing| = 0$. Therefore, $|\varnothing| \leq |S|$.

3 Union and intersection

Just like in arithmetic, where we can add, subtract, or multiply (they are called *operations*) two numbers and obtain another number, we can also combine two sets and make new ones from them. The two most basic set operations are *union* and *intersection*.

3.1 Definition (*Union and intersection*)
Given two sets A and B. The **union** of A and B, denoted by $A \cup B$, is a new set containing all elements in A and all elements in B. The **intersection** of A and B, denoted by $A \cap B$, is a new set containing all elements that are both in A and in B.

3.2 We can sketch up some Venn diagrams for unions and intersections as well.

$A \cup B$ $A \cap B$

If we think of the sets A and B as sheets of paper cut into the oval shapes in the Venn diagram, then the union $A \cup B$ is the region that is covered by *at least one* sheet of paper, and the intersection $A \cup B$ is the region that is covered by *both* sheets of paper.

3.3 Example
The Venn diagrams above represents the most general possible relation between two sets A and B. What would the case be if A and B has a containment relation, say $A \subseteq B$?

The union $A \cup B$ should contain all elements in A and all elements in B. But all elements in A are elements in B as well, so the union simply contains all elements in B. On the other hand, the intersection $A \cup B$ should contain all elements that are both in A and B. We notice that all elements in A are also in B, and elements not in A should not be in the intersection. This implies that the intersection simply contains all elements in A.

Observe the following Venn diagrams for this situation.

$A \cup B$ $A \cap B$

3.4 Example
For a more concrete example, let $A = \{1, 2, 3, 4, 5\}$ and $B = \{2, 3, 5, 7\}$. Then

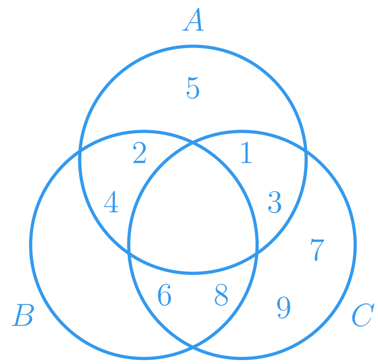
$$A \cup B = \{1, 2, 3, 4, 5, 7\} \text{ and } A \cap B = \{2, 3, 5\}.$$

3.5 Example
Unions and intersections become more interesting (and complicated) when we have more

than two sets at the same time. Consider the following example:

$$A = \{1, 2, 3, 4, 5\}, \ B = \{2, 4, 6, 8\}, \text{ and } C = \{1, 3, 6, 7, 8, 9\}.$$

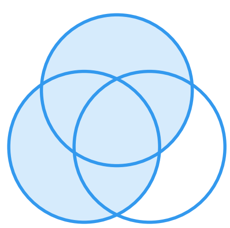
The associated Venn diagram for this example would be the following figure.



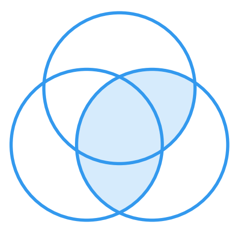
The most general relation between three sets will have 7 disjoint regions in its associated Venn diagram, 8 if we count the region outside of all three sets. When we fill out such a Venn diagram, it is not necessarily the case that every region contains some element(s), as this example has demonstrated.

We can compute the union and intersection of these three sets from the Venn diagram. Be extra careful that when a sequence of three sets are connected with $\cup$ and $\cap$ signs, the order of operation must be specified by parentheses. For example,

- $(A \cup B) \cap C = \{1, 3, 6, 8\}$:

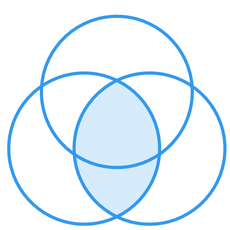


$A \cup B$

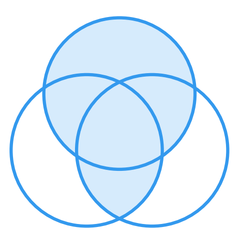


$(A \cup B) \cap C$

- $A \cup (B \cap C) = \{1, 2, 3, 4, 5, 6, 8\}$:



$B \cap C$



$A \cup (B \cap C)$

Different orders of operation may result in different results.

3.6 Remark

The following properties that relate unions and intersections with sizes of sets are the starting points of our next topic – the *inclusion-exclusion principle*.

Given two sets A and B,

- $|A \cap B| \leq |A|$ and $|A \cap B| \leq |B|$;
- $|A \cup B| \geq |A|$ and $|A \cup B| \geq |B|$;
- $|A \cup B| \leq |A| + |B|$.

All properties can be verified with the examples above.

4 The inclusion-exclusion principle

Let's start our discussion with a question.

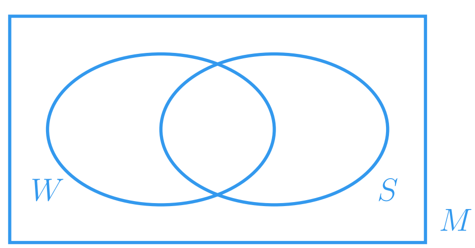
4.1 Question

A total of 23 of you registered for Math 202-0 this quarter. Among you and your classmates, 12 are from Weinberg College of Arts and Sciences; 9 are of senior standing. If we further know that 6 students are seniors in Weinberg College, how many of you are either from Weinberg or seniors?

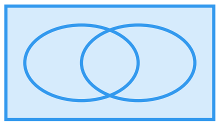
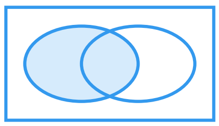
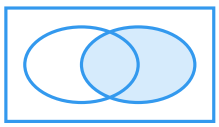
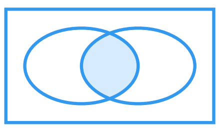
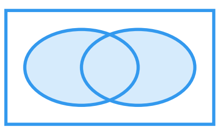
4.2 To put the context of this problem in set-theoretic language, we define the following sets:

$$\begin{aligned} M &= \{\text{students in Math 202-0}\} \\ W &= \{\text{Weinberg students in Math 202-0}\} \\ S &= \{\text{senior students in Math 202-0}\} \end{aligned}$$

The we have the inclusion relations $W \subseteq M$ and $S \subseteq M$. Further, we can draw the following Venn diagram associated with this situation.



Regions of the diagram represent different subsets of our class, as shown in the following table.

Region	Set	Size	Description
	M	23	All students in Math 202-0
	W	12	Weinberg students in Math 202-0
	S	9	Seniors in Math 202-0
	$W \cap S$	6	Senior Weinberg students in Math 202-0
	$W \cup S$	?	Students in Math 202-0 who are *either* from Weinberg *or* seniors

From Remark 3.6, we know that

$$|W \cup S| \leq |W| + |S|,$$

which means we are possibly *over-counting* if we simply add up the sizes of W and S. Upon further inspection, we notice that the part that we are over-counting, double-counting to be exact, is precisely the subset $W \cap S$. Therefore, if we subtract the size of $W \cap S$ from the sum, every person in $W \cup S$ is counted one and only one time. This tells us that

$$|W \cup S| = |W| + |S| - |W \cap S| = 12 + 9 - 6 = 15.$$

This is the number of students who are *either* from Weinberg *or* seniors.

4.3 Remark (*Inclusion-exclusion principle on two sets*)
We can summarize our findings in the previous question. Given two sets A and B, we have

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

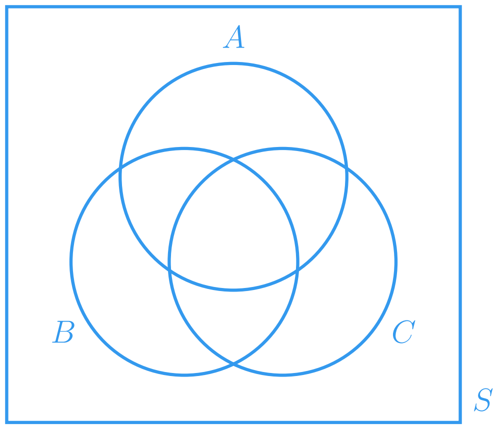
The inclusion-exclusion principle could also be applied on more than two sets. We will study the principle on three sets with a question as well.

4.4 Question
How many integers between 1 and 40 are *not* divisible by 2, 3, or 5?

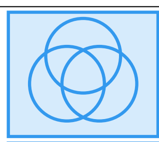
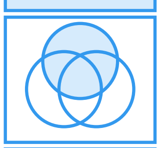
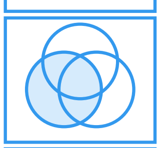
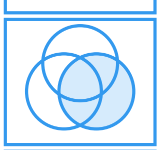
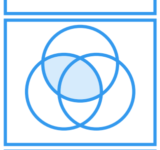
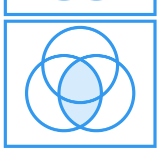
4.5 Once again, let's first put the context into set-theoretic language by defining

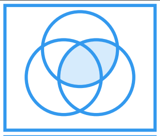
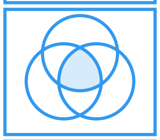
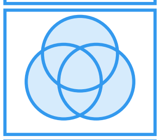
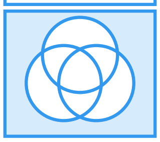
$$S = \{\text{integers between 1 and 40}\}$$
$$A = \{\text{integers between 1 and 40 that are multiples of 2}\}$$
$$B = \{\text{integers between 1 and 40 that are multiples of 3}\}$$
$$C = \{\text{integers between 1 and 40 that are multiples of 5}\}$$

The Venn diagram associated with this situation is



We denote all relevant regions with word and set descriptions.

Region	Set	Size	Description
	S	40	All integers between 1 and 40
	A	?	All integers between 1 and 40 that are divisible by 2
	B	?	All integers between 1 and 40 that are divisible by 3
	C	?	All integers between 1 and 40 that are divisible by 5
	$A \cap B$	?	All integers between 1 and 40 that are divisible by both 2 and 3
	$B \cap C$	?	All integers between 1 and 40 that are divisible by both 3 and 5

Region	Set	Size	Description
	$A \cap C$	?	All integers between 1 and 40 that are divisible by both 3 and 5
	$A \cap B \cap C$	?	All integers between 1 and 40 that are divisible by all of 2, 3, and 5
	$A \cup B \cup C$	?	All integers between 1 and 40 that are divisible by either 2, 3, or 5
	$S - A \cup B \cup C$	?	All integers between 1 and 40 that are divisible by none of 2, 3, and 5

Unlike the previous example, the sizes of the relevant sets are not immediately clear from the statement of the problem. We need to do some computation for these first.

$$\begin{aligned} |A| &= 40 \div 2 = 20, \\ |B| &= 40 \div 3 = 13 \text{ (with remainder 1)}, \\ |C| &= 40 \div 5 = 8. \end{aligned}$$

If we simply add up these numbers, we will reach $20 + 13 + 8 = 41$, which is absurd, because there are only 40 numbers between 1 and 40. Just like in the previous example, we must have over-counted. Further inspection tells us that elements in $A \cap B$, $B \cap C$, and $A \cap C$ are counted *at least* twice, and elements in $A \cap B \cap C$ are counted three times! To rectify over-counting, the inclusion-exclusion principle on three sets says that

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|.$$

Now we need to compute the size of these intersections.

If an integer is divisible by 2 and 3, then it must also be divisible by 6. Therefore,

$$|A \cap B| = 40 \div 6 = 6 \text{ (with remainder 4)}.$$

Likewise,

$$\begin{aligned} |B \cap C| &= 40 \div 15 = 2 \text{ (with remainder 10)}, \\ |A \cap C| &= 40 \div 10 = 4, \\ |A \cap B \cap C| &= 40 \div 30 = 1 \text{ (with remainder 10)}. \end{aligned}$$

Therefore, we can compute

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C| \\ &= 20 + 13 + 8 - 6 - 2 - 4 + 1 = 30. \end{aligned}$$

This is the number of integers between 1 and 40 that are divisible by either 2, 3, or 5. To find the number of integers in this range that are not divisible by any of them, we shall subtract this number from 40, so that $40 - 30 = 10$ is our answer.

If we want to make extra sure that we counted correctly, here is an enumeration of all integers between 1 and 40 that are not divisible by 2, 3, or 5.

$$\{1, 7, 11, 13, 17, 19, 23, 29, 31, 37\}.$$

Indeed, there are 10 of them.

4.6 Remark (*Inclusion-exclusion principle on three sets*)
Given three sets A, B, and C, we have

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|.$$