Math 240 — TA: Shuyi Weng
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Matrix Operations

Addition and Scalar Multiplication

Addition is only defined for matrices of the same size, *i.e.*, matrices with the same number of rows AND the same number of columns. If A and B are both $m \times n$ matrices, their **sum** $A + B$ is the $m \times n$ matrix with each entry the sum of the corresponding entries in A and B.

Example 1. Let $A, B \in \mathbb{R}^{2\times 2}$ such that

$$A = \begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -5 \\ 3 & 2 \end{bmatrix}.$$

Then their sum is

$$A + B = \begin{bmatrix} 2+4 & 5+(-5) \\ (-3)+3 & 1+2 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 3 \end{bmatrix}.$$

The **scalar multiple** of a matrix is a new matrix of the same size with each entry the same multiple of the original matrix.

Example 2. Let A be the same as in Example 1, and let $c = 4$, then

$$cA = \begin{bmatrix} 2\times 4 & 5\times 4 \\ (-3)\times 4 & 1\times 4 \end{bmatrix} = \begin{bmatrix} 8 & 20 \\ -12 & 4 \end{bmatrix}.$$

There are some nice properties of matrix addition and scalar multiplication.

Theorem 1. *Let $A, B, C \in \mathbb{R}^{m\times n}$ be matrices of the same size, and let $r, s \in \mathbb{R}$ be scalars. Then*

1. $A + B = B + A$.
2. $(A + B) + C = A + (B + C)$.
3. $r(A + B) = rA + rB$.
4. $(r + s)A = rA + sA$.
5. $r(sA) = (rs)A$.

Matrix Multiplication

If $A \in \mathbb{R}^{m\times n}$, and $B \in \mathbb{R}^{n\times p}$, write $B = \begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_p \end{bmatrix}$, where $\mathbf{b}_1, \ldots, \mathbf{b}_p$ are the column vectors of B. Then the **product** AB is the $m \times p$ matrix with columns $A\mathbf{b}_1, \ldots, A\mathbf{b}_p$, *i.e.*,

$$AB = \begin{bmatrix} A\mathbf{b}_1 & \cdots & A\mathbf{b}_p \end{bmatrix}$$

Algorithm (Row-Column Rule)**.** If the product AB is defined, then the entry in the i-th row and the j-th column is the sum of the products of corresponding entries from the i-th row of A and the j-th column of B.

Example 3. Let $A, B \in \mathbb{R}^{2\times 2}$ such that

$$A = \begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -5 \\ 3 & 2 \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} 2\times 4 + 5\times 3 & 2\times(-5) + 5\times 2 \\ (-3)\times 4 + 1\times 3 & (-3)\times(-5) + 1\times 2 \end{bmatrix} = \begin{bmatrix} 23 & 0 \\ -9 & 17 \end{bmatrix},$$

and

$$BA = \begin{bmatrix} 4\times 2 + (-5)\times(-3) & 4\times 5 + (-5)\times 1 \\ 3\times 2 + 2\times(-3) & 3\times 5 + 2\times 1 \end{bmatrix} = \begin{bmatrix} 23 & 15 \\ 0 & 17 \end{bmatrix}.$$

Remark. Notice that in Example 3, $AB \neq BA$. In general, if AB and BA are both defined, they are not necessarily equal.

There are also some nice properties of matrix multiplication.

Theorem 2. *Let $A \in \mathbb{R}^{m\times n}$, and let B, C be matrices such that the sums and products are defined in each statement below. Let $r \in \mathbb{R}$ be a scalar. Then*

1. $(AB)C = A(BC)$.
2. $A(B + C) = AB + AC$.
3. $(B + C)A = BA + CA$.
4. $r(AB) = (rA)B = A(rB)$.
5. $I_m A = A = AI_n$.

Matrix Transpose

Given a matrix $A \in \mathbb{R}^{m\times n}$, its **transpose** A^T is the $n \times m$ matrix whose columns are formed from the corresponding rows of A. You may find examples of matrix transpose in the textbook.

Inverse of a Matrix

Definition. An $n \times n$ matrix A is said to be **invertible** if there exists an $n \times n$ matrix B such that $AB = BA = I_n$. In this case, we call B an **inverse** of A. A matrix is said to be **singular** if it is not invertible.

Remark. The inverse of an invertible matrix is essentially unique. Let A be an invertible matrix, and let B, C both be inverses of A. Then

$$B = BI = B(AC) = (BA)C = IC = C.$$

The unique inverse of A is denoted by A^{-1}.

Example 4. Let $A, B \in \mathbb{R}^{2\times 2}$ be matrices given by

$$A = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{ and } BA = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus B is the inverse of A.

Algorithm. To find the inverse of a matrix A, row reduce the augmented matrix $\begin{bmatrix} A & I \end{bmatrix}$. If A is row equivalent to I, then $\begin{bmatrix} A & I \end{bmatrix}$ is row equivalent to $\begin{bmatrix} I & A^{-1} \end{bmatrix}$. Otherwise, A does not have an inverse.

Remark. Assuming A is invertible, and let $B = A^{-1}$. Write

$$B = \begin{bmatrix} \mathbf{x}_1 & \cdots & \mathbf{x}_n \end{bmatrix}.$$

Then the product

$$AB = \begin{bmatrix} A\mathbf{x}_1 & \cdots & A\mathbf{x}_n \end{bmatrix} = \begin{bmatrix} \mathbf{e}_1 & \cdots & \mathbf{e}_n \end{bmatrix} = I_n.$$

Thus the algorithm described above is essentially solving n matrix equations $A\mathbf{x}_i = \mathbf{e}^i$ at the same time. The solutions to these matrix equations forms the columns of the inverse of the invertible matrix A.

Example 5. Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 5 & 6 \\ -2 & -7 & -9 \end{bmatrix}$$

Row reduce the augmented matrix

$$\begin{bmatrix} 1 & 3 & 4 & 1 & 0 & 0 \\ 2 & 5 & 6 & 0 & 1 & 0 \\ -2 & -7 & -9 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & 4 & 1 & 0 & 0 \\ 0 & -1 & -2 & -2 & 1 & 0 \\ 0 & -1 & -1 & 2 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & -1 & 0 \\ 0 & 0 & 1 & 4 & -1 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 3 & 4 & 1 & 0 & 0 \\ 0 & 1 & 0 & -6 & 1 & -2 \\ 0 & 0 & 1 & 4 & -1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 3 & 1 & 2 \\ 0 & 1 & 0 & -6 & 1 & -2 \\ 0 & 0 & 1 & 4 & -1 & 1 \end{bmatrix}$$

Thus the inverse of A is

$$A^{-1} = \begin{bmatrix} 3 & 1 & 2 \\ -6 & 1 & -2 \\ 4 & -1 & 1 \end{bmatrix}$$

Finding the inverse of a matrix is useful in the following scenario.

Theorem 3. *If A is an invertible $n \times n$ matrix, then for each $\mathbf{b} \in \mathbf{R}^n$, the matrix equation $A\mathbf{x} = \mathbf{b}$ has a unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.*

The inversion operation on square matrices satisfies the following two properties.

Theorem 4. *If $A \in \mathbb{R}^{n \times n}$ is invertible, so is A^{-1}, and $(A^{-1})^{-1} = A$.*

Theorem 5. *If $A, B \in \mathbb{R}^{n \times n}$ are both invertible, so is AB, and $(AB)^{-1} = B^{-1}A^{-1}$.*

The Invertible Matrix Theorem

Theorem 6. *Let $A \in \mathbb{R}^{n \times n}$. Then the following statements are equivalent.*

1. *A is invertible.*
2. *A is row equivalent to I_n.*
3. *A has n pivots in its reduced echelon form.*
4. *The matrix equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*
5. *The columns of A are linearly independent.*
6. *The linear transformation T defined by $T(\mathbf{x}) = A\mathbf{x}$ is one-to-one.*
7. *The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for every $\mathbf{b} \in \mathbb{R}^n$.*
8. *The columns of A span $\mathbb{R}^n$.*
9. *The linear transformation T defined by $T(\mathbf{x}) = A\mathbf{x}$ is onto.*
10. *There exists an $n \times n$ matrix B such that $AB = I_n$.*
11. *There exists an $n \times n$ matrix C such that $CA = I_n$.*