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Determinants

Definition

Recall that the **determinant** of a 2×2 matrix is defined by

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

The 2×2 matrix is invertible if and only if its determinant is nonzero. This notion of determinants can be extended to larger matrices.

Definition. The **determinant** of an $n \times n$ matrix $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ is recursively defined by

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(A_{1j}),$$

where A_{ij} is the matrix obtained by removing the i-th row and the j-th column of A.

Example 1. Find the determinant of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$$

Compute $\det(A) = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + a_{13}\det(A_{13})$:

$$\begin{aligned} \det(A) &= 1 \cdot \det \begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix} - 1 \cdot \det \begin{bmatrix} 1 & 3 \\ 1 & 6 \end{bmatrix} + 1 \cdot \det \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \\ &= 1 \cdot (12 - 9) - 1 \cdot (6 - 3) + 1 \cdot (3 - 2) = 1. \end{aligned}$$

Theorem 1. *The determinant of an $n \times n$ matrix can be computed by a cofactor expansion across any row or down any column. In another word,*

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$

for any fixed i-th row, and

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}),$$

for any fixed j-th column.

Remark. To simplify determinant computation, we usually pick the row/column with the least number of nonzero entries for cofactor expansion.

Example 2. Find the determinant of the matrix

$$A = \begin{bmatrix} 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 0 & 5 & 0 & -6 \end{bmatrix}$$

Use cofactor expansion at the third column, we have

$$\det(A) = 2 \cdot \det(A_{13}) = 2 \cdot \det \begin{bmatrix} 0 & 3 & -4 \\ -5 & -8 & 3 \\ 0 & 5 & -6 \end{bmatrix}$$

Then use cofactor expansion at the first column of A_{13}, we get

$$\det(A) = 2 \det(A_{13}) = 2 \cdot (-(-5)) \cdot \det \begin{bmatrix} 3 & -4 \\ 5 & -6 \end{bmatrix} = 10 \cdot (-18 + 20) = 20.$$

The next corollary follows directly from Theorem 1, yet it is very useful.

Corollary. *The determinant of a triangular matrix is the product of the entries on the main diagonal.*

Example 3. The determinant

$$\det \begin{bmatrix} 1 & 9 & 8 & 7 \\ 0 & 2 & 9 & 6 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 4 \end{bmatrix} = 1 \cdot 2 \cdot 3 \cdot 4 = 24.$$

Exercise. Find the determinants of the following matrices.

1. $\begin{bmatrix} 0 & 1 & 2 \\ 7 & 8 & 3 \\ 6 & 5 & 4 \end{bmatrix}$
2. $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$
3. $\begin{bmatrix} 6 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 2 & 1 \end{bmatrix}$
4. $\begin{bmatrix} 4 & 2 & 0 \\ 4 & 6 & 0 \\ 5 & 2 & 3 \end{bmatrix}$

Properties of Determinants

The determinant of a matrix change with row operations performed on it.

Theorem 2. *Let A be an $n \times n$ square matrix.*

1. *If a multiple of one row of A is added to another row to product a matrix B, then* $\det(B) = \det(A)$.
2. *If two rows of A are interchanged to produce B, then* $\det(B) = -\det(A)$.
3. *If one row of A is multiplied by a constant k to produce B, then* $\det(B) = k\det(A)$.

This theorem is very useful in computing determinants.

Example 4. Find the determinant of the matrix

$$A = \begin{bmatrix} 0 & 7 & 5 & 3 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & -1 \\ 2 & 2 & 2 & 4 \end{bmatrix}$$

Perform row operations to obtain its echelon form.

$$A \xrightarrow{\text{i}} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 7 & 5 & 3 \\ 1 & 1 & 2 & -1 \\ 2 & 2 & 2 & 4 \end{bmatrix} \xrightarrow{\text{ii}} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 7 & 5 & 3 \\ 1 & 1 & 2 & -1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \xrightarrow{\text{iii}} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 7 & 5 & 3 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & -1 & 1 \end{bmatrix} \xrightarrow{\text{iv}} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 7 & 5 & 3 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

Denote these four matrices by A_1, A_2, A_3, A_4, respectively.

(i) A_1 is obtained by swapping two rows of A, thus $\det(A_1) = -\det(A)$.

(ii) A_2 is obtained by multiplying the last row of A_1 by $1/2$, thus $\det(A_2) = \det(A_1)/2$.

(iii) A_3 is obtained by row replacement operations on A_2, thus $\det(A_3) = \det(A_2)$.

(iv) A_4 is obtained by swapping two rows of A_3, thus $\det(A_4) = -\det(A_3)$.

Combining these four results, we get $\det(A_4) = \det(A)/2$. Notice that A_4 is upper-triangular, thus $\det(A_4) = 1 \cdot 7 \cdot (-1) \cdot (-2) = 14$. It follows that $\det(A) = 2\det(A_4) = 28$.

The next corollary follows directly from Theorem 2.

Corollary. *Let A be an $n \times n$ square matrix, and let $c \in \mathbb{R}$. Then* $\det(cA) = c^n \det(A)$. *In particular,* $\det(-A) = (-1)^n \det(A)$.

The following theorem is an addition to the invertible matrix theorem.

Theorem 3. *A square matrix is invertible if and only if its determinant is nonzero.*

The determinant also preserves matrix multiplication and transpose operation.

Theorem 4. *Let A, B be $n \times n$ square matrices. Then*

1. $\det(AB) = \det(A)\det(B)$;
2. $\det(A^T) = \det(A)$.

Exercise. If A is invertible, what is $\det(A^{-1})$?

Exercise. If A is a square matrix such that $\det(A^5) = 0$, can A be invertible? Why?

Exercise. Let A and B be $n \times n$ matrices with $\det(A) = -4$ and $\det(B) = 3$. Compute

1. $\det(AB)$
2. $\det(3A)$
3. $\det(A^{-1})$
4. $\det(-A)$
5. $\det(A^3)$
6. $\det(B^T)$
7. $\det(B^{-1}AB)$
8. $\det(A^TBA)$

The corollary above shows that the determinant is not a linear map. However, it does possess a linearity property described by the following theorem.

Theorem 5. *Let A be an $n \times n$ square matrix, and write A in its column vectors*

$$A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}$$

Define a transformation $T\colon \mathbb{R}^n \to \mathbb{R}$ by

$$T(\mathbf{x}) = \det \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_{j-1} & \mathbf{x} & \mathbf{a}_{j+1} & \cdots & \mathbf{a}_n \end{bmatrix}.$$

Then T is a linear transformation.

Exercise. Verify Theorem 5 for any 2×2 matrix A.