Solving Linear Systems

Linear Systems

You have dealt with linear equations and systems of linear equations since you first learn mathematics in elementary school. Here are some rigorous definitions regarding linear systems.

Definition. A **linear equation** in the variables $x_1, \dots, x_n$ is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where b and the **coefficients** $a_1, \dots, a_n$ are constants.

Definition. A **system of linear equations**, or a **linear system**, is a collection of one or more linear equations with the same variables.

Definition. A **solution** to a system is an ordered list $(s_1, \dots, s_n)$ of numbers such that each equation in the linear system is true when $x_1 = s_1, x_2 = s_2, \dots, x_n = s_n$ are substituted in.

Definition. The collection of all possible solutions is the **solution set** to a linear system.

Now we will have some examples of linear systems.

Example 1. The following is a linear system with one variable.

$$x = 1$$

This is a trivial example.

Example 2. The following is a linear system with two variables

$$\begin{array}{rcrcr} x & & & = & 1 \\ x & + & y & = & 1 \end{array}$$

This is a somewhat less trivial example.

Example 3. The following is a linear system with three variables

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & - & x_3 & = & 3 \\ -2x_1 & + & 4x_2 & + & 5x_3 & = & -5 \\ 3x_1 & - & 6x_2 & - & 6x_3 & = & 8 \end{array}$$

This example is so much less trivial that it ended up on your quiz. :)

Row Reduction

Throughout this section, we will look at the linear system

$$\begin{array}{rcrcrcr} 2x_1 & + & 8x_2 & + & 4x_3 & = & 2 \\ 2x_1 & + & 5x_2 & + & x_3 & = & 5 \\ 4x_1 & + & 10x_2 & - & x_3 & = & 1 \end{array}$$

as an example. The precise definitions of words in bold can be found in your textbook.

This linear system can be represented by its **augmented matrix**

$$\begin{bmatrix} 2 & 8 & 4 & 2 \\ 2 & 5 & 1 & 5 \\ 4 & 10 & -1 & 1 \end{bmatrix}$$

There are three basic row operations:

1. Replacement: replace one row by the sum of itself and a multiple of another row.
2. Interchange: interchange two rows.
3. Scaling: multiply all entries in a row by a nonzero constant.

We perform row reduction on the augmented matrix:

$$\begin{bmatrix} 2 & 8 & 4 & 2 \\ 2 & 5 & 1 & 5 \\ 4 & 10 & -1 & 1 \end{bmatrix} \begin{matrix} \div 2 \\ \\ \\ \end{matrix} \Rightarrow \begin{bmatrix} 1 & 4 & 2 & 1 \\ 2 & 5 & 1 & 5 \\ 4 & 10 & -1 & 1 \end{bmatrix} \begin{matrix} \\ -2(\text{I}) \\ -4(\text{I}) \end{matrix} \Rightarrow \begin{bmatrix} 1 & 4 & 2 & 1 \\ 0 & -3 & -3 & 3 \\ 0 & -6 & -9 & -3 \end{bmatrix} \begin{matrix} \\ \div(-3) \\ \div(-3) \end{matrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 4 & 2 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 2 & 3 & 1 \end{bmatrix} \begin{matrix} \\ \\ -2(\text{II}) \end{matrix} \Rightarrow \begin{bmatrix} 1 & 4 & 2 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

This matrix is now in **echelon form**. We can further reduce it into **reduced echelon form**

$$\begin{bmatrix} 1 & 4 & 2 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \begin{matrix} -2(\text{III}) \\ -(\text{III}) \\ \\ \end{matrix} \Rightarrow \begin{bmatrix} 1 & 4 & 0 & -5 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \end{bmatrix} \begin{matrix} -4(\text{II}) \\ \\ \\ \end{matrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Thus the solution to this linear system is

$$\begin{array}{rcr} x_1 & = & 11 \\ x_2 & = & -4 \\ x_3 & = & 3 \end{array}$$

This linear system has a unique solution.

Exercise. Use row operations to solve the linear system

$$\begin{array}{rcrcrcrcr} x_1 & & & + & 2x_3 & + & 4x_4 & = & -8 \\ & & x_2 & - & 3x_3 & - & x_4 & = & 6 \\ 3x_1 & + & 4x_2 & - & 6x_3 & + & 8x_4 & = & 0 \\ & - & x_2 & + & 3x_3 & + & 4x_4 & = & -12 \end{array}$$

Does this system has a unique solution, no solution, or infinitely many solutions?

Solutions of Linear Systems

Example 4. We first solve the linear system

$$\begin{array}{rcrcr} x & + & y & = & 4 \\ x & - & 2y & = & -2 \end{array}$$

We perform row reduction on the matrix

$$\begin{bmatrix} 1 & 1 & 4 \\ 1 & -2 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 4 \\ 0 & -3 & -6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

Thus the solution to this linear system is

$$\begin{array}{rcl} x & = & 2 \\ y & = & 2 \end{array}$$

Example 5. Now we try to solve the linear system

$$\begin{array}{rcrcr} x & + & y & = & 4 \\ -2x & - & 2y & = & -8 \end{array}$$

Perform row reduction

$$\begin{bmatrix} 1 & 1 & 4 \\ -2 & -2 & -8 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, the solution to this linear system is

$$\begin{array}{rcl} x & = & 4 - y \\ y & \text{is free} & \end{array}$$

Example 6. Try another linear system

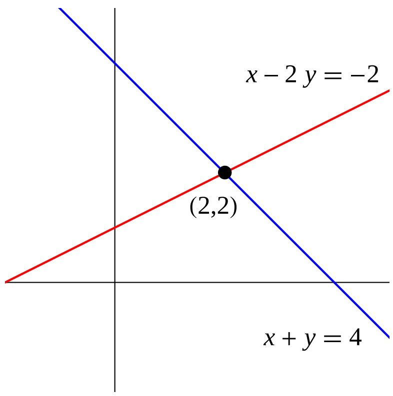
$$\begin{array}{rcrcr} x & + & y & = & 4 \\ 3x & + & 3y & = & 2 \end{array}$$

Perform row reduction

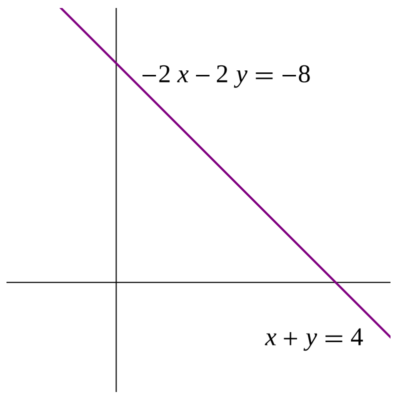
$$\begin{bmatrix} 1 & 1 & 4 \\ 3 & 3 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & -10 \end{bmatrix}$$

The system is inconsistent, thus has no solution.

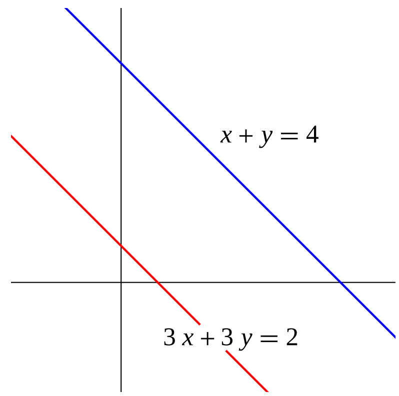
Remark. In examples 4–6, each linear equation represents a line in the xy-plane. The solution to the linear system is the intersection (if there is any) of the lines specified by the equations. The geometric picture of these three examples are as shown in the following figures.



Example 4



Example 5



Example 6

Exercise. Consider the linear system

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & 3x_3 & = & 4 \\ x_1 & + & kx_2 & + & 4x_3 & = & 6 \\ x_1 & + & 2x_2 & + & (k+2)x_3 & = & 6 \end{array}$$

For which values of the constant k does the system has (a) a unique solution? (b) no solution? (c) infinitely many solutions?

A Neat Application

Problem (Balancing chemical formula)**.** Consider the reaction of copper (Cu) with dilute nitric acid (HNO_3)

$$a\ \text{Cu} + b\ \text{HNO}_3 \to c\ \text{Cu(NO}_3)_2 + d\ \text{NO} + e\ \text{H}_2\text{O}$$

To balance the formula, that is, to give positive integers a, b, c, d, e such that the number of atoms of each element must be the same before and after the reaction. Find the smallest such set of integers.

To start, we look at copper (Cu) atoms. There are a copper atoms before the reaction, and b copper atoms after. So we must have

$$a = c.$$

Now we look at hydrogen (H) atoms. b of them are on the left-hand side, and $2e$ of them are on the right. Thus

$$b = 2e.$$

Similarly, from nitrogen (N) atoms, we have

$$b = 2c + d,$$

and from oxygen (O) atoms, we have

$$3b = 6c + d + e.$$

We now have a system of linear equations

$$\begin{array}{rcrcrcrcrcl} a & & & - & c & & & & & = & 0 \\ & & b & & & & & - & 2e & = & 0 \\ & & b & - & 2c & - & d & & & = & 0 \\ & & 3b & - & 6c & - & d & - & e & = & 0 \end{array}$$

The reduced echelon form of this system is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -3/4 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & -3/4 \\ 0 & 0 & 0 & 1 & -1/2 \end{bmatrix}$$

to solve the system with the least integers, let $e = 4$, then we get

$$\begin{array}{rcl} a & = & 3 \\ b & = & 8 \\ c & = & 3 \\ d & = & 2 \\ e & = & 4 \end{array}$$

Hence, the balanced chemical formula is

$$3\ \mathrm{Cu} + 8\ \mathrm{HNO_3} \rightarrow 3\ \mathrm{Cu(NO_3)_2} + 2\ \mathrm{NO} + 4\ \mathrm{H_2O}$$