

# The Deformation Complex of a $d$ -algebra is a ( $d+1$ )-algebra

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## 1 Introduction

In this paper we give a purely algebraic proof of a variant of the theorem of Kontsevich [3] stating that the deformation complex of a  $d$ -algebra shifted by  $d$  is naturally a  $(d + 1)$ -algebra.

By a  $d$ -algebra structure on a vector space  $V$  over a field of characteristic zero we mean the analogue of Poisson algebra with its bracket of degree  $1 - d$ . Our definition coincides with the usual one for  $d \geq 2$ . Nevertheless, in this paper '1-algebra' means 'usual Poisson algebra', not 'associative algebra'.

Kontsevich's definition of  $d$ -algebra [3] is as follows: a  $d$ -algebra is an algebra over the chain operad of the operad of little  $d$ -disks. But in the same paper he proves that over any field of characteristic zero this chain operad is quasi-isomorphic to its homology operad. Thus, our theorem is a particular, characteristic zero case of Kontsevich's theorem.

Let us outline the idea of the proof. First, let us discuss the notion of a deformation Lie algebra of a  $d$ -algebra. As in the case of associative, Lie, commutative etc. algebras, first, we need the definition of a homotopy  $d$ -algebra, which can be found with the help of the theory of Koszul operads [2]. The key point is that the operad governing  $d$ -algebras is Koszul as it was shown in [1]. The definition of the structure of a homotopy  $d$ -algebra on a complex  $V$  says that it is the same as a differential on the cofree  $d$ -coalgebra  $Cofree_d(V[-d])$  cogenerated by  $V[-d]$ . Structures of usual  $d$ -algebra correspond to quadratic differentials on  $Cofree_d(V[-d])$ . Let  $X$  be a  $d$ -algebra. Denote by  $X^\vee$  the  $d$ -coalgebra  $Cofree_d(V[-d])$  equipped with the quadratic differential corresponding to the  $d$ -algebra structure on  $X$ . The reader familiar with the definition of homotopy associative (Lie, commutative, etc.) algebra will see that the definition of homotopy  $d$ -algebra is similar. The analogue of  $X^\vee$  in those cases is called the bar complex of  $X$ .

The deformation complex  $\text{def}(X)$  of a  $d$ -algebra  $X$  is just the differential graded Lie algebra of derivations of  $X^\vee$ . This object admits a more 'geometric' definition in terms of the infinitesimal neighborhood of the identity in the 'algebraic group'  $\text{Aut}X^\vee$ . Let us explain the meaning of this. We will use the technical notion of coproartinian cocommutative coalgebra (Section 2.4.1) which is dual to the notion of proartinian local commutative algebra. Now, note that the tensor product of a  $d$ -coalgebra and a cocommutative coalgebra is naturally a  $d$ -coalgebra. Define a functor  $F'$  from the opposite to the category of coproartinian cocommutative coalgebras to the category of sets by setting  $F'(a) = \text{Hom}_{d\text{-coalg}}(X^\vee \otimes a, X^\vee)$ . One sees that the composition of any two elements from  $F'(a)$  is well defined. Thus,  $F'$  is actually a functor taking values in the category of monoids. Since we need the neighborhood of identity, define a subfunctor  $F$  of  $F'$  by taking as  $F(a)$  only those morphisms  $X^\vee \otimes a \rightarrow X^\vee$  for which the through map  $X^\vee \rightarrow X^\vee \otimes a \rightarrow X^\vee$  is identity. Here the first arrow is induced by the canonical inclusion of the ground field  $k$  to  $a$ . One sees that  $F$  takes values in groups. It turns out that  $F$  viewed as a functor to the category of sets is representable: there exists a cocommutative coalgebra  $A$  such that  $F(a) \cong \text{Hom}(a, A)$  naturally in  $a$ . The associative composition law  $F \times F \rightarrow F$  defines an associative map of cocommutative coalgebras  $A \otimes A \rightarrow A$  making  $A$  a bialgebra. One sees that  $A \cong U(\text{def}(X))$ ,  $U$  meaning the universal enveloping algebra.

One can modify the above construction. Let  $X, Y$  be  $d$ -algebras and  $\phi : X^\vee \rightarrow Y^\vee$  a morphism. Define a functor

$$F_{X,Y}^\phi : \text{Coproartalg}^{\text{op}} \rightarrow \text{Sets}$$

by setting  $F_{X,Y}^\phi(a) \subset \text{Hom}_{d\text{-coalg}}(X^\vee \otimes a, Y^\vee)$ , where we take only those morphisms for which the through map  $X^\vee \rightarrow X^\vee \otimes a \rightarrow Y^\vee$  is  $\phi$ . Again, this functor is representable. Denote the corresponding coproartinian coalgebra by  $\underline{\text{Hom}}_c^\phi(X, Y)$ . Let

$$X^\vee \xrightarrow{\phi} Y^\vee \xrightarrow{\psi} Z^\vee$$

be the sequence of morphisms. Then we have a natural composition

$$\underline{\text{Hom}}_c^\phi(X, Y) \otimes \underline{\text{Hom}}_c^\psi(Y, Z) \rightarrow \underline{\text{Hom}}_c^{\psi \circ \phi}(X, Z). \quad (1)$$

Now we can take the advantage of the fact that there is a natural tensor product on the category of  $d$ -coalgebras. This means that the functor  $F_{X,Y}^\phi$  can be extended to the category of coproartinian  $d$ -coalgebras (the definition remains the

same but we allow  $a$  to be any coproartinian  $d$ -coalgebra). This functor is also representable. Denote the corresponding coproartinian  $d$ -coalgebra by  $\underline{\text{Hom}}^\phi(X, Y)$ . One sees that this construction is similar to the one of internal homomorphisms. The only difference is that we work in the formal neighborhood of a given morphism. The most interesting case for us is  $\underline{\text{Hom}}^{Id}(X, X)$ . The analogue of the composition morphism (1) provides us with an associative map of  $d$ -coalgebras  $\underline{\text{Hom}}^{Id}(X, X) \otimes \underline{\text{Hom}}^{Id}(X, X) \rightarrow \underline{\text{Hom}}^{Id}(X, X)$ . We call this structure  $d$ -bialgebra. One sees that as a  $d$ -coalgebra (the differential is ignored)  $\underline{\text{Hom}}^{Id}(X, X)$  is isomorphic to the cofree  $d$ -coalgebra cogenerated by  $\text{def}(X)$  (viewed as a graded vector space). The differential and the associative product look more sophisticated.

**Remark** Note that the same construction is applicable to the category of associative algebras. In this case  $\underline{\text{Hom}}^{Id}(X, X)$  is the Hopf algebra isomorphic to the Hopf algebra on the tensor coalgebra  $T(C^\bullet(X, X)[1])$  defined in [1] (the so-called  $B_\infty$ -structure).

Our next step is to show that the structure of a  $d$ -bialgebra on a cofree  $d$ -coalgebra cogenerated by a complex  $K$  implies the structure of  $(d + 1)$ -algebra on  $K[-d]$ . Contrary to the case of associative algebras this can be done in an easy purely algebraic way. This result applied to  $\underline{\text{Hom}}^{Id}(X, X)$  gives the desired  $(d + 1)$ -algebra structure on  $\text{def}(X)[-d]$ .

Here is the content of the sections. We start with the quick review of  $d$ -algebras, their homotopy theory, and their deformations. Also we introduce the notion of the coproartinian coalgebra which is the dual to the notion of proartinian algebra. In Section 2 we construct the functor  $F_{X,Y}^\phi$  and show that it is representable. The corollary of this is the fact that we have a structure of  $d$ -bialgebra on a cofree  $d$ -coalgebra cogenerated by the deformation complex of a  $d$ -algebra. In section 3 we prove that this structure implies the desired structure of  $(d + 1)$ -algebra on the shifted deformation complex.

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## 2 $d$ -Algebras, homotopy $d$ -algebras, and their deformations

In this section we will recall the notions outlined in its title. All the definitions are parallel to the ones for associative, commutative, or Lie algebras.

## 2.1 $d$ -Algebras

A structure of  $d$ -algebra on a complex  $V$  of vector spaces over a field  $k$  of characteristic zero is given by

1 a dg commutative associative product  $\cdot : S^2V \rightarrow V$  ;

2 a map

$$\{\} : \Lambda^2(V[d-1]) \rightarrow V[d-1], \quad (2)$$

which turns  $V[d-1]$  into DGLA (Differential Graded Lie Algebra).

By abuse of notation we will also denote by  $\{\cdot, \cdot\}$  the degree  $1-d$  map  $S^2V \rightarrow V$  if  $d$  is odd ( $\Lambda^2V \rightarrow V$  if  $d$  is even) corresponding to the map (2).

These operations must satisfy the Leibnitz identity

$$\{ab, c\} = a\{b, c\} + (-1)^{|b|(|c|+d-1)}\{a, c\}b.$$

A  $d$ -algebra with unit is a  $d$ -algebra with a marked element 1 which is the unit with respect to the product and its bracket with any element vanishes. The structure of  $d$ -algebra (resp.  $d$ -algebra with unit) is governed by an operad which is denoted by  $e'_d$  (resp.  $e_d$ ). This means that a  $d$ -algebra (resp.  $d$ -algebra with unit) is the same as an algebra over the operad  $e'_d$  (resp.  $e_d$ ). As usual, we define a morphism of  $d$ -algebras (with unit)  $V$  and  $W$  as a morphism of complexes  $V \rightarrow W$  which respects all the operations (and the units). Therefore, we have the categories  $d\text{-alg}$  of  $d$ -algebras and  $d\text{-alg}_1$  of  $d$ -algebras with unit.

## 2.2 $d$ -Coalgebras

This structure is dual to the structure of  $d$ -algebra. A structure of  $d$ -coalgebra (with counit) on a complex  $V$  is specified by a dg cocommutative coassociative coproduct  $\Delta : V \rightarrow S^2V$  (with counit)  $\epsilon : V \rightarrow k$  and a Lie cobracket  $\delta : V[1-d] \rightarrow \Lambda^2V[1-d]$ , satisfying the conditions dual to the ones for  $d$ -algebras.  $d$ -Coalgebras (resp.  $d$ -coalgebras with counit) form a category  $d\text{-coalg}$  (resp.  $d\text{-coalg}_1$ ).

## 2.3 Free $d$ -algebras

Let  $Complexes$  be the category of complexes and their morphisms of degree zero. We have the forgetful functor  $Obl : d\text{-alg}_1 \rightarrow complexes$  which takes a  $d$ -algebra to

its underlying complex. This functor has the left adjoint. We will denote it  $Free_d$  or simply  $Free$ . The theory of operads says that

$$Free_d(V) = \bigoplus_{n=0}^{\infty} e_d(n) \otimes_{S_n} V^{\otimes n}.$$

For a complex  $V$ , let  $FreeLie(V)$  denote the free graded Lie algebra generated by  $V$  with the differential induced by the one on  $V$ . Denote by  $OblFreeLie(V)$  the underlying complex of  $FreeLie(V)$ . Then on the level of complexes we have the canonical isomorphism

$$S^\bullet(FreeLie(V[d])[-d]) \cong OblFree_d(V).$$

## 2.4 Cofree coproartinian $d$ -coalgebras

Similarly, we have the forgetful functor  $Obl : d\text{-coalg} \rightarrow \text{complexes}$ . Unfortunately, it does not have an adjoint functor. The reason is that the linear dual to the free  $d$ -algebra generated by even finite dimensional space  $V$  is not a  $d$ -coalgebra. This can be cured either by passage to the category of topological  $d$ -coalgebras or by passage to coproartinian coalgebras. We choose the second option.

### 2.4.1 Coproartinian $d$ -coalgebras

A  $d$ -coalgebra with counit  $V$  is called *coproartinian* if there exists an exhaustive increasing filtration  $F$  of  $V$  by sub-coalgebras with counit and the following conditions are satisfied.

- 1  $F^0V$  is a one-dimensional coalgebra.
- 2 Let  $1 \in F^0V$  be a unique element such that  $\epsilon(1) = 1$ , where  $\epsilon$  is the counit. Then for any  $x \in F^1V$  we have

$$\Delta x = 1 \otimes x + x \otimes 1; \tag{3}$$

$$\delta x = 0, \tag{4}$$

and for any  $x \in F^iV$ ,  $i \geq 2$  we have

$$\Delta x - 1 \otimes x - x \otimes 1 \in F^{i-1}V \otimes F^{i-1}V \tag{5}$$

and

$$\delta x \in F^{i-1}V \otimes F^{i-1}V. \tag{6}$$

**LEMMA 2.1** *If  $V$  is a coproartinian coalgebra, it has the only grouplike element  $e$ . Therefore, all filtrations on  $V$  that make it coproartinian have the same term  $F^0V = ke$ .*

*Proof.* Let  $F$  be a filtration on  $V$  that makes it coproartinian. Then we take the element  $1 \in F^0V$  such that  $\epsilon(1) = 1$ , as we did it in the definition of coproartinian coalgebras. It is grouplike. Let  $i$  be the minimal number such that  $F^i(V)$  contains a grouplike element  $x \neq 1$ . Clearly,  $i > 0$ . We have from (5) that

$$(x - 1) \otimes (x - 1) = \Delta x - 1 \otimes x - x \otimes 1 + 1 \otimes 1 \in T^2F^{i-1}(V).$$

Therefore,  $x - 1 \in F^{i-1}V$  and  $x \in F^{i-1}V$ . Contradiction.

**LEMMA 2.2** *Let  $W$  be a coproartinian coalgebra and let  $s_n : W \rightarrow \text{Hom}_{S_n}(e_d(n), W^{\otimes n})$  be the structure maps of  $W$  as a  $d$ -coalgebra. Let*

$$s'_n : W \rightarrow \text{Hom}_{S_n}(e_d(n), (W/F^0W)^{\otimes n})$$

*be the composition of  $s_n$  with the  $n$ -th tensor power of the projection  $W \rightarrow W/F^0W$ . Then  $s'_n(F^k W) = 0$  whenever  $n > k$ .*

Denote by  $d\text{-coart}$  the full subcategory of  $d\text{-coalg}_1$  formed by coproartinian coalgebras. Define the forgetful functor  $Obl : d\text{-coart} \rightarrow \text{complexes}$  by setting  $Obl(V) = V/F^0V$ . This is a well-defined functor by Lemma 2.1.

**PROPOSITION 2.3** *The forgetful functor  $Obl$  has the right adjoint*

*Proof.* Define the right adjoint functor  $Cofree_d$  as

$$Cofree_d(V) = \bigoplus_{n=0}^{\infty} \text{Hom}_{S_n}(e_d(n), V^{\otimes n}).$$

The structure morphisms of  $e_d$  define the canonical  $d$ -coalgebra structure on it. Also, we have a filtration on it by the tensor powers of  $V$ . So, it is a coproartinian coalgebra. Denote  $p : \bigoplus_{n=0}^{\infty} \text{Hom}_{S_n}(e_d(n), V^{\otimes n}) \rightarrow V$  the projection onto  $\text{Hom}_{S_1}(e_d(1), V) \cong V$ .

Let  $W$  be another coproartinian coalgebra. Consider  $\text{Hom}_{d\text{-coart}}(W, Cofree_d(V))$ . The composition with  $p$  defines the map

$$p_* : \text{Hom}_{d\text{-coart}}(W, Cofree_d(V)) \rightarrow \text{Hom}_{\text{Complexes}}(W, V).$$

By Lemma 2.1  $F^0W$  goes to  $F^0Cofree_d(V)$  under any coalgebra morphism. Therefore, the map  $p_*$  induces a map

$$p'_* : Hom_{d\text{-coart}}(W, Cofree_d(V)) \rightarrow Hom_{Complexes}(W/F^0(W), V),$$

and we need to prove that this map is an isomorphism. For this, we define a map

$$q : Hom_{Complexes}(W/F^0(W), V) \rightarrow Hom_{d\text{-coart}}(W, Cofree_d(V))$$

and show that it is inverse to  $p'_*$ . By Lemma 2.2, we have a well defined structure map  $s : W \rightarrow \bigoplus_{n=1}^{\infty} Hom_{S_n}(e_d(n), (W/F^0W)^{\otimes n})$ . Let  $r \in Hom_{Complexes}(W/F^0(W), V)$ . Define  $q'(r)$  as the composition of  $s$  and tensor powers of  $r$  and set  $q(r)(w) = q'(r)(w) + \epsilon(w) \cdot 1$ , where  $w \in W$  and  $1$  is the grouplike element in  $Cofree_d(V)$ . One sees that  $q$  is a coalgebra morphism and that  $q$  is inverse to  $p'_*$ .  $\triangle$

### 2.4.2 Differentials on $d$ -coalgebras

Let  $W$  be a  $d$ -coalgebra. A graded map  $D : W \rightarrow W$  is called a *derivation of  $W$* , if it is a derivation of  $W$  with respect to its cocommutative and Lie coalgebra structures (for a moment, we forget the differential on  $W$ ). If the grading  $|D| = 1$ , and  $D^2=0$ , then  $D$  is nothing else but a differential on  $W$ . All derivations of  $W$  form a graded Lie algebra  $Der(W)$  with the commutator  $[D, E] = D \circ E - (-1)^{|D||E|} E \circ D$ ,  $\circ$  meaning the composition of maps  $W \rightarrow W$ . The differential  $d$  on  $W$  is an element of this Lie algebra, and  $[d, d] = 0$ . Define the differential  $\delta$  on  $Der(W)$  by  $\delta x = [d, x]$ . This turns  $Der(W)$  into a DGLA.

### 2.4.3 Derivations on cofree coalgebras

Let  $Cofree_d(V)$  be the cofree  $e_d$ -coalgebra cogenerated by a graded space  $V$  with zero differential. Let  $p : Cofree_d(V) \rightarrow V$  be the canonical projection onto cogenerators. Let  $D \in Der(Cofree_d(V))$ . Then we have the corestriction  $cor(D) = p \circ D : Cofree_d(V) \rightarrow V$ .

**PROPOSITION 2.4** *The map*

$$cor : Der(Cofree_d(V)) \rightarrow Hom_k(Cofree_d(V), V)$$

*is an isomorphism of graded vector spaces.*

The  $Cofree_d(V)$  is graded by the tensor powers of  $V$  so that

$$gr_i Cofree_d(V) \cong \text{Hom}_{S_i}(e_d(i), V^{\otimes i}).$$

Define a subspace  $\text{Der}_{1,2} \subset \text{Der}(Cofree_d(V))$  consisting of the elements  $x$  such that  $cor(x)(gr_i(Cofree_d(V))) = 0$  for all  $i$  except 1 and 2.

**PROPOSITION 2.5** *The set of the elements  $x \in \text{Der}_{1,2}$  such that  $|x| = 1$  and  $[x, x] = 0$  is in 1-1 correspondence with the structures of differential  $d$ -algebras on the shifted graded vector space  $V[-d]$ .*

*Proof.* Take the components of  $cor(x)$   $x_1 : V \rightarrow V$  and  $x_2 : \text{Hom}_{S_2}(e(2), V^{\otimes 2}) \rightarrow V$ . One sees that  $x_1$  is a differential on  $V$  and, hence, on  $V[-d]$ . Recall that  $e_d(2)$  is generated by the elements  $m, b$  corresponding to the commutative product and the bracket. Therefore,  $j : \text{Hom}_{S_2}(e(2), V^{\otimes 2}) \cong S^2V \oplus \Lambda^2(V[1-d])[d-1]$ . Since  $x_2$  has degree 1, it defines under  $j$  a map of degree zero  $k : \Lambda^2(V[-1])[1] \oplus S^2(V[-d])[d] \rightarrow V$ . The condition  $[x, x] = 0$  is equivalent to the following:

- 1 each of the restrictions of  $k$

$$b : \Lambda^2(V[-1])[1] \rightarrow V \text{ and } m : S^2(V[-d])[d] \rightarrow V$$

are compatible with the differential on  $V$  defined by  $x_1$ ;

- 2 The map  $m$  is the commutative product on  $V[-d]$  and  $b$  is the Lie bracket on  $V[-1]$ ; the maps  $m, b$  define a structure of  $d$ -algebra on  $V[-d]$ .

Whence the statement of the proposition  $\triangle$

**DEFINITION 2.6** *For a  $d$ -algebra  $V$  define the coproartinian  $d$ -coalgebra  $V^\vee$  as the coalgebra  $Cofree_d(V[d])$  with the differential corresponding to the  $d$ -algebra structure on  $V \cong V[d][-d]$  by Proposition 2.5*

## 2.5 Homotopy $d$ -algebras

**DEFINITION 2.7** *A structure of homotopy  $d$ -algebra on a graded vector space  $V$  is a differential of the coalgebra  $Cofree_d(V[d])$  vanishing on  $1 \in Cofree_d(V[d])$ .*

For a homotopy  $d$ -algebra  $V$  we denote by  $V^\vee$  the corresponding differential cofree coalgebra.



**DEFINITION 2.8** *A morphism of homotopy  $d$ -algebras is a morphism of the corresponding differential cofree coalgebras.*

Thus, homotopy  $d$ -algebras form a category.

We see that any  $d$ -algebra  $V$  defines a homotopy  $d$ -algebra  $V^\vee$ . But the set of morphisms between two  $d$ -algebras viewed as homotopy  $d$ -algebras is wider than the set of usual morphisms between them. In other words, we have an injection  $\text{Hom}_{d\text{-alg}}(V, W) \rightarrow \text{Hom}_{d\text{-coart}}(V^\vee, W^\vee)$ .

For a homotopy  $d$ -algebra  $V$  the *linear part* of the differential  $d$  on  $V^\vee$  is the restriction of  $d$  onto  $gr_1(V^\vee) \stackrel{\text{def}}{=} gr_1(\text{Cofree}_d(V[d])) \cong V[d]$ . It takes values in  $gr_1 V^\vee$  and defines a differential on  $V[d]$ . A structure of a homotopy  $d$ -algebra on a complex  $V$  is by definition a differential on  $\text{Cofree}_d(V[d])$  such that its linear part coincides with the differential on  $V$ .

From the operadic point of view, the structure of a homotopy  $d$ -algebra on a complex is governed by a dg -operad. Denote it by  $he_d$ . Let  $e'_d$  be the operad governing  $d$ -algebras without unit. The fact that any usual  $d$ -algebra is also a homotopy  $d$ -algebra reflects in a map  $p : he_d \rightarrow e'_d$ . It is known that  $he_d$  is a free operad and that  $p$  is a quasiisomorphism of operads. Thus  $he_d$  is a free resolution of  $e'_d$ .

### 2.5.1 Deformation Lie algebra

**DEFINITION 2.9** *let  $V$  be a homotopy  $e_d$ -algebra. Define its deformation Lie algebra  $\text{def}(V) = \text{Der}(V^\vee)$  with the differential being the bracket with the differential on  $V^\vee$ .*

## 3 Infinitesimal Internal Homomorphisms

In this section first we define the tensor product of (coartinian)  $e_d$ -coalgebras, and then construct a substitute for the internal homomorphisms.

### 3.1 Tensor product of $e_d$ -coalgebras with counit

Let  $V, W$  be  $d$ -coalgebras with counit. Define the  $d$ -coalgebra structure on  $V \otimes W$  as follows. The differential on  $V \otimes W$  is the differential on the tensor product of complexes. The coproduct is defined by  $\Delta(v \otimes w) = \epsilon \Delta(v) \otimes \Delta(w)$  and the

cocommutator  $\delta(v \otimes w) = \epsilon(\delta(v) \otimes \Delta(w)) + \epsilon((-1)^{(d-1)|v|} \Delta(v) \otimes \delta(w))$ , where  $\epsilon$  means the sign corresponding to the permutation (1324) of the graded tensor factors:

$$\epsilon : V \otimes V \otimes W \otimes W \rightarrow V \otimes W \otimes V \otimes W.$$

The counit is the tensor product of counits. One sees that the tensor product of coproartinian coalgebras is a coproartinian coalgebra.

## 3.2 Internal homomorphisms

### 3.2.1 Useful Lemma

Let  $A$  be a coproartinian  $d$ -coalgebra;  $B$  a  $d$ -algebra without unit. We have an injection of  $d$ -coalgebras  $k \cong F^0 A \rightarrow A$ . Therefore, the factor  $A/k$  is naturally a  $d$ -coalgebra. Then  $\text{Hom}_k(A/k, B)$  is a  $d$ -algebra, hence,  $\text{Hom}_k(A/k, B)[d-1]$  is a Lie algebra. For a Lie algebra  $\mathfrak{g}$  denote  $MC(\mathfrak{g}) = \{\mathfrak{x} \in \mathfrak{g}^1 : \mathfrak{d}\mathfrak{x} + [\mathfrak{x}, \mathfrak{x}]/2 = \mathfrak{o}\}$ .

**LEMMA 3.1** *There is a natural bijection between the sets  $\text{Hom}_{d\text{-coart}}(A, B^\vee)$  and  $MC(\text{Hom}_k(A/k, B)[d-1])$ .*

*Proof.* If we forget about the differentials, then

$$\text{Hom}_{d\text{-coart}}(A, B^\vee) \cong \text{Hom}_k(A/k, B[d]) \cong \text{Hom}_k(A/k, B)[d-1]^1.$$

A direct computation shows that the morphisms compatible with the differential correspond under this identification to the  $MC(\text{Hom}_k(A/k, B)[d-1])$ .  $\triangle$

### 3.2.2 Coalgebra $\underline{\text{Hom}}^\phi(V, W)$

Let  $S$  be a coproartinian dg coalgebra. We have a retraction

$$k \cong F^0 S \rightarrow S \xrightarrow{\epsilon} k.$$

Let  $\phi \in \text{Hom}_{d\text{-coart}}(V^\vee, W^\vee)$ . The canonical inclusion  $k \rightarrow S$  defines a map

$$h : \text{Hom}_{d\text{-coart}}(V^\vee \otimes S, W^\vee) \rightarrow \text{Hom}_{d\text{-coart}}(V^\vee, W^\vee).$$

Set  $F_{VW}^\phi(S) = h^{-1}(\phi)$ .  $F_{VW}^\phi$  is a functor  $d\text{-coart}^0 \rightarrow \text{Sets}$ .

**PROPOSITION 3.2**  $F_{VW}^\phi$  is representable.

*Proof.* Denote the coalgebra which represents  $F_{VW}^\phi$  by  $\underline{\text{Hom}}^\phi(V, W)$ . Let us construct it. Take a  $d$ -algebra  $a = \text{Hom}_k(V^\vee, W)$ . By Lemma 3.1 the morphism  $\phi$  defines an element  $\phi' \in MC(a[d-1])$  via the inclusion  $\text{Hom}(V^\vee/k, W) \rightarrow \text{Hom}(V^\vee, W)$ . We will denote by the same letter the corresponding element of degree  $d$  in  $a$ . Let  $a'$  be a  $d$ -algebra whose operations are the same as in  $a$  but the differential is  $d'x = dx + \{\phi', x\}$ , where  $d$  is the differential on  $a$ . We claim that  $\underline{\text{Hom}}^\phi(V, W) = a'$ . Indeed, Let  $b = \text{Hom}_k(V^\vee \otimes S/k, W)$  and  $c = \text{Hom}(V^\vee/k, W)$ . We have

$$\text{Hom}_{d\text{-coart}}(V^\vee \otimes S, W^\vee) \cong MC(b[d-1]).$$

We have a retraction

$$c \xrightarrow{G} b \xrightarrow{H} c$$

induced by the canonical retraction  $k \rightarrow S \rightarrow k$ . Therefore, we have a semidirect sum of  $d$ -algebras

$$b \cong c + \text{Hom}_k(S/k, a). \quad (7)$$

The map of Maurer-Cartan elements induced by  $H$  is the map  $h$ . Therefore,  $F_{VW}^\phi(S)$  can be alternatively described as the set of  $x \in MC(b[d-1])$ ,  $H(x) = \phi'$ . Using the splitting (7), we write  $x = \phi' + s$ ,  $s \in \text{Hom}_k(S/k, a)$ . The Maurer-Cartan equation reads as  $ds + \{\phi', s\} + \{s, s\}/2 = 0$ . This is the same as to say that  $s$  viewed as an element of  $\text{Hom}_k(S/k, a')$  is a morphism of  $d$ -coalgebras.  $\triangle$

**COROLLARY 3.3** *There is a natural map*

$$\circ : \underline{\text{Hom}}^\phi(U, V) \otimes \underline{\text{Hom}}^\psi(V, W) \rightarrow \underline{\text{Hom}}^{\psi \circ \phi}(U, W). \quad (8)$$

*This map is associative, meaning that the maps  $\circ(\circ \otimes Id)$  and  $\circ(Id \otimes \circ)$  from  $\underline{\text{Hom}}^\phi(U, V) \otimes \underline{\text{Hom}}^\psi(V, W) \otimes \underline{\text{Hom}}^\chi(W, X)$  to  $\underline{\text{Hom}}^{\chi \circ \psi \circ \phi}(U, X)$  coincide.*

*Proof.* We have a natural composition map

$$\begin{aligned} \text{Hom}_{d\text{-coart}}(U^\vee \otimes S, V^\vee) \times \text{Hom}_{d\text{-coart}}(V^\vee \otimes T, W^\vee) \\ \rightarrow \text{Hom}_{d\text{-coart}}(U^\vee \otimes S \otimes T, W^\vee). \end{aligned}$$

This map induces a morphism of the functors  $d\text{-coart}^0 \times d\text{-coart}^0 \rightarrow \text{Ens}$  :

$$F_{U,V}^\phi \times F_{V,W}^\psi \rightarrow F_{UW}^{\psi \circ \phi} \circ \bigotimes, \quad (9)$$

where  $\bigotimes : d\text{-coart} \times d\text{-coart} \rightarrow d\text{-coart}$  is the tensor product. We have the element

$$Id \in F_{U,V}^\phi(\underline{\text{Hom}}^\phi(U, V)) \cong \text{Hom}(\underline{\text{Hom}}^\phi(U, V), \underline{\text{Hom}}^\phi(U, V)).$$

Similarly, we have the element  $Id \in F_{V,W}^\psi(\underline{\text{Hom}}^\psi(V, W))$ . Define the morphism (8) as the image of  $Id \times Id$  under the morphism (9).  $\triangle$

## 4 $\underline{\text{Hom}}^{Id}(V, V)$ and $\text{def}(V)$ .

### 4.1 $\underline{\text{Hom}}^{Id}(V, V)$ is a $d$ -bialgebra

**DEFINITION 4.1** A structure of coproartinian  $d$ -bialgebra on a complex  $V$  is

- 1 a structure of a coproartinian  $d$ -coalgebra on  $V$ ;
- 2 a morphism of coalgebras  $m : V \otimes V \rightarrow V$  such that it defines an associative product on  $V$  with unit being the grouplike element of  $V$ .

**PROPOSITION 4.2** Let  $V$  be a  $d$ -algebra. Then  $\underline{\text{Hom}}^{Id}(V, V)$  is naturally a coproartinian  $d$ -bialgebra.

*Proof.* The product is given by the composition morphism (8)  $\Delta$

### 4.2 Restriction of $F_{VV}^{Id}$ onto the subcategory of coproartinian cocommutative coalgebras

**DEFINITION 4.3** A coproartinian cocommutative coalgebra is an object of  $d$ -coart such that its cocommutator is 0. Denote by  $\text{coart}$  the corresponding full subcategory of  $d$ -coart

**PROPOSITION 4.4** The inclusion functor  $I : \text{coart} \rightarrow d\text{-coart}$  has the right adjoint  $C$ . The coalgebra  $C(a)$  is the biggest cocommutative subcoalgebra of  $\text{Ker}\delta$ , where  $\delta$  is the cocommutator on  $a$

Note that for a coproartinian  $d$ -bialgebra  $X$ ,  $C(X)$  is naturally a cocommutative Hopf algebra.

**COROLLARY 4.5** The restriction of the functor  $F_{VV}^{Id}$  to the category of the coalgebras is represented by  $C(\underline{\text{Hom}}^{Id}(V, V))$ . The associative product on  $C(\underline{\text{Hom}}^{Id}(V, V))$  corresponds to the composition of functors (9).

On the other hand, since  $\text{def}(V)$  is the Lie algebra of the group of automorphisms of  $V^\vee$ , we have

**PROPOSITION 4.6** *The restriction of the functor  $F_{VV}^{Id}$  to the category of the coalgebras is presented by the universal enveloping algebra  $U(\text{def}(V))$  viewed as a cocommutative coalgebra. The associative product on  $U(\text{def}(V))$  corresponds to the morphism of functors (9).*

Thus,

**COROLLARY 4.7** *We have a canonical isomorphism of Hopf algebras  $C(\underline{\text{Hom}}^{Id}(V, V)) \rightarrow U(\text{def}(V))$*

Let us express  $C(\underline{\text{Hom}}^{Id}(V, V))$  explicitly. Recall that as a  $d$ -coalgebra  $\underline{\text{Hom}}^{Id}(V, V) \cong a^{\vee}$ , where  $a'$  is the  $d$ -algebra  $\text{Hom}_k(V^{\vee}, V)$  with the differential twisted by the corestriction of  $Id: V^{\vee} \xrightarrow{Id} V^{\vee} \xrightarrow{cor} V$  which is, of course, just the map  $cor \in a'$ . One sees that for a complex  $X$ ,  $C(\text{Cofree}_d(X)) \cong S(X)$ , where  $S(X)$  is a cofree coproartinian cocommutative coalgebra cogenerated by  $X$ . Therefore,  $C(\underline{\text{Hom}}^{Id}(V, V)) \cong S(a'[d])$ . One sees that as a complex  $a'[d]$  is isomorphic to  $\text{def}(V)$  and that the composition  $S(a'[d]) \cong C(\underline{\text{Hom}}^{Id}(V, V)) \rightarrow U(\text{def}(V))$  is the Poincaré-Birkhoff-Witt isomorphism.

## 5 A homotopy $(d+1)$ -algebra structure on $\text{def}(V, V)$

We can summarize our findings in the following way. We have a  $d$ -algebra  $a'$ , which as a complex is isomorphic to  $\text{def}(V)[-d]$ . Also we have the associative product  $a^{\vee} \otimes a^{\vee} \rightarrow a^{\vee}$  which turns it into  $d$ -bialgebra. Also we know that the restriction of this product onto  $S(a'[d]) = C(a^{\vee}) \subset a^{\vee}$  turns  $S(a'[d])$  into a Hopf algebra and

$$S(a'[d]) \cong U(\text{def}(V)). \quad (10)$$

Let us investigate these structures. First, note that  $a^{\vee}$  together with the cocommutative coproduct and the associative product is a cocommutative cofree Hopf algebra. It is well known that any such an algebra is isomorphic to  $U(\mathfrak{g})$  for a certain Lie algebra  $\mathfrak{g}$ . This Lie algebra is formed by the primitive elements of  $a^{\vee}$ , its commutator is the commutator with respect to the associative product and its differential is the restriction of the differential on  $a^{\vee}$ . One sees that for any space  $X$  the set of primitive elements of  $\text{Cofree}_d(X)$  is isomorphic to  $\text{CofreeLie}(X[1-d])[d-1]$ , where  $\text{CofreeLie}$  means 'the cofree Lie coalgebra cogenerated by'. Thus, we have a Lie algebra structure on the Lie coalgebra  $\text{CofreeLie}(a'[1])[d-1]$ . The fact that  $S(a'[d])$  is a Hopf algebra translates into the fact that  $\text{CofreeLie}(a'[1])[d-1]$  is

a differential Lie bialgebra, meaning that all operations are compatible with the differential and that the cocycle condition is fulfilled:

$$\delta([x, y]) = [\delta x, y] + (-1)^{|x|(d-1)}[x, \delta y]. \quad (11)$$

Note that the only difference between our bialgebra and usual bialgebras is that the coproduct has a nonzero grading. The isomorphism (10) means that the restriction of the commutator on the primitive elements

$$\text{def}(V) \cong a'[d] \subset \text{CofreeLie}(a'[1])[d-1] \quad (12)$$

coincides with the commutator on  $\text{def}(V)$ . Thus, (12) is a morphism of DGLA.

Now take the chain complex  $S(\text{CofreeLie}(a'[1])[d])$  of  $\text{CofreeLie}(a'[1])[d-1]$  as a Lie algebra.

Note that we have an isomorphism of graded spaces  $S(\text{CofreeLie}(a'[1])[d]) \cong \text{Cofree}_{d+1}(a'[1+d])$ . The cocycle condition (11) means that the differential on  $S(\text{CofreeLie}(a'[1])[d])$  is compatible with the coproduct and the cocommutator on  $\text{Cofree}_{d+1}(a'[1+d])$ . This means that we have a structure of homotopy  $(d+1)$ -algebra on a complex  $a' \cong \text{def}(V)[-d]$ . The morphism (12) means that the homotopy Lie algebra structure on  $\text{def}(V)$  induced from the homotopy  $(d+1)$ -structure on  $\text{def}(V)[-d]$  coincides with the Lie algebra structure on  $\text{def}(V)$  as the deformation Lie algebra. As for the commutative binary operation on  $\text{def}(V)$ , it is the same as the commutative product on  $a'$  as a  $d$ -algebra. One checks also that the commutator of  $a'$  as a  $d$ -algebra is homotopy trivial.

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