

**NONCOMMUTATIVE DIFFERENTIAL
CALCULUS, HOMOTOPY BV ALGEBRAS
AND FORMALITY CONJECTURES**

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ABSTRACT. We define a notion of a strong homotopy BV algebra and apply it to deformation theory problems. Formality conjectures for Hochschild cochains are formulated. We prove several results supporting these conjectures.

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0. INTRODUCTION

In non commutative geometry, a manifold M is replaced by a possibly non commutative algebra A over a commutative ground ring k ($[C]$, $[G]$). Many constructions of differential calculus on M (vector fields, polyvector fields, forms, etc.) have their non commutative generalizations. For example, in case of vector fields, for any algebra A one can define the Lie algebra $Der(A)$ of derivations of A . If $A = C^\infty(M)$, one recovers the Lie algebra of vector fields.

If one passes to the graded space $\Gamma(M, \wedge^\bullet(TM))$ of polyvector fields on M , the non commutative analogue of this construction is not just a graded space but a

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complex of Hochschild cochains $C^\bullet(A, A)$ equipped with the Hochschild differential $\delta : C^\bullet(A, A) \rightarrow C^{\bullet+1}(A, A)$ defined for any algebra A (cf. [CE], [Ger]). By definition, $C^n(A, A)$ is the space of linear maps from $A^{\otimes n}$ to A . If $A = C^\infty(M)$ then the subcomplex of those cochains which are multidifferential operators has $\Gamma(M, \wedge^\bullet(TM))$ as its cohomology ([HKR]). In this case, for us $C^\bullet(A, A)$ will always stand for this subcomplex.

Now assume that a classical object has an algebraic structure (for example, a structure of a graded algebra of some kind). Then one could expect that its non commutative analogue (which is, as we see, a complex) should have a structure of a differential graded algebra of the same kind. We will see that this is true in many cases, and is expected to be true in many more, if one replaces differential graded algebras by more general objects, strong homotopy algebras (as suggested by [GJ1], [HJ], [K]; cf. also [DT]). The theory of strong homotopy structures was developed by Stasheff, Moore and others, mainly for topological applications (cf., for example, [S], [LS], [Mo]). One can generalize the notion of a differential graded associative algebra to define a strong homotopy associative algebra (or A_∞ algebra); in the same spirit one defines strong homotopy commutative algebras (C_∞ algebras) and strong homotopy Lie algebras (L_∞ algebras). Below we will have more examples of strong homotopy algebras.

A differential graded algebra is a strong homotopy algebra. However, in a strong homotopy category morphisms are defined differently even for ordinary algebras. A crucial property of strong homotopy structures is that for them being quasi-isomorphic is an equivalence relation. (One says that \mathcal{A}_1^\bullet is quasi-isomorphic to \mathcal{A}_2^\bullet if there is a morphism from \mathcal{A}_1^\bullet to \mathcal{A}_2^\bullet which induces an isomorphism on cohomology).

The space $\Gamma(M, \wedge^{\bullet+1}(TM))$ possesses a structure of a graded Lie algebra given by the Schouten - Nijenhuis bracket $[\ , \]_S$. On the non commutative side, the Gerstenhaber bracket $[\ , \]_G$ makes $C^{\bullet+1}(A, A)$ a differential graded Lie algebra for any algebra A . By the formality theorem of Kontsevich, if $A = C^\infty(M)$, then the differential graded Lie algebras $(C^{\bullet+1}(A, A), \delta, [\ , \]_G)$ and $(\Gamma(M, \wedge^{\bullet+1}(TM)), \delta = 0, [\ , \]_S)$ are quasi-isomorphic as strong homotopy Lie algebras, or L_∞ algebras (cf. [K]).

From this example one sees what should be the principles of non commutative calculus:

- A). An object from classical calculus should have its non commutative analogue.
- B). If there is an algebraic structure on a classical object, the corresponding non commutative object should possess a similar structure up to strong homotopy.
- C). Thus, if $A = C^\infty(M)$, one gets two strong homotopy structures, one coming from classical calculus and another from non commutative one. Those two structures should be equivalent (formality).

As an algebraic structure gets richer, B), obviously, gets more difficult to prove. On the other hand, C) becomes easier. Indeed, as a general rule, the structure coming from non commutative calculus is equivalent to a deformation of the classical structure; but the richer the structure, the harder it is to deform. Let us illustrate this principle by the following example from [T].

A graded space \mathcal{A}^\bullet is a Gerstenhaber algebra if two operations are defined:

$$\cdot : \mathcal{A}^i \otimes \mathcal{A}^j \rightarrow \mathcal{A}^{i+j}$$

and

$$[\ , \] : \mathcal{A}^{i+1} \otimes \mathcal{A}^{j+1} \rightarrow \mathcal{A}^{i+j+1}$$

such that \mathcal{A}^\bullet is a graded commutative algebra with respect to the product \cdot , $\mathcal{A}^{\bullet+1}$ is a graded Lie algebra with respect to the bracket $[\ , \]$, and the two operations satisfy the Leibnitz identity

$$[a, b \cdot c] = [a, b] \cdot c + (-1)^{(|a|-1)|b|} b \cdot [a, c]$$

for any homogeneous elements a, b, c of \mathcal{A}^\bullet . It is well known that the Hochschild cohomology $HH^\bullet(A, A)$, which is by definition the cohomology of the Hochschild complex $C^\bullet(A, A)$, has a canonical structure of a Gerstenhaber algebra ([Ger]).

In [T], the following is proved.

Theorem 0.1. *For any algebra A the space $C^\bullet(A, A)$ is a strong homotopy Gerstenhaber algebra. Its underlying L_∞ structure is given by the Gerstenhaber bracket. The induced structure of a Gerstenhaber algebra on cohomology is given by the wedge product and the Schouten bracket.*

Cf. Section 1 for the definition of a strong homotopy Gerstenhaber algebra. We call such algebras G_∞ algebras.

Theorem 0.2. *For any algebra A , the G_∞ algebra $C^\bullet(A, A)$ is quasi-isomorphic to a deformation of the G_∞ algebra $HH^\bullet(A, A)$.*

Theorem 0.3. *If $M = \mathbb{R}^n$ and $A = C^\infty(M)$, the G_∞ algebra $\Gamma(M, \wedge^\bullet(TM))$ has no non-trivial G_∞ deformations. For any M , $C^\bullet(A, A)$ and $\Gamma(M, \wedge^\bullet(TM))$ are quasi-isomorphic as G_∞ algebras.*

Theorem 0.3 generalizes the formality theorem of Kontsevich. We would like to apply the above scheme to other constructions of classical differential calculus.

We sketch the proofs of the above theorems in Section 3. For Theorem 0.1 a new version of the proof is presented, more streamlined as compared to [T]. This version is based on using Etingof-Kazhdan dequantization instead of quantization, an idea suggested to us by P. Etingof.

Now let us discuss non commutative analogues of other constructions from calculus. A non commutative analogue of the space of differential forms is well known to be the Hochschild chain complex $C_\bullet(A, A)$ with the differential $b : C_\bullet(A, A) \rightarrow C_{\bullet-1}(A, A)$ ([CE], [HKR]). One sees in this example that the principle B) above has its obvious limitations: for example, the exterior product of forms does not have an analogue on the Hochschild chain complex, unless A is commutative. (A non commutative analogue of the product can be restored if one passes from the algebra of forms to the algebra of all differential operators on forms, cf. [NT]). In this paper we will mainly discuss not an algebra structure but a module structure on forms.

There are two standard pairings between polyvector fields and forms: for $\pi \in \Gamma(M, \wedge^k(TM))$, one defines

$$i_\pi : \Omega^\bullet(M) \rightarrow \Omega^{\bullet-k}(M)$$

and

$$L_\pi : \Omega^\bullet(M) \rightarrow \Omega^{\bullet-k+1}(M)$$

where i_π is the contraction of a form by a polyvector and

$$L_\pi = [d, i_\pi]$$

The following identities hold (we denote the Schouten bracket $[\ , \]_S$ simply by $[\ , \]$):

$$(0.1) \quad [L_\pi, L_\varphi] = L_{[\pi, \varphi]}; \quad [i_\pi, L_\varphi] = i_{[\pi, \varphi]}; \quad [i_\pi, i_\varphi] = 0$$

as well as

$$(0.2) \quad i_{\pi \cdot \varphi} = i_\pi \cdot i_\varphi; \quad L_{\pi \cdot \varphi} = L_\pi \cdot i_\varphi + (-1)^{|\pi|} i_\pi \cdot L_\varphi$$

How to rephrase these identities in terms of Gerstenhaber algebras?

A Gerstenhaber module over a Gerstenhaber algebra \mathcal{A}^\bullet is a graded space \mathcal{M}^\bullet together with a structure of a Gerstenhaber algebra on $\mathcal{A}^\bullet \oplus \mathcal{M}^\bullet$ such that \mathcal{A}^\bullet is a Gerstenhaber subalgebra and $\mathcal{M}^\bullet \cdot \mathcal{M}^\bullet = [\mathcal{M}^\bullet, \mathcal{M}^\bullet] = 0$.

For any Gerstenhaber algebra \mathcal{A}^\bullet one constructs canonically another Gerstenhaber algebra $\mathcal{A}^\bullet[\epsilon]$ as follows: let ϵ be a formal parameter of degree one, $\epsilon^2 = 0$. Define

$$\mathcal{A}^\bullet[\epsilon]^n = \mathcal{A}^n + \epsilon \mathcal{A}^{n-1}$$

Define the product \star and the bracket $\{ \ , \ }$ on $\mathcal{A}^\bullet[\epsilon]$ to be $k[\epsilon]$ -bilinear operations such that

$$a \star b = ab + \epsilon(-1)^{|a|}[a, b]$$

and

$$\{a, b\} = [a, b]$$

for any homogeneous elements a and b of \mathcal{A}^\bullet . We will call the Gerstenhaber algebra $\mathcal{A}^\bullet[\epsilon]$ the canonical deformation of \mathcal{A}^\bullet (with the odd parameter ϵ). It is convenient to describe the formulas (0.1, 0.2) above in the language of canonical deformations as follows:

Lemma 0.4. *Formulas*

$$\begin{aligned} (\pi + \epsilon\varphi) \star \alpha &= (-1)^\pi i_\pi \alpha \\ \{\pi + \epsilon\varphi, \alpha\} &= L_\pi \alpha + i_\varphi \alpha \end{aligned}$$

define a structure of a $\Gamma(M, \wedge^\bullet(TM))[\epsilon]$ -module on $\Omega^\bullet(M)$.

How to formulate a non commutative analogue of the above situation? First of all, for any algebra A , for any $D \in C^k(A, A)$ one can define operators

$$i_D : C_\bullet(A, A) \rightarrow C_{\bullet-k}(A, A)$$

and

$$L_D : C_\bullet(A, A) \rightarrow C_{\bullet-k+1}(A, A);$$

operators L_D define on $C_\bullet(A, A)$ a structure of a graded module over the differential graded Lie algebra $C^{\bullet+1}(A, A)$. We show in subsection 1.2 how to extend the construction of the canonical deformation $\mathcal{A}^\bullet[\epsilon]$ to strong homotopy Gerstenhaber algebras; more precisely, we will construct a canonical strong homotopy Gerstenhaber algebra structure on $\mathcal{A}^\bullet[\epsilon]$ for any strong homotopy Gerstenhaber algebra \mathcal{A}^\bullet .

Conjecture 0.5. *For any algebra A , on $C^\bullet(A, A)$ there is a canonical structure of a G_∞ module over the canonical deformation $(C^\bullet(A, A)[\epsilon], \delta)$. This structure induces the structure of a module over the differential graded Lie algebra $C^{\bullet+1}(A, A)$ given by operators L_D . The induced structure of a Gerstenhaber module on cohomology is as in Lemma 0.4.*

Corollary 0.5.1. *For any algebra A , on $C^\bullet(A, A)$ there is a canonical structure of a L_∞ module over the differential graded Lie algebra $C^\bullet(A, A)[\epsilon]$ which extends the structure of a module over the differential graded Lie algebra $C^{\bullet+1}(A, A)$ given by operators L_D .*

This last statement was proved in [DT].

One can prove a statement which is weaker than Conjecture 0.5:

Theorem 0.6. *For any algebra A , there is a G_∞ algebra structure on $C^\bullet(A, A)[\epsilon]$ such that $C_\bullet(A, A)$ is a G_∞ module over $C^\bullet(A, A)[\epsilon]$. The induced structure on $C_\bullet(A, A)$ of a module over the differential graded Lie algebra $C^{\bullet+1}(A, A)$ is given by operators L_D . The induced structure of a Gerstenhaber module on cohomology is as in Lemma 0.4.*

When $A = C^\infty(M)$, one can show that the G_∞ algebra $C^\bullet(A, A)[\epsilon]$ from Theorem 0.6 is quasi-isomorphic to the canonical deformation $\Gamma(M, \wedge^\bullet(TM))[\epsilon]$. Modulo ϵ , at the level of L_∞ algebras, this quasi-isomorphism is given by the formality theorem of Kontsevich. Therefore one has two G_∞ modules over $\Gamma(M, \wedge^\bullet(TM))[\epsilon]$: one is $\Omega^\bullet(M)$ and the other is $C_\bullet(A, A)$. One can prove as in [T] that those G_∞ modules are quasi-isomorphic.

By virtue of the formality theorem of Kontsevich, the differential graded Lie algebra $\Gamma(M, \wedge^{\bullet+1}(TM))$ is quasi-isomorphic to $C^{\bullet+1}(A, A)$ which acts on $C_\bullet(A, A)$; therefore the latter is an L_∞ module over $\Gamma(M, \wedge^{\bullet+1}(TM))$.

Theorem 0.7. *(Formality theorem for Hochschild chains). As L_∞ modules over $\Gamma(M, \wedge^{\bullet+1}(TM))$, $C_\bullet(A, A)$ and $\Omega^\bullet(M)$ are quasi-isomorphic.*

We show in [Ts] that Theorem 0.7 allows to compute the Hochschild homology of $A(\pi)$, where π is a Poisson stricture and $A(\pi)$ is the deformation of $C^\infty(M)$ given by the Kontsevich theorem ([K]). $A(\pi)$ is an algebra over $\mathbb{C}[[t]]$ whose reduction modulo t is isomorphic to $C^\infty(M)$. In particular, one gets

Corollary 0.7.1. *The space of traces on the deformed algebra $A(\pi)$ is isomorphic to the space of $\mathbb{C}[[t]]$ -valued distributions that are zero on all Poisson brackets.*

Now we would like to extend the above constructions of non commutative calculus to include the de Rham differential d . Recall that a Batalin-Vilkovisky (or BV) algebra is a Gerstenhaber algebra \mathcal{A}^\bullet together with an operator $\Delta : \mathcal{A}^\bullet \rightarrow \mathcal{A}^{\bullet-1}$ such that $\Delta^2 = 0$, degree of Δ is 1, and

$$\Delta(ab) = \Delta(a)b + (-1)^{|a|}a\Delta(b) + (-1)^{|a|}[a, b]$$

for any homogeneous elements a and b of \mathcal{A}^\bullet . The canonical deformation $\mathcal{A}^\bullet[\epsilon]$ has a canonical BV operator $\Delta = \partial/\partial\epsilon$.

One defines a BV module $(\mathcal{M}^\bullet, \Delta_{\mathcal{M}^\bullet})$ over a BV algebra \mathcal{A}^\bullet as a Gerstenhaber module \mathcal{M}^\bullet such that $\mathcal{A}^\bullet \oplus \mathcal{M}^\bullet$ is a BV operator on the corresponding Gerstenhaber algebra $\mathcal{A}^\bullet \oplus \mathcal{M}^\bullet$.

Lemma 0.8. *The de Rham complex $(\Omega^\bullet(M), d)$ is a BV module over the BV algebra $(\Gamma(M, \wedge^\bullet(TM))[\epsilon], \partial/\partial\epsilon)$.*

How to define a notion of a strong homotopy BV algebra (or BV_∞ algebra)? This question is of certain importance because it is relevant for studying mirror symmetries, generalized period maps for Calabi - Yau manifolds, Frobenius manifolds, and related topics ([BK], [B], [M]). In Section 2, we propose an answer which is strongly suggested by our context. This answer combines the standard technique of strong homotopy structures with Kravchenko's construction from [Ko]. A BV_∞ operator in our sense is an odd differential operator of square zero on a certain infinite dimensional graded space subject to additional properties. One of those properties is a restriction on the order (and the principal symbols) of its graded components, the other is that it has to preserve some natural filtration. A BV_∞ algebra in our sense is an algebra not over an operad but over a PROP.

Let us mention the reasons why we prefer our definition of a BV_∞ algebra. First, it is substantially more relaxed than the standard definition using Koszul operads. In the context of this definition there is still a natural way to define a BV_∞ morphism, such that being quasi-isomorphic is an equivalence relation. One can prove an analogue of another standard property of strong homotopy algebras, namely that any BV_∞ algebra is isomorphic to a direct sum of a minimal BV_∞ algebra and a linearly contractible BV_∞ algebra (Section 2; compare to [K]). Another reason is the following. Note that the G_∞ structure on $C^\bullet(A, A)$ which is given by Theorem 0.1 is of a special kind: it arises from the chain complex of a differential graded Lie bialgebra (cf. subsection 1.2). For such algebras, the obstructions to being a BV_∞ algebra in our sense lie in the cohomology of the deformation complex of this Lie bialgebra [LR]. Conjecturally, the constructions from [EK] extend to relate this cohomology to that of the deformation complex of the Hopf algebra which is the canonical quantization of the above Lie bialgebra; this last Hopf algebra is the cobar construction of the algebra of Hochschild cochains equipped with the product from [GJ], [GV]; cf. also [T].

We show in Section 2 that for any G_∞ algebra \mathcal{A}^\bullet , the canonical deformation $\mathcal{A}^\bullet[\epsilon]$ has a canonical BV_∞ structure.

Conjecture 0.9. *For any algebra A , the cyclic differential B on the Hochschild chain complex $C_\bullet(A, A)$ extends canonically to a structure of a BV_∞ module over the BV_∞ algebra $C^\bullet(A, A)[\epsilon]$, the canonical deformation of the Gerstenhaber algebra $C^\bullet(A, A)$ from Theorem 0.1.*

If the above is true, then one easily proves the following

Corollary 0.9.1. *For any algebra A , the negative cyclic complex $CC_\bullet^-(A) = (C_\bullet(A, A)[[u]], b + uB)$ has a canonical structure of an L_∞ module over the differential graded Lie algebra $(C^\bullet(A, A)[\epsilon][[u]], \delta + u\partial/\partial\epsilon)$. Modulo u this structure is the same as in Corollary 0.5.1.*

A statement which is very close to the above corollary was proven in [DT]. Note that the fact that was used in proving index theorems (for example, in [BNT]) is the existence of the fundamental homomorphism χ from [DT], Section 8. This is a corollary of 0.9.1. Another, more effective way to construct this homomorphism is contained in [NT], Section 4.

Now let us return to formality theorems for cyclic chains. Note that, because of Theorem 0.3, for $A = C^\infty(M)$ BV_∞ algebras $\Gamma(M, \wedge^\bullet(TM))[\epsilon]$ and $C^\bullet(A, A)[\epsilon]$

are quasi-isomorphic. Thus, $C_\bullet(A, A)$ and $\Omega^\bullet(M)$ are both BV_∞ modules over $\Gamma(M, \wedge^\bullet(TM))[\epsilon]$.

Conjecture 0.10. *For $A = C^\infty(M)$, $C_\bullet(A, A)$ and $\Omega^\bullet(M)$ are quasi-isomorphic as BV_∞ modules over $\Gamma(M, \wedge^\bullet(TM))[\epsilon]$.*

In particular, the following formality conjecture for cyclic chains would be true (cf. [Ts]):

Conjecture 0.11. *For $A = C^\infty(M)$, $C_\bullet(A, A)$ and $\Omega^\bullet(M)$ are quasi-isomorphic as L_∞ modules over $\Gamma(M, \wedge^{\bullet+1}(TM))[\epsilon]$.*

Let us mention a geometric application (cf. [Ts]). Let π be a Poisson manifold M . By definition π is regular if its symplectic leaves form a foliation. Denote this foliation by \mathcal{F}_π . The tangent bundle to \mathcal{F}_π has $Sp(2n)$ as its structure group. Reducing the structure group to the maximal compact subgroup, one can view \mathcal{F}_π as a $U(n)$ -bundle; let $\hat{A}(\mathcal{F}_\pi)$ be the \hat{A} polynomial of Chern classes of this bundle. The following is a corollary of the conjecture above:

Conjecture 0.12. *The construction of $\hat{A}(\mathcal{F}_\pi)$ can be generalized to the non-regular case.*

Now let us turn to another structure from classical calculus. Assume that we are given an n -dimensional manifold M with a volume form Ω . One defines the divergence operator

$$\Delta : \Gamma(M, \wedge^\bullet(TM)) \rightarrow \Gamma(M, \wedge^{\bullet-1}(TM))$$

to be $\Delta = I^{-1}dI$ where $I : \Gamma(M, \wedge^\bullet(TM)) \rightarrow \Omega^{n-\bullet}(M)$ sends a polyvector π to the form $i_\pi\Omega$. It is well known (cf., for example, [BK]) that Δ is a BV operator on the Gerstenhaber algebra $\Gamma(M, \wedge^\bullet(TM))$.

As a non commutative analogue of the above situation, one can consider an algebra A with a trace Tr . We assume the pairing $\langle a, b \rangle = Tr(ab)$ to be non degenerate, meaning, essentially, that for any Hochschild one-cochain there exists unique one-cochain D^* such that $\langle a, Db \rangle = \langle D^*a, b \rangle$ for all a and b in A . Strictly speaking, we need a somewhat stronger condition: for any Hochschild m -cochain D there exists a unique m -cochain D^* such that

$$Tr(a_0D(a_1, \dots, a_m)) = Tr(a_mD^*(a_0, \dots, a_{m-1}))$$

for all $a_i \in A$ (cf. [Sh]; ; note that our definition of Hochschild cochains sometimes depends on an algebra A , like in the case $A = C^\infty(M)$).

Given such a trace, one can define a map of complexes

$$J : C^\bullet(A, A) \rightarrow C_\bullet(A, A)^*$$

and a differential

$$B_0 : C^\bullet(A, A) \rightarrow C^{\bullet-1}(A, A)$$

such that, if B is Connes' cyclic differential on $C_\bullet(A, A)$, one has $B^*J = JB_0$; cf. [CFS]

Conjecture 0.13. *The operator B_0 extends to a BV_∞ structure on the G_∞ algebra $C^\bullet(A, A)$.*

From this, along the lines of [T], one would be able to obtain the cyclic formality conjecture of Kontsevich (cf. [Sh]).

Conjecture 0.13 would imply a weaker form of Conjecture 0.9 which is a cyclic analogue of Theorem 0.6. Indeed, one can apply Conjecture 0.13 to the algebra R which is a trivial extension of A by the square zero ideal A^* (the space dual to A , viewed as an A -bimodule). The trace on R is given by

$$Tr(a, \lambda) = \lambda(1)$$

for $a \in A$, $\lambda \in A^*$.

Finally, let us mention three topics of non commutative calculus that are left out of this paper. First, one can try to prove another version of Conjecture 0.9 using “an infinitesimal analogue” of the BV operad. Second, a question very closely related to Conjecture 0.13 is the study of algebraic structures on the Hochschild and cyclic complexes of Hopf algebras. These complexes were defined recently by Connes and Moscovici in [CM], [CM1] with the motivation to be able to prove index theorems by direct calculations (as was, in a different context, the motivation for [DT], [NT]). Third, as we mentioned above, there is a non commutative analogue of the algebra $D(\Omega^\bullet(M))$ of differential operators on differential forms ([NT]). This is an A_∞ algebra whose cohomology is $D(\Omega^\bullet(M))$, with an A_∞ derivation acting on cohomology as $[d_{DR}, _]$ where d_{DR} is the de Rham differential. As a complex, this A_∞ algebra is equal to $C_\bullet(C^\bullet(A, A), C^\bullet(A, A))$, the chain complex of the algebra of cochains of A . The product on it is very closely related to the product from [GJ] and [GV] that we are using in Section 3. The first term of the A_∞ derivation is the cyclic differential B (the higher terms are very closely related to [GJ1] and [HJ]).

The algebra $D(\Omega^\bullet(M))$ has an additional structure, an “order” filtration F_\bullet where, for a function f and a vector field X , operators f and i_X are of “order” zero and operators L_X and df are of “order” one. Thus we have the following algebraic structure: a graded algebra D with an increasing filtration F_\bullet such that $gr_F D$ is commutative, together with an extra derivation d of degree one which satisfies the transversality property that dF_p is inside F_{p+1} (as in the theory of variations of Hodge structures).

Such an algebraic structure admits a strong homotopy version (a D_∞ algebra). Conjecturally, for any A , $C_\bullet(C^\bullet(A, A), C^\bullet(A, A))$ has a canonical D_∞ structure. This statement generalizes Theorem 0.1, as well as the main result of [NT]. It is closely related to the theorems and conjectures above, as well as to constructions of [B]. The exact nature of these relationships is not yet clear.

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1. Homotopy Gerstenhaber algebras. For a graded Lie algebra \mathfrak{g}^\bullet put

$$\wedge^k \mathfrak{g}^\bullet = S^k(\mathfrak{g}^\bullet[-1])$$

(as usual, for a complex, $V^\bullet, V^\bullet[m]^i = V^{i+m}$). There is unique bracket $[\ , \]$ on $\bigoplus_{k \geq 0} \wedge^k \mathfrak{g}^\bullet$ such that:

1. The restriction of $[\ , \]$ to $\wedge^1 \mathfrak{g}^\bullet = \mathfrak{g}^\bullet$ is the Lie bracket on \mathfrak{g}^\bullet , and $[\wedge^0 \mathfrak{g}^\bullet, \wedge^1 \mathfrak{g}^\bullet] = 0$.

2. With this bracket and with the exterior product, $\wedge^\bullet \mathfrak{g}^\bullet$ becomes a Gerstenhaber algebra.

If $\mathfrak{g}^\bullet = \mathfrak{g}$, the Lie algebra of a Lie group G , then this Gerstenhaber algebra is the algebra of left invariant polyvector fields on G .

We want to give a definition of a G_∞ structure on a graded space \mathcal{A}^\bullet . In what follows, we will take a liberty of talking about algebras, not coalgebras; therefore, strictly speaking, our definition will be correct only in the case when \mathcal{A}^\bullet is finite dimensional. The definition we are about to give translates into a collection of maps from $(\mathcal{A}^\bullet)^*$ to $(\mathcal{A}^\bullet)^{* \otimes n}$ subject to some identities ($*$ stands for dual); to make it correct, one has to replace those maps by maps from $(\mathcal{A}^\bullet)^{\otimes n}$ to \mathcal{A}^\bullet , subject to dual identities. Furthermore, all our strong homotopy algebra structures will be described in terms of differential operators on some algebra; that algebra will be always complete with respect to a filtration, and we will always assume our operators to be continuous in the corresponding topology.

For any graded space V^\bullet consider the free graded Lie algebra $\text{Lie}(V^\bullet)$ generated by V^\bullet . Let $\text{Lie}^k(V^\bullet)$ be defined inductively by

$$\text{Lie}^1(V^\bullet) = V^\bullet; \text{Lie}^k(V^\bullet) = [V^\bullet, \text{Lie}^{k-1}(V^\bullet)]$$

One has

$$\text{Lie}(V^\bullet) = \bigoplus_{k=1}^{\infty} \text{Lie}^k(V^\bullet)$$

By \mathcal{F}^\bullet we will denote the filtration on $\bigoplus \wedge^k(\text{Lie}(V^\bullet))$ induced by the filtration $\mathcal{F}^\bullet = \text{Lie}^{\geq k}(V^\bullet)$, and by $\wedge^\bullet(\text{Lie}(V^\bullet))$ the completion of $\bigoplus \wedge^k$ with respect to this filtration.

As usual, we define the grading on the dual space to a graded space V^\bullet by $(V^\bullet)^{*i} = (V^{-i})^*$.

Definition 1.1. A G_∞ algebra is a graded space \mathcal{A}^\bullet together with an operator ∂ of degree 1 on $\wedge(\text{Lie}^\bullet(\mathcal{A}^\bullet[1]^*))$ such that $\partial^2 = 0$ and ∂ is a derivation with respect to both the product and the bracket.

Remark 1.2. Sometimes a different grading on $\wedge^\bullet(\text{Lie}(\mathcal{A}^\bullet[1]^*))$ is used, in which the bracket has degree 0 and the product has degree 1. The reason for this is to make $\wedge^\bullet(\text{Lie}(\mathcal{A}^\bullet[1]^*))$ an algebra over the operad dual to e_2 where e_2 is the operad algebras over which are Gerstenhaber algebras.

A derivation ∂ from Definition 1.1 preserves two multiplicative ideals of $\wedge^\bullet(\text{Lie}(\mathcal{A}^\bullet[1]^*))$: I_1 , generated by the commutant of $\text{Lie}(\mathcal{A}^\bullet[1]^*)$, and $I_2 = \wedge^{\geq 2}(\text{Lie}(\mathcal{A}^\bullet[1]^*))$. The quotient of $\wedge^\bullet(\text{Lie}(\mathcal{A}^\bullet[1]^*))$ by I_1 is equal to $\wedge^\bullet(\text{Lie}(\mathcal{A}^\bullet[1]^*))$; since

$$\wedge^k(\mathcal{A}^\bullet[1]^*) \simeq S^k(\mathcal{A}^\bullet[1]^*[-1]) = S^k(\mathcal{A}^\bullet[2]^*),$$

the differential induced by ∂ is by definition an L_∞ structure on the space $\mathcal{A}^\bullet[1]$. On the other hand, the quotient of $\wedge^\bullet(\text{Lie}(\mathcal{A}^\bullet[1]^*))$ by I_2 is isomorphic to $\text{Lie}(\mathcal{A}^\bullet[1]^*)$, thus ∂ induces on it a differential which is by definition a structure of a C_∞ algebra on \mathcal{A}^\bullet .

For a G_∞ algebra \mathcal{A}^\bullet , a derivation ∂ of $\wedge^\bullet(\text{Lie}(\mathcal{A}^\bullet[1]^*))$ is uniquely determined by its restriction to $\mathcal{A}^\bullet[1]^*$. The components of this restriction are maps

$$\mathcal{A}^\bullet[1]^* \rightarrow \text{Lie}^{k_1}(\mathcal{A}^\bullet[1]^*) \wedge \dots \wedge \text{Lie}^{k_n}(\mathcal{A}^\bullet[1]^*)$$

We view $\text{Lie}(\mathcal{A}^\bullet[1]^*)$ as the subspace of primitive elements of $U(\text{Lie}(\mathcal{A}^\bullet[1]^*) = T(\mathcal{A}^\bullet[1]^*)$, therefore its dual is a quotient of $T(\mathcal{A}^\bullet[1])$. This quotient is taken by the linear span of shuffle products of tensors of positive degree.

We denote by m_{k_1, \dots, k_n} the map conjugate to the map above:

$$m_{k_1, \dots, k_n} : (\mathcal{A}^\bullet)^{\otimes k_1} \otimes \dots \otimes (\mathcal{A}^\bullet)^{\otimes k_n} \rightarrow \mathcal{A}^\bullet$$

These operations have the following properties:

1. Degree of m_{k_1, \dots, k_n} is equal to $3 - n - k_1 - \dots - k_n$.
2. Operations $m_{k_1, \dots, k_n}(a_{11}, \dots, a_{1, k_1}; \dots; a_{n1}, \dots, a_{n, k_n})$ are Harrison cochains in every group of variables $a_{i1}, \dots, a_{i, k_i}$ (recall that a Harrison cochain is a multi-linear map annihilating all shuffle products).
3. Operations m_{k_1, \dots, k_n} are invariant under permutations of blocks $(a_{i1}, \dots, a_{i, k_i})$ with appropriate signs.

The condition $\partial^2 = 0$ now can be translated into a quadratic identity for m_{k_1, \dots, k_n} . In particular, $\delta = m_1$ is a differential of degree 1 on \mathcal{A}^\bullet ; $ab = (-1)^{|a|}m_2(a, b)$ is a product of degree zero and, more generally, operations m_k define a C_∞ structure on \mathcal{A}^\bullet ; $[a, b] = (-1)^{|a|-1}m_{1,1}(a, b)$ is a bracket of degree -1 and, more generally, the operations $m_{1, \dots, 1}$ define a L_∞ structure on $\mathcal{A}^\bullet[1]$; the operation $m_{1,2}(a; b, c)$ has degree -2 and is up to a sign a homotopy for the Leibnitz identity, etc.

Example 1.2. Let \mathcal{A}^\bullet be a Gerstenhaber algebra. Then one makes it a G_∞ algebra by defining the only non-zero operations m_{k_1, \dots, k_n} to be $m_{1,1}(a, b) = (-1)^{|a|}ab$, $m_2(a, b) = (-1)^{|a|-1}[a, b]$.

Example 1.3. Let \mathfrak{g}^\bullet be a graded Lie bialgebra [Dr]. On $\wedge^\bullet(\mathfrak{g}^\bullet)$, let ∂^{coLie} be the derivation whose restriction to \mathfrak{g}^\bullet is the cobracket $\mathfrak{g}^\bullet \rightarrow \wedge^2(\mathfrak{g}^\bullet)$. This is the standard cochain differential of the Lie coalgebra \mathfrak{g}^\bullet . If \mathfrak{g}^\bullet is free as a Lie algebra, $\mathfrak{g}^\bullet = \text{Lie}(V^\bullet)$, then ∂^{coLie} defines a G_∞ structure on the graded space $\mathcal{A}^\bullet = (V^\bullet)^*[-1]$. More generally, if \mathfrak{g}^\bullet is a differential graded Lie bialgebra, i.e. if a differential δ is given on \mathfrak{g}^\bullet which is a derivation of the Lie algebra and a coderivation of the Lie coalgebra, then we can extend δ to a derivation of $\wedge^\bullet(\mathfrak{g}^\bullet)$, and $\partial = \delta + \partial^{\text{coLie}}$ defines a G_∞ structure on \mathcal{A}^\bullet .

1.1. A canonical deformation of a homotopy Gerstenhaber algebra.

First of all, note that for any graded Lie algebra \mathfrak{g}^\bullet the Gerstenhaber algebra $\wedge^\bullet(\mathfrak{g}^\bullet)$ has a canonical BV structure. Indeed, one can define Δ_{Lie} to be the only BV operator satisfying

$$\Delta_{\text{Lie}}(v \wedge w) = [v, w]$$

for v, w in \mathfrak{g}^\bullet . This operator is the usual chain differential $\wedge^\bullet(\mathfrak{g}^\bullet) \rightarrow \wedge^{\bullet-1}(\mathfrak{g}^\bullet)$.

Recall that for any Gerstenhaber algebra \mathcal{A}^\bullet one can define a canonical deformation $(\mathcal{A}^\bullet[\epsilon], \star, \{ \ , \ \})$ (cf. Introduction). Along with it one can define the trivial deformation $\mathcal{A}^\bullet[\epsilon]$ by extending the product and the bracket $k[\epsilon]$ - bilinearly.

Lemma 1.2.1. *Let Δ be a BV operator on a Gerstenhaber algebra \mathcal{A}^\bullet . Then the operator $1 - \epsilon\Delta$ is an isomorphism between the trivial and the canonical deformations of \mathcal{A}^\bullet .*

The proof is straightforward.

Now, for a G_∞ algebra \mathcal{A}^\bullet let $\wedge^\bullet(\text{Lie}(\mathcal{A}^\bullet[1]^*))[\epsilon]$ be the canonical deformation of the Gerstenhaber algebra $\wedge^\bullet(\text{Lie}(\mathcal{A}^\bullet[1]^*))$. By Lemma above, the BV operator Δ_{Lie} defines an isomorphism between the trivial and the canonical G_∞ algebra structures on $\wedge^\bullet(\text{Lie}(\mathcal{A}^\bullet[1]^*))[\epsilon]$. The derivation ∂ acts on the canonical deformation, therefore it acts on the trivial deformation; this derivation of the trivial deformation defines a G_∞ structure on $\mathcal{A}^\bullet[\epsilon]$; this is, by definition, the canonical deformation of \mathcal{A}^\bullet .

Let us give an explicit formula for the canonical deformation of a G_∞ structure, i.e. for the canonical G_∞ structure on $\mathcal{A}^\bullet[\epsilon]$.

Let $D = [\partial, \Delta_{\text{Lie}}]$. This is a derivation, and not just a differential operator of order two, precisely because ∂ is a derivation with respect to the bracket. Then the canonical deformation is defined by the derivation

$$(1.1) \quad \partial_{\text{can}} = \partial + \epsilon D$$

of $\wedge^\bullet(\text{Lie}(\mathcal{A}^\bullet[1]^*))[\epsilon]$. Strictly speaking, we need a derivation of $\wedge^\bullet(\mathcal{A}^\bullet[\epsilon]^*)$, not of $\wedge^\bullet(\text{Lie}(\mathcal{A}^\bullet[1]^*))[\epsilon]$. An easy way to get one from another is to write down the $k[\epsilon]$ -multilinear operations m_{k_1, \dots, k_n} defined by ∂_{can} and to notice that they satisfy all the needed identities.

In the Example 1.3, note that any Lie bialgebra has a canonical derivation $D : \mathfrak{g}^\bullet \rightarrow \wedge^2(\mathfrak{g}^\bullet) \rightarrow \mathfrak{g}^\bullet$ which is the cobracket followed by the bracket. This is a derivation and a coderivation. When extended to a derivation of $\wedge^\bullet(\mathfrak{g}^\bullet)$, it coincides with the derivation D above.

Lemma 1.2.3. *If \mathcal{A}^\bullet is a Gerstenhaber algebra (cf. Example 1.2) then the above definition coincides with the one in Introduction.*

2. HOMOTOPY BV ALGEBRAS

Recall that a BV operator Δ_{Lie} was defined in the beginning of 1.2.

Definition 2.1. A BV_∞ structure on a G_∞ algebra \mathcal{A}^\bullet is an operator Δ on $\wedge^\bullet(\text{Lie}(\mathcal{A}^\bullet[1]^*))$ such that:

1. $\Delta^2 = 0$; $\Delta = \partial + \Delta_1 + \Delta_3 + \dots$
where Δ_i is an operator of degree $-i$ and ∂ is the derivation defining the G_∞ structure on \mathcal{A}^\bullet ; $\Delta_i(1) = 0$.
2. $\Delta_1 - \Delta_{\text{Lie}}$ is a differential operator of order 1.
3. Δ_{2i-1} is a differential operator of order i for $i > 1$.
4. All operators Δ_k preserve the filtration \mathcal{F}^\bullet .

We use the term differential operators in the sense of Grothendieck's inductive definition. We can extend this definition as follows:

Definition 2.2. Let $f : R^\bullet \rightarrow S^\bullet$ be a morphism of graded commutative algebras. A homogeneous operator $D : R^\bullet \rightarrow S^\bullet$ is a differential operator of order k with respect to f if the operator $x \mapsto D(ax) - (-1)^{|D||a|} f(a)D(x)$ is of order $k-1$ with respect to f for any homogeneous a in R^\bullet ; the zero operator is of order -1 with respect to f .

Definition 2.3. A morphism of two BV_∞ algebras $(\mathcal{A}^\bullet, \Delta)$ and $(\mathcal{B}^\bullet, \Delta')$ is an operator

$$F : \wedge^\bullet(\text{Lie}(\mathcal{B}^\bullet[1]^*)) \rightarrow \wedge^\bullet(\text{Lie}(\mathcal{A}^\bullet[1]^*))$$

such that:

1.
$$F = \sum_{i=0}^{\infty} F_{2i}$$

where F_{2i} is of degree $-2i$.
2. F_0 is a morphism of Gerstenhaber algebras.
3. For $i > 0$ F_{2i} is a differential operator of order i with respect to F_0 and $F_{2i}(1) = 0$.
4. All the operators F_k preserve the filtration \mathcal{F}^\bullet .

A BV_∞ algebra is called minimal if the corresponding G_∞ algebra is minimal, i.e. if the differential $\delta = m_1$ is equal to zero. A BV_∞ algebra $(\mathcal{A}^\bullet, \Delta)$ is linear contractible if the differential ∂ is induced by a contractible differential on \mathcal{A}^\bullet and $\Delta = \partial + \Delta_{\text{Lie}}$.

Theorem 2.4. *Any BV_∞ algebra is isomorphic to direct sum of a minimal BV_∞ algebra and a linear contractible BV_∞ algebra.*

Corollary 2.5. *If there is a quasi-isomorphism $\mathcal{A}^\bullet \rightarrow \mathcal{B}^\bullet$ then there is a quasi-isomorphism $\mathcal{B}^\bullet \rightarrow \mathcal{A}^\bullet$.*

The proofs follow the usual scheme, as in [K], 4.5.

Remark 2.6. Note that a BV_∞ structure in sense of Definition 2.1 is in fact a homotopy BV operator from [Ko] acting on the algebra $\wedge^\bullet(\text{Lie}(\mathcal{A}^\bullet[1]^*))$. In [Ko], however, the order of Δ_{2i-1} is supposed to be $i+1$. Thus we can say that our BV_∞ operator is Kravchenko's operator on $\wedge^\bullet(\text{Lie}(\mathcal{A}^\bullet[1]^*))$ where the principal symbol of Δ_1 is that of Δ_{Lie} and the principal symbols of Δ_i are equal to zero for all odd $i > 1$.

Proposition 2.7. *The canonical deformation of a G_∞ algebra has a canonical BV_∞ structure given by $\Delta_{\text{can}} = \partial_{\text{can}} + \Delta_{\text{Lie}} + \frac{\partial}{\partial \epsilon}$*

The proof is straightforward.

Remark 2.8. One can compare our definition to the usual operadic definition. BV algebras are algebras over an operad which is defined by relations of degree ≤ 2 (the BV operad). One can easily define Koszul operads in this context, following [GK] and [Pr]. It is essentially proven in [Get] and [GJ] that the BV operad is Koszul. Now, the standard operadic definition of a BV_∞ algebra is that of an algebra over the bar construction of the Koszul dual. This definition is a partial case of ours for which all the operators Δ_i have to be derivations of $\wedge^\bullet(\text{Lie}(\mathcal{A}^\bullet[1]^*))$ (they will have automatically to be derivations of the bracket).

3. FORMALITY THEOREMS FOR HOCHSCHILD COCHAINS

In this section we outline the proofs of Theorems 0.1 - 0.3 from Introduction.

Sketch of the proof of Theorem 0.1 In order to show that the complex of Hochschild cochains $C^\bullet(A, A)$ is a G_∞ algebra, one shows first that it is a B_∞ algebra, i.e. its cobar construction is a differential graded Hopf algebra (see below). Then, by a theorem of Etingof - Kazhdan, one passes to the dequantization of this Hopf algebra which is a Lie bialgebra. This Lie bialgebra is cofree as a Lie coalgebra, and to such a bialgebra one can associate a G_∞ algebra by a construction dual to that of Example 1.3. This gives the required G_∞ structure on $C^\bullet(A, A)$.

More precisely, for a differential graded associative algebra \mathcal{C}^\bullet , let $T(\mathcal{C}^\bullet[1])$ be the cobar construction of \mathcal{C}^\bullet , i.e. the differential graded coalgebra with the comultiplication

$$\Delta(D_1 \otimes \dots \otimes D_n) = \sum_{k=1}^{n-1} (D_1 \otimes \dots \otimes D_k) \otimes (D_{k+1} \otimes \dots \otimes D_n)$$

and the differential

$$\begin{aligned} d(D_1 \otimes \dots \otimes D_n) &= \sum_{k=1}^{n-1} (-1)^{\sum_{i \leq k} (|D_i|+1)-1} D_1 \otimes \dots \otimes D_k D_{k+1} \otimes \dots \otimes D_n + \\ &\quad \sum_{k=1}^n (-1)^{\sum_{i < k} (|D_i|+1)} D_1 \otimes \dots \otimes \delta D_i \otimes \dots \otimes D_n \end{aligned}$$

Here δ is the differential in \mathcal{C}^\bullet and D_i are homogeneous elements of \mathcal{C}^\bullet .

Definition 3.1. We say that \mathcal{C}^\bullet is a B_∞ algebra if there is a multiplication on the cobar construction of \mathcal{C}^\bullet which makes it a differential graded bialgebra. We require this multiplication to preserve the filtration $f_p = T^{\leq p}(\mathcal{C}^\bullet)$.

Consider the following example ([GJ], [GV]). Let \mathcal{C}^\bullet be the Hochschild cochain complex of an algebra A . The algebra A may be itself graded. For cochains D_i in $C^{d_i}(A, A)$ define a cochain

$$\begin{aligned} D_0\{D_1, \dots, D_m\}(a_1, \dots, a_n) &= \\ &= \sum (-1)^{\sum_{q < p \leq m} (|a_{i_q}|-1)(|D_p|-1)} D_0(a_1, \dots, a_{i_1}, \\ &\quad D_1(a_{i_1+1}, \dots), \dots, D_m(a_{i_m+1}, \dots), \dots) \end{aligned}$$

(the total degree $|D_i|$ of D_i is its degree of homogeneity plus d_i). Let m be the two-cochain of A defined by

$$m(a, b) = (-1)^{|a|} ab$$

Put

$$\begin{aligned} D \smile E &= (-1)^{|D|} m\{D, E\}; \\ [D, E] &= D\{E\} - (-1)^{(|D|-1)(|E|-1)} E\{D\}; \quad \delta D = [m, D] \end{aligned}$$

This defines respectively the cup product, the Gerstenhaber bracket and the Hochschild differential on $C^\bullet(A, A)$. The space $C^\bullet(A, A)$ with the differential δ and the cup product is a differential graded associative algebra, the space $C^{\bullet+1}(A, A)$ with the bracket and the differential is a differential graded Lie algebra, and the above operations induce on $HH^\bullet(A, A)$ a structure of a Gerstenhaber algebra.

To define a multiplication on $T(C^\bullet(A, A))$ note that, since it has to be compatible with the comultiplication, it has to be determined by its composition with the projection from $T(C^\bullet(A, A))$ to $C^\bullet(A, A)$. We define this projection of the product $(D_1 \otimes \dots \otimes D_m) \bullet (E_1 \otimes \dots \otimes E_n)$ to be equal to $D\{E_1, \dots, E_n\}$ if $m = 1$ and 0 if $m > 1$.

Lemma 3.2. *The product \bullet defines on the differential graded algebra $(C^\bullet(A, A), \smile, \delta)$ a B_∞ structure.*

Proof: cf. [GJ], [GV].

For a B_∞ algebra C^\bullet , the bialgebra $T(C^\bullet)$ has an antipode S which can be defined explicitly:

$$\begin{aligned} S(D_1 \otimes \dots \otimes D_n) + D_1 \otimes \dots \otimes D_n = \\ \sum_{0 < k_1 < n} (D_1 \otimes \dots \otimes D_{k_1}) \bullet (D_{k_1+1} \otimes \dots \otimes D_n) - \\ \sum_{0 < k_1 < k_2 < n} (D_1 \otimes \dots \otimes D_{k_1}) \bullet (D_{k_1+1} \otimes \dots \otimes D_{k_2}) \bullet (D_{k_2+1} \otimes \dots \otimes D_n) + \dots \end{aligned}$$

Thus $T(C^\bullet[1])$ becomes a differential graded Hopf algebra. By the result from section 2.4 of [EK], suitably adapted to our situation, this Hopf algebra admits a canonical dequantization which is a graded Lie bialgebra. Since the construction of a dequantization is natural, the differential on $T(C^\bullet[1])$ defines a differential on this Lie bialgebra. One can show that this bialgebra is cofree as a Lie coalgebra, with the space of cogenerators $C^\bullet[1]$. Applying the construction from Example 1.3 to the dual Lie bialgebra, we get the required G_∞ structure on C^\bullet . Note that for all G_∞ algebras which are obtained as in Example 1.3 all the operations m_{k_1, \dots, k_n} are zero if $n > 2$. Therefore, in particular, the induced L_∞ structure on $C^\bullet[1]$ does not have non-zero higher brackets.

This proves Theorem 0.1. Theorem 0.2 follows easily from Theorem 2.4, or rather from its version for G_∞ algebras. To prove Theorem 0.3, one notices that the deformation complex of any G_∞ algebra \mathcal{A}^\bullet is the complex of derivations of the Gerstenhaber algebra $\wedge^\bullet(\text{Lie}(\mathcal{A}^\bullet[1]^*))$, with the differential $[\partial, _]$. For $M = \mathbb{R}^n$ and $\mathcal{A}^\bullet = HH^\bullet(A, A) = \Gamma(M, \wedge^\bullet(TM))$, the cohomology of this complex could be computed explicitly: there is a spectral sequence converging to it, with the first term equal to the Poisson cohomology (similar to the one defined in [Br]) of the odd symplectic space $T^*(M) = \mathbb{R}^{n|n}$. Since this cohomology is essentially the de Rham cohomology of M , one concludes that in case of $M = \mathbb{R}^n$ any deformation of the G_∞ algebra $\Gamma(M, \wedge^\bullet(TM))$ must be trivial. Thus, by virtue of Theorem 0.2, Theorem 0.3 is true for $M = \mathbb{R}^n$. For general M , one needs an additional argument with Gelfand - Fuks cohomology as in [K], [T].

4. FORMALITY THEOREMS FOR HOCHSCHILD CHAINS

Sketch of the proof of Theorem 0.6 For an algebra A , consider the dual space A^* as an A - bimodule. Let $R = A \dot{+} A^*$ be the algebra with the product

$$(a + \lambda)(b + \mu) = ab + a\mu + \lambda b$$

for $a, b \in A$, $\lambda, \mu \in A^*$. Any Hochschild cochain is a multi-linear form on $A \dot{+} A^*$; it can be decomposed into components which are linear forms on $A^{\otimes p} \otimes (A^*)^{\otimes q}$ with values in A or in A^* . The full Hochschild complex contains a subcomplex of those cochains whose components are given, respectively, by linear maps $A^{\otimes p} \rightarrow A^{\otimes q} \otimes A$ or $A^{\otimes p} \otimes A \rightarrow A^{\otimes q}$. Our definition of tensor powers of A may depend on the ring A ; for example, if $A = C^\infty(M)$ then

$$A^{\otimes p} = \text{jets}_{\text{diagonal}} C^\infty(M^p)$$

Define a grading on R by $R^0 = A$ and $R^{-1} = A^*$. Then the complex $C^\bullet(R, R)$ becomes graded; all its components of degree < -1 are equal to zero and

$$C^\bullet(R, R)^{-1} = C_\bullet(A, A)^*$$

(the dual complex). The subcomplex $C^\bullet(R, R)^0$ consists of cochains whose components (see above) are all zero except for $p = q = 1$ or $p = q = 0$. The cochains with $p = q = 1$ form a subcomplex; denote it by $C^\bullet(R, R)_1^1$. The quotient of $C^\bullet(R, R)^0$ by this subcomplex is exactly $C^\bullet(A, A)$. It is easy to see that the exact sequence

$$0 \rightarrow C^\bullet(R, R)_1^1 \rightarrow C^\bullet(R, R)^0 \rightarrow C^\bullet(A, A) \rightarrow 0$$

splits and that $C^\bullet(R, R)_1^1$ is quasi-isomorphic to $C^{\bullet-1}(A, A)$. Thus, the complex $C^\bullet(R, R)^0$ is quasi-isomorphic to $C^\bullet(A, A)[\epsilon]$.

Next step is to show that the operations $D_0\{D_1, \dots, D_m\}$ from Section 3 all preserve the grading on $C^\bullet(R, R)$. After that, applying the results of Section 3, one sees that $C^\bullet(R, R)^0$ is a G_∞ algebra and $C^\bullet(R, R)^{-1}$ is a G_∞ module over it. Since every G_∞ module admits a dual, we see that $C_\bullet(A, A)$ is a G_∞ module over $C^\bullet(R, R)^0$ and therefore over some odd deformation $C^\bullet(A, A)[\epsilon]$ of the G_∞ algebra $C^\bullet(A, A)$.

This implies Theorem 0.6. After that, using the same reasoning as in the end of Section 3, one deduces Theorem 0.7.

5. DEFORMATION COMPLEXES OF BIALGEBRAS AND HOMOTOPY BV OPERATORS

Let \mathfrak{g}^\bullet be a graded Lie bialgebra. Put (cf. [LR])

$$C^{p-1, q-1}(\mathfrak{g}^\bullet) = \text{Hom}(\wedge^p(\mathfrak{g}^\bullet), \wedge^q(\mathfrak{g}^\bullet))$$

for $p, q \geq 1$. Recall that, in our notation, $\wedge^q(\mathfrak{g}^\bullet) = S^p(\mathfrak{g}^\bullet[-1])$; for $D \in C^{p-1, q-1}(\mathfrak{g}^\bullet)$ put $|D| = 2p - 2 +$ (homogeneity degree of D). For example, if $\mathfrak{g}^\bullet = \mathfrak{g}^0 = \mathfrak{g}$ is a Lie bialgebra, then $C^{p-1, q-1}(\mathfrak{g}^\bullet)$ is concentrated in degree $p + q - 2$. There are two differentials

$$d^{\text{Lie}} : C^{p-1, q-1}(\mathfrak{g}^\bullet) \rightarrow C^{p, q-1}(\mathfrak{g}^\bullet)$$

and

$$d_{\text{coLie}} : C^{p-1, q-1}(\mathfrak{g}^\bullet) \rightarrow C^{p-1, q}(\mathfrak{g}^\bullet)$$

The first is the standard cochain differential of the Lie algebra \mathfrak{g}^\bullet with coefficients in the module $\wedge^q(\mathfrak{g}^\bullet)$, the second is a dual differential for the coalgebra \mathfrak{g}^\bullet (or the differential d^{Lie} for the dual Lie algebra $(\mathfrak{g}^\bullet)^*$). If δ is a differential of the bialgebra \mathfrak{g}^\bullet which is a derivation with respect to both structures, then it induces a differential

$$\delta : C^{p-1, q-1}(\mathfrak{g}^\bullet) \rightarrow C^{p-1, q-1}(\mathfrak{g}^\bullet)$$

The above three differentials commute with each other. We put $d = \delta + d^{\text{Lie}} + d_{\text{coLie}}$. The total complex (C^\bullet, d) is called the deformation complex of the differential graded Lie bialgebra \mathfrak{g}^\bullet .

Let us interpret this complex in terms of differential operators on the algebra $\wedge^\bullet(\mathfrak{g}^\bullet)$. For any such differential operator of order p , we say that it is of pure order p if its restriction to $\wedge^{<p}(\mathfrak{g}^\bullet)$ is zero. Let \mathcal{D}^p be the space of differential operators of pure order p . Define the Poisson bracket (cf. [KS])

$$\{ \quad , \quad \} : \mathcal{D}^p \otimes \mathcal{D}^q \rightarrow \mathcal{D}^{p+q-1}$$

to be the component of pure order $p + q - 1$ of the commutator of operators.

Clearly, an operator of pure order p is uniquely determined by its restriction to $\wedge^p(\mathfrak{g}^\bullet)$, so we identify \mathcal{D}^p with $C^{p-1, \bullet}(\mathfrak{g}^\bullet)$. Note that the derivations δ and ∂^{coLie} from Example 1.3 are in \mathcal{D}^1 and the chain differential Δ_{Lie} is in \mathcal{D}^2 .

Lemma 5.1. *Under the identification of $C^\bullet(\mathfrak{g}^\bullet)$ with the space of differential operators, one has*

$$\delta = \{\delta, -\}; \quad \partial^{\text{Lie}} = \{\Delta_{\text{Lie}}, -\}; \quad \partial_{\text{coLie}} = \{\partial^{\text{coLie}}, -\}$$

where the left hand sides refer to the three differentials on $C^\bullet(\mathfrak{g}^\bullet)$ and the right hand sides to differential operators on $\wedge^\bullet(\mathfrak{g}^\bullet)$.

The proof is straightforward.

If a Lie bialgebra \mathfrak{g}^\bullet has a filtration preserved by the bracket, the cobracket and the differential, then by the filtered deformation complex of \mathfrak{g}^\bullet we will mean the subcomplex of $C^\bullet(\mathfrak{g}^\bullet)$ consisting of maps preserving the induced filtration on $\wedge^\bullet(\mathfrak{g}^\bullet)$.

Recall from Section 3 that a BV_∞ operator on the cochain complex $C^\bullet(A, A)$ is a differential operator on $\wedge(\mathfrak{g}^\bullet)$

$$\Delta = \delta + \partial^{\text{coLie}} + \Delta_1 + \Delta_3 + \dots$$

satisfying

$$\Delta^2 = 0,$$

where $\mathfrak{g}^\bullet = \mathfrak{g}^\bullet(A)$ is the Etingof - Kazhdan dequantization of the Hopf algebra $T(C^\bullet(A, A)[1])$. The order of $\Delta_1 - \Delta_{\text{Lie}}$ has to be equal to 1, the order of Δ_{2i-1} to i for $i > 1$. All Δ_i have to preserve the increasing filtration on $\wedge(\mathfrak{g}^\bullet)$ which is induced by a certain filtration on \mathfrak{g}^\bullet . (The filtration is increasing, not decreasing as in Section 3, because we pass to the dual bialgebra).

Theorem 5.2. *All obstructions to existence of a BV_∞ structure on $C^\bullet(A, A)$ lie in the cohomology of the filtered deformation complex of the differential graded Lie bialgebra $\mathfrak{g}^\bullet(A)$.*

Proof Rewrite Δ as

$$\Delta = \sum_{p \geq 0} \Delta(p),$$

where $\Delta(0) = \delta + \partial^{\text{coLie}}$ and $\Delta(p)$ is the sum of components of pure order $i - p$ in Δ_{2i-1} for $p > 0$. Solving the equation $\Delta^2 = 0$ is equivalent to solving a chain of equations $\{\Delta(0), \Delta(p)\} = c_p$ where c_p is a known cochain.

Corollary 5.3. *The first obstruction to existence of a BV_∞ operator on $C^\bullet(A, A)$ is the canonical derivation D of $\mathfrak{g}^\bullet(A)$ (the cobracket followed by the bracket).*

For a graded associative bialgebra H^\bullet , there is a parallel version of a deformation complex (cf. [GS]). Put

$$C^{p-1, q-1}(H^\bullet) = \text{Hom}((H^\bullet)^{\otimes p}, (H^\bullet)^{\otimes q})$$

For $D \in C^{p-1, q-1}$ set $|D| = p + q - 2$ (homogeneity degree of D). Define

$$\partial^{\text{alg}} : C^{p-1, q-1}(H^\bullet) \rightarrow C^{p, q-1}(H^\bullet)$$

$$\partial_{\text{coalg}} : C^{p-1, q-1}(H^\bullet) \rightarrow C^{p-1, q}(H^\bullet)$$

to be respectively the Hochschild cochain differential of the algebra H^\bullet with coefficients in the bimodule $(H^\bullet)^{\otimes p}$ and the dual differential for the coalgebra H^\bullet (or, in other words, the differential ∂^{alg} for the dual algebra $(H^\bullet)^*$). If δ is a differential on H^\bullet which is a derivation and a coderivation, then it induces a differential

$$\delta : C^{p-1, q-1}(H^\bullet) \rightarrow C^{p-1, q-1}(H^\bullet)$$

Put $d = \delta + \partial^{\text{alg}} + \partial_{\text{coalg}}$. The total complex $C^\bullet(H^\bullet)$ with the differential d is called the deformation complex of H^\bullet . If H^\bullet is a filtered bialgebra then the filtered deformation complex consists of those maps which preserve the induced filtration on tensor powers.

Conjecture 5.4. *There is a canonical quasi-isomorphism between $C^\bullet(\mathfrak{g}^\bullet)$ and $C^\bullet(H^\bullet)$ where \mathfrak{g}^\bullet is a Lie bialgebra and H^\bullet is its canonical quantization of Etingof - Kazhdan.*

Conjecture 5.5. *The image of the canonical derivation D under the canonical dequantization of Etingof - Kazhdan is $\frac{1}{2} \log(S^2)$ where S is the antipode.*

The deformation complex of a Lie bialgebra has the usual algebraic structure, namely the Lie bracket (in our grading it has degree zero). As we see, it also has an associative product coming from its identification with the space of differential operators. What are the analogues of these structures for Hopf algebras, and how to translate a BV_∞ operator on $C^\bullet(A, A)$ into their terms? The product $C^{p-1, q-1} \otimes C^{p'-1, q'-1} \rightarrow C^{p+p'-1, q+q'-1}$ exists; it is the analogue of the commutative multiplication of degree two on the deformation complex of a bialgebra (the multiplication of principal symbols).

It is more or less clear that a Hopf algebra analogue of a BV_∞ operator should be multiplicative, not a “connection” like Δ but rather its “holonomy”.

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