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Matrix Equations and Linear Independence

Matrix Equations

Definition. A **matrix equation** is an equation of the form

$$A\mathbf{x} = \mathbf{b},$$

where A is an $m \times n$ matrix, and $\mathbf{b}$ is a vector in $\mathbb{R}^m$. The solution(s) $\mathbf{x}$, assuming existence, is a vector in $\mathbb{R}^n$.

Remark (Row-Vector Rule)**.** To compute $A\mathbf{x}$, the i-th entry is the sum of the products of corresponding entries from row i of A and from the vector $\mathbf{x}$.

Example 1. An example of the product $A\mathbf{x}$ would be

$$\begin{bmatrix} 2 & -3 & 5 \\ -1 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \times 1 + (-3) \times (-2) + 5 \times 0 \\ (-1) \times 1 + 0 \times (-2) + (-4) \times 0 \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \end{bmatrix}$$

Exercise. Compute the matrix products for the following pairs.

1. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

2. $A = \begin{bmatrix} 1 & 3 & 5 \\ -1 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$

3. $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

4*. $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, X = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

Remark. Let A be an $m \times n$ matrix. We can view A as a row matrix with n column vectors as its entries, and each vector is in $\mathbb{R}^m$. In another word, we can write

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}.$$

Now, if $\mathbf{x}$ is a vector in $\mathbb{R}^n$, the matrix product $A\mathbf{x}$ can be expressed as

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

This is a **linear combination** of the column vectors of A. Hence, a solution to the matrix equation $A\mathbf{x} = \mathbf{b}$ is a way to express $\mathbf{b}$ as a linear combination of the column vectors of A. This remark also shows that every system of linear equations can be written in the form of a matrix equation.

Homogeneous and Nonhomogeneous Systems

Definition. A linear system is **homogeneous** if its corresponding matrix equation is in the form $A\mathbf{x} = \mathbf{0}$. A homogeneous system always have the **trivial solution**, namely $\mathbf{x} = \mathbf{0}$.

Example 2. Find all solutions to the matrix equation $A\mathbf{x} = \mathbf{0}$, where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

We approach this problem by the usual row reduction method. However, notice that the last column of the augmented matrix starts with the zero vector, and stays zero under all row operations. So we can just reduce the **coefficient matrix** A.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 6 & 6 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

So a vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

is a solution to the matrix equation if it satisfies

$$\begin{aligned} x_1 - x_3 &= 0 \\ x_2 + 2x_3 &= 0 \end{aligned}$$

Hence, $\mathbf{x}$ must be in the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = x_3\mathbf{v}, \text{ where } \mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Thus all solutions to this matrix equation is a scalar multiple of the vector $\mathbf{v}$.

Remark. Notice that if we have $\mathbf{x}_1$ and $\mathbf{x}_2$ two solutions (not necessarily distinct) to the homogeneous system $A\mathbf{x} = \mathbf{0}$, *i.e.*, both $A\mathbf{x}_1 = \mathbf{0}$ and $A\mathbf{x}_2 = \mathbf{0}$ are satisfied, then addition of these two equations would give

$$A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

Hence, $x_1 + x_2$ is also a solution to this homogeneous system. Further, multiplying one equation by a constant $c \in \mathbb{R}$ gives

$$A(c\mathbf{x}_1) = cA\mathbf{x}_1 = c \cdot \mathbf{0} = \mathbf{0}.$$

Therefore, any linear combination of solutions to a homogeneous system is also a solution to the same system.

Definition. A linear system is **nonhomogeneous** if its corresponding matrix equation is in the form $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} \neq \mathbf{0}$.

Remark. Comparing to the previous remark on homogeneous systems, if we have $\mathbf{x}_1$ and $\mathbf{x}_2$ two solutions to the nonhomogeneous system $A\mathbf{x} = \mathbf{b}$, the vector sum $\mathbf{x}_1 + \mathbf{x}_2$ is not a solution because

$$A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = \mathbf{b} + \mathbf{b} = 2\mathbf{b} \neq \mathbf{b}.$$

However, if we have a solution $\mathbf{x}_0$ to the corresponding homogeneous system, *i.e.*, $A\mathbf{x}_0 = \mathbf{0}$, and a particular solution $\mathbf{x}_p$ to the nonhomogeneous system, *i.e.*, $A\mathbf{x}_\mathrm{p} = \mathbf{b}$, then their vector sum $\mathbf{x}_0 + \mathbf{x}_\mathrm{p}$ is again a solution to the nonhomogeneous system, because

$$A(\mathbf{x}_0 + \mathbf{x}_\mathrm{p}) = A\mathbf{x}_0 + A\mathbf{x}_\mathrm{p} = \mathbf{0} + \mathbf{b} = \mathbf{b}.$$

In fact, this method gives all solutions to the nonhomogeneous system $A\mathbf{x} = \mathbf{b}$.

Example 3. Find all solutions to the matrix equation $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

From Example 2, we know that the homogeneous solutions has the form

$$\mathbf{x}_0 = t\mathbf{v}, \text{ where } \mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

To find a particular solution, we need to express $\mathbf{b}$ as a linear combination of the column vectors of A. It is easy to see that

$$-\begin{bmatrix}1\\4\\7\end{bmatrix}+\begin{bmatrix}2\\5\\8\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix}.$$

Thus a particular solution to this nonhomogeneous system is

$$\mathbf{x}_{\mathrm{p}}=\begin{bmatrix}-1\\1\\0\end{bmatrix}.$$

Therefore, all solutions to this system are in the form

$$\mathbf{x}=\mathbf{x}_{\mathrm{p}}+t\mathbf{v}.$$

Exercise. Find all solutions to $A\mathbf{x}=\mathbf{b}$ for the following pairs of A and $\mathbf{b}$.

1. $A=\begin{bmatrix}1&1&0&2\\-1&1&2&0\\0&-1&-1&-1\end{bmatrix}$, $\mathbf{b}=\begin{bmatrix}2\\0\\-1\end{bmatrix}$

2. $A=\begin{bmatrix}2&8&4\\2&5&1\\4&10&-1\end{bmatrix}$, $\mathbf{b}=\begin{bmatrix}2\\5\\1\end{bmatrix}$

3. $A=\begin{bmatrix}1&0&2&4\\0&1&-3&-1\\3&4&-6&8\\0&-1&3&4\end{bmatrix}$, $\mathbf{b}=\begin{bmatrix}-8\\6\\0\\-12\end{bmatrix}$

Linear Independence

Definition. A set of vectors $\{\mathbf{v}_1,\dots,\mathbf{v}_n\}$ is **linearly dependent** if there exists $c_1,\dots,c_n\in\mathbb{R}$ not all zero such that

$$c_1\mathbf{v}_1+\cdots+c_n\mathbf{v}_n=\mathbf{0}.$$

A set of vectors $\{\mathbf{v}_1,\dots,\mathbf{v}_n\}$ is **linearly independent** if it is not linearly dependent.

Remark. To determine linear independence, we need to solve the linear system

$$c_1\mathbf{v}_1+\cdots+c_n\mathbf{v}_n=\mathbf{0},$$

or equivalently, the matrix equation

$$\begin{bmatrix}\mathbf{v}_1&\cdots&\mathbf{v}_n\end{bmatrix}\begin{bmatrix}c_1\\\vdots\\c_n\end{bmatrix}=\mathbf{0}$$

Notice that this is a homogeneous system, so it always has the trivial solution

$$c_1 = \cdots = c_n = 0.$$

We are interested in whether it has any other solutions. If the trivial solution is the only solution to this matrix equation, then the vectors are linearly independent. Otherwise, they are linearly dependent.

Example 4. Determine if the following vectors are linearly independent.

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}.$$

We perform row reduction on the coefficient matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The only solution to this matrix is the trivial solution. Hence, the vectors are linearly independent.

Example 5. Determine if the following vectors are linearly independent.

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Perform row reduction on the coefficient matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 0 \\ 1 & 3 & 6 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 2 & 5 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

There is one free variable in the system. Thus there are infinitely many solutions, and the vectors are linearly dependent.

Remark. In Example 5, we are given 4 vectors in $\mathbb{R}^3$. These vectors must be linearly dependent because the number of vectors is larger than the dimension of the underlying space.

Exercise. Without scratchwork, determine whether the following sets of vectors are linearly independent.

1. $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

2. $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

3. $\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 6 \\ 9 \end{bmatrix}$.

4. $\begin{bmatrix} -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

5. $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

6. $\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ -3 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$.

7*. $\begin{bmatrix} 7 \\ 0 \\ 4 \\ 0 \\ 1 \\ 9 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ 7 \\ 1 \\ 4 \\ 8 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 6 \\ 2 \\ 3 \\ 1 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 3 \\ 3 \\ 2 \\ 2 \\ 4 \end{bmatrix}$.

Exercise. Using row operation, determine whether the following sets of vectors are linearly independent.

1. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix}$.

2. $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 7 \\ 11 \end{bmatrix}$.