Eigenvalues, Eigenvectors, and Diagonalization

The concepts of eigenvalues, eigenvectors, and diagonalization are best studied with examples. We will use some specific matrices as examples here.

Example 1. Consider the 2×2 matrix

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}.$$

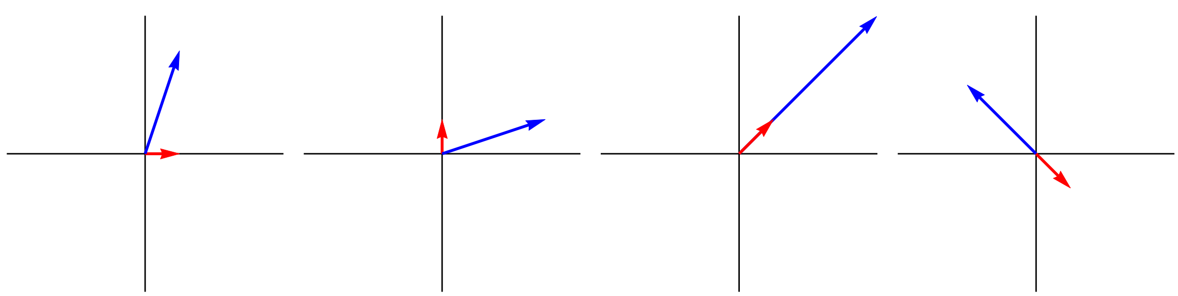
First, this matrix corresponds to a linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\mathbf{x}) = A\mathbf{x}$ for any vector $\mathbf{x} \in \mathbb{R}^2$. Let's first investigate this linear transformation. Consider the vectors

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \mathbf{x}_4 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

The images of these vectors under T are

$$A\mathbf{x}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \ A\mathbf{x}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \ A\mathbf{x}_3 = \begin{bmatrix} 4 \\ 4 \end{bmatrix}, \ A\mathbf{x}_4 = \begin{bmatrix} -2 \\ 2 \end{bmatrix},$$

If we draw the original vectors and their images on the same picture, respectively, we obtain the following.



Notice that the vector $\mathbf{x}_1$ forms an angle with its image $A\mathbf{x}_1$, so does $\mathbf{x}_2$. However, $\mathbf{x}_3$ and its image $A\mathbf{x}_3$ are colinear, and so are $\mathbf{x}_4$ and its image $A\mathbf{x}_4$. In another word, $A\mathbf{x}_3$ is a scalar multiple of $\mathbf{x}_3$, and $A\mathbf{x}_4$ is a scalar multiple of $\mathbf{x}_4$. Here, we call $\mathbf{x}_3$ and $\mathbf{x}_4$ the **eigenvectors** of the matrix A, and the scalars $\lambda_3, \lambda_4 \in \mathbb{R}$ such that $A\mathbf{x}_3 = \lambda_3\mathbf{x}_3$ and $A\mathbf{x}_4 = \lambda_4\mathbf{x}_4$ are called the **eigenvalues** of the matrix A. The precise definitions of these two concepts are given below.

Definition. Let A be an $n \times n$ matrix. If there is a scalar λ and a nonzero vector $\mathbf{v}$ such that they satisfy the matrix equation $A\mathbf{v} = \lambda\mathbf{v}$, then we call λ an **eigenvalue** of A, and $\mathbf{v}$ an **eigenvector** of A corresponding to the eigenvalue λ.

Remark. Given an eigenvalue λ of the matrix A, the eigenvector corresponding to λ is not unique. In fact, if $\mathbf{v}$ is such an eigenvector, then any nontrivial scalar multiple of v is also an eigenvector corresponding to λ. By linearity,

$$A(c\mathbf{v}) = c(A\mathbf{v}) = c(\lambda\mathbf{v}) = \lambda(c\mathbf{v})$$

for any scalar $c \neq 0$.

We want a systematic procedure to find the eigenvalues and eigenvectors of any given matrix A. In another word, we want to describe all possible solutions to the matrix equation $A\mathbf{v} = \lambda\mathbf{v}$, where both λ and $\mathbf{v}$ are unknown to us. However, notice that $\lambda\mathbf{v} = \lambda I\mathbf{v}$, where I is the identity matrix. Furthermore, we can move every term to the left-hand side of the equation, hence, equivalently, we solve for the homogeneous system

$$(A - \lambda I)\mathbf{v} = \mathbf{0}. \tag{1}$$

By the invertible matrix theorem, we know that this system has non-trivial solutions if and only if $\det(A - \lambda I) = 0$. This is called the **characteristic equation**. Notice that $\det(A - \lambda I)$ is a polynomial of degree n in the variable λ, and this polynomial is also called the **characteristic polynomial** of A. The problem of finding all eigenvalues now reduces to finding all roots of the characteristic polynomial, which we (usually) know how to do. After getting the eigenvalues, we can now solve the homogeneous system (1), or equivalently, the null space of the matrix $A - \lambda I$, to obtain the eigenvectors corresponding to each eigenvalue.

Remark. By the construction above, all eigenvectors corresponding to a specific eigenvalue λ form a linear subspace. This subspace is called the **eigenspace** of A corresponding to λ.

Example 2. We still consider the matrix

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}.$$

The characteristic polynomial of A is

$$\chi_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 9 = \lambda^2 - 2\lambda - 8 = (\lambda + 2)(\lambda - 4).$$

Hence, the eigenvalues of A are $\lambda_1 = -2$ and $\lambda_2 = 4$.

For the eigenvalue $\lambda_1 = -2$, its corresponding eigenspace is all solution to the matrix equation $(A + 2I)\mathbf{v} = \mathbf{0}$, *i.e.*, this eigenspace is

$$E_1 = \mathrm{nul}(A + 2I) = \mathrm{nul}\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}.$$

Similarly, the eigenspace corresponding to the eigenvalue $\lambda_2 = 4$ is

$$E_1 = \mathrm{nul}(A - 4I) = \mathrm{nul}\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

As verified in Example 1, the vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are eigenvectors of A.

One nice application of the eigenvalues and eigenvectors is to **diagonalize** a matrix. But before that, we need to introduce the concept of **similarity**.

Definition. Let A anb B be $n \times n$ matrices. We say that A and B are **similar** if there exists an invertible matrix P such that $A = PBP^{-1}$. Sometimes, the matrix P is referred to as the **change-of-coordinate matrix**.

Theorem 1. *If A and B are similar, then they have the same characteristic polynomial, and hence the same eigenvalues.*

Remark. The converse of this theorem is not true.

The motivation behind diagonalization of a matrix is to come up with a simpler way of computing matrix powers rather than arduously performing matrix multiplication. Given a matrix A with an eigenvalue λ and corresponding eigenspace E. We have a pretty good understanding of the action of A^k on the eigenspace E. Each iteration of A multiplies each vector in E by the scalar λ, and there are a total of k iterations, thus each vector in E is multiplied by λ^k after the action of A^k. We want extend this relatively simple multiplication to the whole vector space $\mathbb{R}^n$.

If we assume that the $n \times n$ matrix A has n eigenvalues $\lambda_1, \ldots, \lambda_n$, counting multiplicity, and n linearly independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$, with each $\mathbf{v}_i$ an eigenvector corresponding to the eigenvalue λ_i, then, by the invertible matrix theorem, the matrix

$$P = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix}$$

is invertible. Furthermore, since the columns are eigenvectors, we have

$$AP = \begin{bmatrix} \lambda_1 \mathbf{v}_1 & \cdots & \lambda_n \mathbf{v}_n \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} = PD,$$

where D is the diagonal matrix with eigenvalues on the diagonal entries. Therefore, we have

$$A = PDP^{-1},$$

and we have just **diagonalized** the matrix A.

Definition. A matrix A is **diagonalizable** if it is similar to a diagonal matrix. The **diagonalization** of a diagonalizable matrix A is the process described above, which achieves

$$A = PDP^{-1},$$

where P is invertible, and D is diagonal.

Example 3. We go back to the examples with the matrix

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}.$$

In Example 2, we computed the eigenvalues and their corresponding eigenvectors

$$\lambda_1 = -2,\ \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \lambda_2 = 4,\ \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

By the diagonalization process described above, let

$$D = \begin{bmatrix} -2 & 0 \\ 0 & 4 \end{bmatrix}, \text{ and } P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Then $A = PDP^{-1}$ is a diagonalization of A.

Example 4. If we want to compute the matrix power A^5, We do not have to perform tedious matrix multiplication seven times. Notice that

$$A^5 = (PDP^{-1})^5 = PD(P^{-1}P)D(P^{-1}P)D(P^{-1}P)D(P^{-1}P)DP^{-1} = PD^5P^{-1}$$

It is much simpler to do this computation, since the matrix power of a diagonal matrix is just the power of each diagonal entry. Therefore,

$$A^8 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} (-2)^5 & 0 \\ 0 & 4^5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -32 & 0 \\ 0 & 1024 \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 496 & 528 \\ 528 & 496 \end{bmatrix}$$

The following theorem characterizes the diagonalizable matrices.

Theorem 2. *Let A be an $n \times n$ matrix. The followings are equivalent.*

1. *A is diagonalizable.*
2. *A has n linearly independent eigenvectors.*
3. *The eigenvectors of A span $\mathbb{R}^n$.*
4. *The sum of the dimensions of the eigenspaces of A equals n.*

Remark. In other words, A is diagonalizable if and only if there are enough eigenvectors to form a basis of $\mathbb{R}^n$. We call such a basis an **eigenbasis** of $\mathbb{R}$.