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Linear Transformations

Linear Transformations

Definition. A transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is **linear** if

$$T(x+y) = T(x) + T(y)$$

for all $x, y \in \mathbb{R}^n$, and

$$T(cx) = cT(x)$$

for all $c \in \mathbb{R}$ and $x \in \mathbb{R}^n$.

Examples of linear transformations include matrix transformations, linear functions, and differentiation operations.

Example 1. Let $f(x) = ax$ and $g(x) = ax + b$ for some $a \in \mathbb{R}$ and some $b \in \mathbb{R}\backslash\{0\}$. Then f is a linear transformation, but g is not.

Exercise. Using the definition of linear transformation, verify the claim in Example 1.

Example 2. Let $A \in \mathbb{R}^{m\times n}$ be a matrix. Then $T(x) = Ax$ defines a linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$.

Example 3. Taking derivative is a linear transformation. Given $f, g\colon \mathbb{R} \to \mathbb{R}$ two differentiable functions, and $c \in \mathbb{R}$ any scalar, we always have

$$\frac{\mathrm{d}}{\mathrm{d}x}f(x) + \frac{\mathrm{d}}{\mathrm{d}x}g(x) = \frac{d}{dx}(f+g)(x), \text{ and } c \cdot \frac{\mathrm{d}}{\mathrm{d}x}f(x) = \frac{\mathrm{d}}{\mathrm{d}x}\big(cf(x)\big).$$

Notice that the domain of this linear transformation is the collection of differentiable functions on $\mathbb{R}$.

Exercise. Consider the transformations from $\mathbb{R}^3$ to $\mathbb{R}^3$ defined below. Which of these are linear?

1. $\begin{aligned} y_1 &= 2x_2 \\ y_2 &= x_2 + 2 \\ y_3 &= 2x_2 \end{aligned}$
2. $\begin{aligned} y_1 &= 2x_2 \\ y_2 &= 3x_3 \\ y_3 &= x_1 + x_3 \end{aligned}$
3. $\begin{aligned} y_1 &= x_2 - x_3 \\ y_2 &= x_1 x_3 \\ y_3 &= x_1 - x_2 \end{aligned}$

Theorem 1. *Given a transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$, the followings are equivalent.*

1. *T is a linear transformation.*
2. *$T(c_1v_1 + \cdots + c_kv_k) = c_1T(v_1) + \cdots + c_kT(v_k)$ for all scalars $c_1, \ldots, c_k \in \mathbb{R}$ and all vectors $v_1, \ldots, v_k \in \mathbb{R}^n$.*
3. *There exists a unique matrix $A \in \mathbb{R}^{m\times n}$ such that $T(x) = Ax$ for all $x \in \mathbb{R}^n$.*

Remark. The matrix $A \in \mathbb{R}^{m\times n}$ has the form

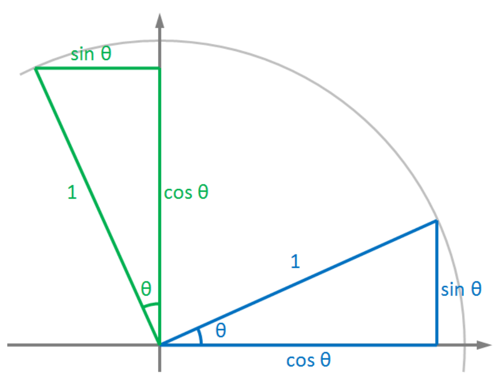
$$A = \begin{bmatrix} T(e_1) & \cdots & T(e_n) \end{bmatrix},$$

where $e_1, \ldots, e_n \in \mathbb{R}^n$ are the standard basis vectors of $\mathbb{R}^n$. This matrix A is called the **standard matrix** of this linear transformation T.

Example 4. A rotation of θ in the plane, which is a linear transformation $r\colon \mathbb{R}^2 \to \mathbb{R}^2$, has the standard matrix

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

The geometric meaning of this linear transformation is shown in the following figure.



Exercise. Find the standard matrix of the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ that corresponds to a reflection across the xy-plane.

Example 5. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation that satisfies

$$T(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 4 \\ 3 \\ -5 \end{bmatrix}, \quad T(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \quad T(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

To find the standard matrix of this linear transformation, we need to know the images of the standard vectors of $\mathbb{R}^3$. We already know one of the three images from the third equation. We can use linearity to find the other two.

$$T(\begin{bmatrix}1\\0\\0\end{bmatrix}) = T(\begin{bmatrix}1\\0\\1\end{bmatrix}) - T(\begin{bmatrix}0\\0\\1\end{bmatrix}) = \begin{bmatrix}0\\2\\1\end{bmatrix} - \begin{bmatrix}-1\\0\\1\end{bmatrix} = \begin{bmatrix}1\\2\\0\end{bmatrix},$$

$$T(\begin{bmatrix}0\\1\\0\end{bmatrix}) = T(\begin{bmatrix}1\\1\\0\end{bmatrix}) - T(\begin{bmatrix}1\\0\\0\end{bmatrix}) = \begin{bmatrix}4\\3\\-5\end{bmatrix} - \begin{bmatrix}1\\2\\0\end{bmatrix} = \begin{bmatrix}3\\1\\-5\end{bmatrix}.$$

Thus the standard matrix in this case is

$$A = \begin{bmatrix}T(e_1) & T(e_2) & T(e_3)\end{bmatrix} = \begin{bmatrix}1 & 3 & -1\\2 & 1 & 0\\0 & -5 & 1\end{bmatrix}.$$

Exercise. Find the standard matrix for each of the following linear transformation.

1. $T(x_1) = 2x_1$
2. $T(x_1, x_2) = (x_1, x_2)$
3. $T(x_1, x_2) = (x_2, 2x_1 - x_2, 0, x_1 + 4x_2)$
4. $T(x_1, x_2, x_3) = (0, 0, x_3)$
5. $T(x_1, x_2, x_3) = x_1 + 2x_2 + 3x_3$

One-to-one, Onto, and Bijective

Definition. Given a function $f\colon X \to Y$. We have the following definitions.

1. The function f is **one-to-one** (or **injective**) if $f(x_1) = f(x_2)$ implies that $x_1 = x_2$.
2. The function f is **onto** (or **surjective**) if for every $y \in Y$, there exists $x \in X$ such that $f(x) = y$.
3. The function f is **bijective**, or a **one-to-one correspondence** if f is both one-to-one and onto.

Example 6. The function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is neither one-to-one nor onto.

Example 7. The function $g\colon \mathbb{R} \to \mathbb{R}$ given by $g(x) = x^3$ is both one-to-one and onto.

Example 8. The function $\hat{f}\colon \mathbb{R} \to [0, \infty)$ given by $f(x) = x^2$ is onto, but not one-to-one.

Example 9. The function $\tilde{g}\colon \mathbb{R} \to \mathbb{R}$ given by $g(x) = x^3 - x$ is onto, but not one-to-one.

Example 10. The function $h\colon \mathbb{R} \to \mathbb{R}$ given by $h(x) = e^x$ is one-to-one, but not onto.

Exercise. Graph the functions given in Examples 6–10, verify and compare the claims.

The criteria for injectivity and surjectivity of linear transformations are much more elegant. Here are two theorems taken from the book. These theorems will be the tools to determine whether a linear transformation is one-to-one, onto, both, or neither.

Theorem 2. *A linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if and only if the equation $T(x) = 0$ has only the trivial solution.*

Theorem 3. *Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let $A \in \mathbb{R}^{m\times n}$ be its standard matrix. Then*

1. *T is one-to-one if and only if the columns of A are linearly independent;*
2. *T is onto if and only if the columns of A span $\mathbb{R}^m$.*

Remark. Given Theorem 3, we can perform row reduction on the standard matrix A to determine whether the corresponding linear transformation T is one-to-one or onto. We obtain the reduced echelon form of the matrix A, and find pivots of the matrix. If there is a pivot in each column of the matrix, then the columns of the matrix are linearly independent, hence the linear transformation is one-to-one; if there is a pivot in each row of the matrix, then the columns of A span the codomain $\mathbb{R}^m$, hence the linear transformation is onto. Therefore, we have the following corollary.

Corollary. *Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.*

1. *If $m < n$, then T cannot be one-to-one;*
2. *If $m > n$, then T cannot be onto.*

Example 11. Consider the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x) = Ax$, where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

The reduced echelon form of the standard matrix A is

$$\begin{bmatrix} \boxed{1} & 0 & -1 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

The matrix is missing a pivot in the third row and in the third column. Thus this linear transformation is neither one-to-one nor onto.

Exercise. Consider the linear transformation defined in Example 5. Is it one-to-one, onto, both, or neither?

Exercise. For each of the following standard matrix, determine if the corresponding linear transformation is (a) one-to-one, (b) onto. What can you say about the geometric meaning in each case?

1. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$
2. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
3. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$
4. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Exercise (hard)**.** Consider the linear transformation of taking derivative (see Example 3). Is it one-to-one?