

# Deformations of Chiral Algebras

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## Abstract

We start studying chiral algebras (as defined by A. Beilinson and V. Drinfeld) from the point of view of deformation theory. First, we define the notion of deformation of a chiral algebra on a smooth curve  $X$  over a bundle of local artinian commutative algebras on  $X$  equipped with a flat connection (whereas ‘usual’ algebraic structures are deformed over a local artinian algebra) and we show that such deformations are controlled by a certain  $\ast$ -Lie algebra  $\mathfrak{g}$ . Then we try to contemplate a possible additional structure on  $\mathfrak{g}$  and we conjecture that this structure up to homotopy is a chiral analogue of Gerstenhaber algebra, i.e. a coisson algebra with odd coisson bracket (in the terminology of Beilinson-Drinfeld). Finally, we discuss possible applications of this structure to the problem of quantization of coisson algebras.

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## 1. Introduction

Chiral algebras were introduced in [1]. In the same paper the authors introduced the classical limit of a chiral algebra which they call a coisson algebra and posed the problem of quantization of coisson algebras. The goal of this paper is to show how the theory of deformation quantization (=the theory of deformations of associative algebras of a certain type) in the spirit of [3] can be developed in this situation.

Central object in the theory of deformations of associative algebras is the differential graded Lie algebra of Hochschild cochains. It turns out that in our situation it is more appropriate to use what we call pro- $\ast$ -Lie-algebras rather than usual Lie algebras (the notion of  $\ast$ -Lie algebra was also introduced in [1]). Next, we compute the cohomology of the pro- $\ast$ -Lie-algebra controlling chiral deformations of a free commutative  $\mathcal{D}_X$ -algebra  $SK$ , where  $K$  is a locally free  $\mathcal{D}_X$ -module.

Next, we state an analogue of Gerstenhaber theorem which says that the cohomology of the deformation complex of an associative algebra carries the structure of a Gerstenhaber algebra. We give a definition of a chiral analogue of Gerstenhaber

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algebra and define the operations of this structure on deformation pro- $*$ -Lie algebra of a chiral algebra.

Finally, mimicking Kontsevich's formality theorem, we formulate the formality conjecture for the deformation pro- $*$ -Lie algebra of the chiral algebra  $SK$  mentioned above and claim that this conjecture implies a 1-1 correspondence between deformations of  $SK$  and coisson brackets on  $SK$ .

## 2. Chiral algebras and their deformations

### 2.1. Chiral operations

In [1] chiral operations are defined as follows. Let  $X$  be a smooth curve and  $M_i, N$   $\mathcal{D}_X$ -modules. Denote by  $i_n : X \rightarrow X^n$  the diagonal embedding and by  $j_n : U_n \rightarrow X^n$  the open embedding of the complement to all diagonals in  $X^n$ . Set

$$P_{\mathbf{ch}}(M_1, \dots, M_n; N) = \text{hom}_{\mathcal{D}_{X^n}}(j_* j^*(M_1 \boxtimes \dots \boxtimes M_n), i_{n*} N). \quad (1)$$

In the case  $n = 0$  set

$$P_{\mathbf{ch}}(M) = H^0(M \otimes_{\mathcal{D}_X} \mathcal{O}_X).$$

Let  $M$  be a fixed  $\mathcal{D}_X$ -module. Write

$$P_{\mathbf{ch}M}(n) = P_{\mathbf{ch}}(M, M, \dots, M; M).$$

It is explained in [1] that  $P_M$  is an operad.

#### 2.1.1. Chiral algebras

Let  $\mathbf{lie}$  be the operad of Lie algebras. A *chiral algebra* structure on  $M$  is a homomorphism  $\mathbf{lie} \rightarrow P_M$ . We have a standard chiral algebra structure on  $M = \omega_X$ . A chiral algebra  $M$  is called unital if it is endowed with an injection  $\omega_X \rightarrow M$  of chiral algebras.

## 2.2. Deformations

### 2.2.1. Agreements

To simplify the exposition, we will only consider unital chiral algebras  $M$  with the following restrictions: we assume that  $X$  is affine and the  $\mathcal{D}_X$ -module  $M$  can be represented as  $M = \omega_X \oplus N$ , where  $N \cong E \otimes_{\mathcal{O}_X} \mathcal{D}_X$  for some locally free coherent sheaf  $E$ .

### 2.2.2. Nilpotent $\mathcal{D}_X$ -algebras

Let  $E$  be a left  $\mathcal{D}_X$ -module equipped with a commutative associative unital product  $E \otimes E \rightarrow E$ . Let  $u : \mathcal{O}_X \rightarrow E$  be the unit embedding. Call  $E$  nilpotent if there exists a  $\mathcal{D}_X$ -module splitting  $s : E \cong \mathcal{M} \oplus \mathcal{O}_X$  and a positive integer  $N$  such that the  $N$ -fold product vanishes on  $\mathcal{M}$ .  $\mathcal{M}$  is then a unique maximal  $\mathcal{D}_X$ -ideal in  $E$ .

### 2.2.3. Deformations over a nilpotent $\mathcal{D}_X$ -algebra

Let  $E$  be a nilpotent  $\mathcal{D}_X$ -algebra with maximal ideal  $\mathcal{M}$ . We have a notion of  $E$ -module and of an  $E$ -linear chiral algebra. For any  $\mathcal{D}_X$ -module  $M$ ,  $M_E := M \otimes_{\mathcal{O}_X} E$  is an  $E$ -module.

Let  $M$  be a chiral algebra. An  $E$ -linear unital chiral algebra structure on  $M_E$  is called *deformation of  $M$  over  $E$*  if the induced structure on  $M_E/\mathcal{M}.M_E \cong M$  coincides with the one on  $M$ . Denote by  $G_M(E)$  the set of all isomorphism classes of such deformations.

## 2.3. The functor $G_M$ and its representability

It is clear that  $E \mapsto G_M(E)$  is a functor from the category of nilpotent  $\mathcal{D}_X$ -algebras to the category of sets. In classical deformation theory one usually has a functor from the category of (usual) local Artinian (=nilpotent and finitely dimensional) algebras to the category of sets and one tries to represent it by a differential graded Lie algebra. In this section we will see that in our situation a natural substitute for a Lie algebra is a so-called  $*$ -Lie algebra in the sense of [1]. More precise, given a  $*$ -Lie algebra  $\mathfrak{g}$ , we are going to construct a functor  $F_{\mathfrak{g}}$  from the category of nilpotent  $\mathcal{D}_X$ -algebras to the category of sets. In the next section we will show that the functor  $G_M$  is 'pro-representable' in this sense. We will construct a pro- $*$ -Lie algebra  $\mathbf{def}_M$  (exact meaning will be given below) and an isomorphism of functors  $G_M$  and  $F_{\mathbf{def}_M}$ .

### 2.3.1. $*$ -Lie algebras

[1] Let  $\mathfrak{g}_i, N$  be right  $\mathcal{D}_X$ -modules. Set

$$P_*(\mathfrak{g}_1, \dots, \mathfrak{g}_n; N) := \text{hom}_{\mathcal{D}_X^n}(\mathfrak{g}_1 \boxtimes \dots \boxtimes \mathfrak{g}_n, i_n^* N),$$

and  $P_{*\mathfrak{g}}(n) := P(\mathfrak{g}, \dots, \mathfrak{g}; \mathfrak{g})$ . It is known that  $P_{*\mathfrak{g}}$  is an operad. A  $*$ -Lie algebra structure on  $\mathfrak{g}$  is by definition a morphism of operads  $f : \mathbf{lie} \rightarrow P_{*\mathfrak{g}}$ . Let  $b \in \mathbf{lie}(2)$  be the element corresponding to the Lie bracket. We call  $f(b) \in P_{*\mathfrak{g}}(2)$  the  $*$ -Lie bracket.

### 2.3.2.

Let  $\mathfrak{g}$  be a  $*$ -Lie algebra and  $A$  be a commutative  $\mathcal{D}_X$ -algebra. introduce a vector space  $\mathfrak{g}(A) = \mathfrak{g} \otimes_{\mathcal{D}_X} A$ . This space is naturally a Lie algebra. Indeed, we have a  $*$ -Lie bracket  $\mathfrak{g} \boxtimes \mathfrak{g} \rightarrow i_{2*}\mathfrak{g}$ . Multiply both parts by  $A \boxtimes A$ :

$$(\mathfrak{g} \boxtimes \mathfrak{g}) \otimes_{\mathcal{D}_X \times X} (A \boxtimes A) \rightarrow i_{2*}\mathfrak{g} \otimes_{\mathcal{D}_X \times X} (A \boxtimes A). \quad (*)$$

The left hand side is isomorphic to  $\mathfrak{g}(A) \otimes \mathfrak{g}(A)$ . The right hand side is isomorphic to  $\mathfrak{g} \otimes_{\mathcal{D}_X} (A \otimes_{\mathcal{O}_X} A)$ . Thus, (\*) becomes:

$$\mathfrak{g}(A) \otimes \mathfrak{g}(A) \rightarrow \mathfrak{g} \otimes_{\mathcal{D}_X} (A \otimes_{\mathcal{O}_X} A).$$

The product on  $A$  gives rise to a map

$$\mathfrak{g} \otimes_{\mathcal{D}_X} (A \otimes_{\mathcal{O}_X} A) \rightarrow \mathfrak{g} \otimes_{\mathcal{D}_X} A \cong \mathfrak{g}(A),$$

and we have a map  $\mathfrak{g}(A) \otimes \mathfrak{g}(A) \rightarrow \mathfrak{g}(A)$ . It is straightforward to check that this map is a Lie bracket.

### 2.3.3.

Let now  $\mathfrak{g}$  be a differential graded  $*$ -Lie algebra and let  $A$  be a differential graded commutative  $\mathcal{D}_X$ -algebra. Then  $\mathfrak{g}(A) := \mathfrak{g} \otimes_{\mathcal{D}_X} A$  is a differential graded Lie algebra.

### 2.3.4.

Let  $A$  be a nilpotent  $\mathcal{D}_X$  algebra and  $\mathcal{M}_A \subset A$  be the maximal nilpotent ideal. Then  $\mathfrak{g}(\mathcal{M}_A)$  is a nilpotent differential graded Lie algebra.

### 2.3.5.

Recall that given a differential graded nilpotent Lie algebra  $\mathfrak{n}$ , one can construct the so called Deligne groupoid  $\mathcal{G}_{\mathfrak{n}}$ . Its objects are all  $x \in \mathfrak{n}^1$  satisfying  $dx + [x, x]/2 = 0$  (so called Maurer-Cartan elements). The group  $\exp(\mathfrak{n}^0)$  acts on the set of Maurer-Cartan elements by gauge transformations.  $\mathcal{G}_{\mathfrak{n}}$  is the groupoid of this action. Denote by  $\mathcal{D}_{\mathfrak{n}}$  the set of isomorphism classes of this groupoid. If  $f : \mathfrak{n} \rightarrow \mathfrak{m}$  is a map of differential graded Lie algebras such that the induced map on cohomology  $H^i(f)$  is an isomorphism for all  $i \geq 0$ , then the induced map  $\mathcal{D}_{\mathfrak{n}} \rightarrow \mathcal{D}_{\mathfrak{m}}$  is a bijection. If  $\mathfrak{n}, \mathfrak{m}$  are both centered in non-negative degrees, then the induced map  $\mathcal{G}_{\mathfrak{n}} \rightarrow \mathcal{G}_{\mathfrak{m}}$  is an equivalence of categories. Since in our situation we will deal with Lie algebras centered in arbitrary degrees, we will use  $\mathcal{D}_{\mathfrak{n}}$  rather than groupoids.

### 2.3.6.

Set  $F_M(A) = \mathcal{D}_{\mathfrak{g}(\mathcal{M}_A)}$ . It is a functor from the category of nilpotent  $\mathcal{D}_X$ -algebras to the category of sets.

## 2.4. Pro- $*$ -Lie- algebras

$*$ -Lie algebras are insufficient for description of deformations of chiral algebras. We will thus develop a generalization. We need some preparation

### 2.4.1. Procategory

For an Abelian category  $C$  consider the category  $\mathbf{pro} C$  whose objects are functors  $I \rightarrow C$ , where  $I$  is a small filtered category. Let  $F_k : I_k \rightarrow C$ ,  $k = 1, 2$  be objects. Set

$$\mathrm{hom}(F_1, F_2) := \lim_{\mathrm{inv}}_{i_2 \in I_2} \lim_{\mathrm{dir}}_{i_1 \in I_1} (F_1(i_1), F_2(i_2)).$$

The composition of morphisms is naturally defined. One can show that  $\mathbf{pro} C$  is an Abelian category. Objects of  $\mathbf{pro} C$  are called pro-objects.

### 2.4.2. Direct image of pro- $\mathcal{D}$ -modules

Let  $M : I \rightarrow \mathcal{D}_Y\text{-mod}$  be a pro-object, where  $Y$  is a smooth algebraic variety and let  $f : Y \rightarrow Z$  be a locally closed embedding. Denote the composition  $f_* \circ M : I \rightarrow \mathcal{D}_Z\text{-mod}$  simply by  $f_* M$ . We will get a functor  $f_* : \mathbf{pro} \mathcal{D}_Y\text{-mod} \rightarrow \mathbf{pro} \mathcal{D}_Z\text{-mod}$ .

### 2.4.3. Chiral and \*-operations

For  $N, M_i \in \mathbf{pro} \mathcal{D}_X\text{-mod}$  we define  $P_*(M_1, \dots, M_n, N), P_{\mathbf{ch}}(M_1, \dots, M_n, N)$  by exactly the same formulas as for usual  $\mathcal{D}_X$ -modules.

### 2.4.4. pro-\*-Lie algebras

\*-Lie algebra structure on a pro- $\mathcal{D}_X$ -module is defined in the same way as for usual  $\mathcal{D}_X$ -modules.

### 2.4.5.

For a pro-right  $\mathcal{D}_X$ -module  $I \rightarrow M$  and a left  $\mathcal{D}_X$ -module  $L$  define a vector space  $M \otimes_{\mathcal{D}_X} L = \liminf_I (M \otimes_{\mathcal{D}_X} L)$ . For a \*-Lie algebra  $\mathfrak{g}$  and a commutative  $\mathcal{D}_X$ -algebra  $a$ ,  $\mathfrak{g} \otimes_{\mathcal{D}_X} a$  is a Lie algebra. Construction is the same as for usual \*-Lie algebras. Similarly, we can define the functor  $F_{\mathfrak{g}}$  from the category of nilpotent  $\mathcal{D}_X$ -algebras to the category of sets.

## 2.5. Representability of $G_M$ by a pro-\*-Lie algebra

We are going to construct a differential graded \*-pro-Lie algebra  $\mathfrak{g}$  such that  $F_{\mathfrak{g}}$  is equivalent to  $G_M$ . We need a couple of technical lemmas.

### 2.5.1.

Let  $Y$  be a smooth affine algebraic varieties and  $U, V$  be right  $\mathcal{D}_Y$ -modules. Let  $U_{\alpha}, \alpha \in A$  be the family of all finitely generated submodules of  $U$ . Denote  $\mathbf{prohom}(U, V) = \liminf_{\alpha} (U_{\alpha}, V)$  the corresponding pro-vector space.

### 2.5.2.

Let  $i : X \rightarrow Y$  be a closed embedding, let  $B$  be a  $\mathcal{D}_Y$ -module and  $M$  be a  $\mathcal{D}_X$ -module. Then

$$\mathbf{prohom}_{\mathcal{D}_Y}(B, i_*(M \otimes_{\mathcal{O}_X} \mathcal{D}_X))$$

is a pro- $\mathcal{D}_X$ -module. Denote it by  $P(B, M)$ . Let now  $Y = X^n$ .

**Lemma 2.1** *Assume that  $B = j_{n*} j_n^*(E \otimes_{\mathcal{O}_{X^n}} \mathcal{D}_{X^n})$ , where  $E$  is locally free and coherent. For any left  $\mathcal{D}_X$ -module  $L$  we have*

$$\mathbf{prohom}(B, i_{n*}(M \otimes_{\mathcal{O}_X} L)) \cong P(B, M) \otimes_{\mathcal{D}_X} L.$$

**Proof.** Let  $F = j_{n*}j_n^*E$ . We have  $B = F \otimes_{\mathcal{O}_{X^n}} \mathcal{D}_{X^n}$ . Note that  $F = \text{limdir } F_\alpha$ , where  $F_\alpha$  runs through the set of all free coherent submodules of  $F$ .

We have

$$\begin{aligned} P(B, M) &= \text{liminv } \text{hom}_{\mathcal{D}_{X^n}}(F_\alpha \otimes_{\mathcal{O}_{X^n}} \mathcal{D}_{X^n}, i_{n*}(M \otimes_{\mathcal{O}_X} L)) \\ &\cong \text{liminv } F_\alpha^* \otimes_{\mathcal{O}_{X^n}} i_{n*}(M \otimes_{\mathcal{O}_X} \mathcal{D}_X) \otimes_{\mathcal{D}_X} L \\ &\cong \text{liminv } \text{hom}_{\mathcal{O}_{X^n}}(F_\alpha, i_{n*}(M \otimes_{\mathcal{O}_X} \mathcal{D}_X)) \otimes_{\mathcal{D}_X} L \\ &\cong \mathbf{prohom}(B, i_{n*}(M \otimes_{\mathcal{O}_X} \mathcal{D}_X)) \otimes_{\mathcal{D}_X} L. \end{aligned}$$

### 2.5.3.

Let  $B, M$  be as above. We have a natural morphism

$$i : i_{n*}P(B, M) \cong P(B, M) \otimes_{\mathcal{D}_X} \mathcal{D}_X^{\otimes_n} \rightarrow \mathbf{prohom}(B, M \otimes_{\mathcal{D}_X} (\mathcal{D}_X)^{\otimes_n}).$$

The above lemmas imply that  $i$  is an isomorphism.

### 2.5.4.

Let  $M$  be a right  $\mathcal{D}_X$ -module. Set

$$U_M(n) = \mathcal{P}_{\mathbf{ch}}(M, M, \dots, M; M \otimes \mathcal{D}_X) := \mathbf{prohom}(j_{n*}j_n^*M^{\boxtimes n}, i_{n*}(M \otimes \mathcal{D}_X)),$$

it is a right pro- $\mathcal{D}_X$ -module. We will endow the collection  $U_M$  with the structure of an operad in \*-pseudotensor category. This means that we will define the composition maps

$$\circ_i \in P_*(U_M(n), U_M(m); U_M(n+m-1)),$$

$i = 1, \dots, n+m-1$ , satisfying the operadic axioms. We need a couple of technical facts.

### 2.5.5.

Let  $i_n : X \rightarrow X^n$  be the diagonal embedding and  $p_n^i : X^n \rightarrow X$  be the projections. Lemma 2.5.3. implies that

#### Lemma 2.2

$$i_{n*}U_M(k) \cong \mathcal{P}_{\mathbf{ch}}(M, \dots, M; M \otimes_{\mathcal{D}_X} \mathcal{D}_X^{\otimes n}).$$

**Lemma 2.3** For any  $\mathcal{D}_X$ -modules  $M, S$  we have an isomorphism

$$i_{n*}(M) \otimes p_n^{i*}S \cong i_{n*}(M \otimes S).$$

**2.5.6.**

We are now ready to define the desired structure. In virtue of 2.3 we have natural maps:

$$\mathcal{P}_{\mathbf{ch}}(M_1, \dots, M_n; N) \rightarrow \mathcal{P}_{\mathbf{ch}}(M_1, \dots, M_i \otimes \mathcal{D}_X, \dots, M_n; (N \otimes \mathcal{D}_X)).$$

Thus, we have maps:

$$\begin{aligned} & \mathcal{P}_{\mathbf{ch}}(M_1, \dots, M_n; (N_i \otimes \mathcal{D}_X)) \boxtimes \mathcal{P}_{\mathbf{ch}}(N_1, \dots, N_m; (K \otimes \mathcal{D}_X)) \\ \rightarrow & \mathcal{P}_{\mathbf{ch}}(M_1, \dots, M_n; (N_i \otimes \mathcal{D}_X)) \\ & \boxtimes \mathcal{P}_{\mathbf{ch}}(N_1, \dots, N_i \otimes \mathcal{D}_X, \dots, N_m; (K \otimes \mathcal{D}_X) \otimes \mathcal{D}_X) \\ \rightarrow & \mathcal{P}_{\mathbf{ch}}(N_1, \dots, N_{i-1}, M_1, \dots, M_n, N_{i+1}, \dots, N_m; K \otimes \mathcal{D}_X \otimes \mathcal{D}_X) \\ \cong & i_{2*} \mathcal{P}_{\mathbf{ch}}(N_1, \dots, N_{i-1}, M_1, \dots, M_n, N_{i+1}, \dots, N_m, K \otimes \mathcal{D}_X). \end{aligned}$$

By substituting  $M$  instead of all  $N_i, M_j, K$ , we get the desired insertion map

$$\circ_i : U_M(n) \boxtimes U_M(m) \rightarrow i_{2*} U_M(n+m-1).$$

**2.5.7.**

Similarly, we have insertion maps

$$\circ_i : U_M(n) \otimes \mathcal{P}_{\mathbf{ch}_M}(m) \rightarrow U_M(n+m-1),$$

and

$$\circ_i : \mathcal{P}_{\mathbf{ch}_M}(n) \otimes U_M(m) \rightarrow U_M(n+m-1).$$

**2.5.8.**

Let  $\mathcal{O}$  be a differential graded operad. Set

$$\mathfrak{g}_{\mathcal{O},n} := \mathcal{O}(n)^{S_n},$$

and  $\mathfrak{g}_{\mathcal{O}} = \bigoplus_n \mathfrak{g}_{\mathcal{O},n}[1-n]$ .

Let  $p_n : \mathcal{O}(n) \rightarrow \mathfrak{g}_{\mathcal{O},n}$  be the natural projection, which is the symmetrization map. Define the brace  $(x, y) \mapsto x\{y\}$ ,  $\mathfrak{g}_{\mathcal{O},n} \otimes \mathfrak{g}_{\mathcal{O},m} \rightarrow \mathfrak{g}_{\mathcal{O},n+m-1}$  as follows.

$$x\{y\} = np_n(\circ_1(x, y)) \quad (2)$$

and the bracket

$$[x, y] = x\{y\} - (-1)^{|x||y|} y\{x\}. \quad (3)$$

We see that  $[\cdot, \cdot]$  is a Lie bracket. Thus,  $\mathfrak{g}_{\mathcal{O}}$  is a differential graded Lie algebra. For an operad  $\mathcal{O}$  denote by  $\mathcal{O}'$  the shifted operad such that the structure of an  $\mathcal{O}'$ -algebra on a complex  $V$  is equivalent to the structure of an  $\mathcal{O}$ -algebra on a complex  $V[1]$ . Thus,  $\mathcal{O}'(n) = \mathcal{O}(n) \otimes \epsilon_n[1-n]$ , where  $\epsilon_n$  is the sign representation of  $S_n$ .

Let  $\mathcal{O}$  be an operad of vector spaces. The set of Maurer-Cartan elements of  $\mathfrak{g}_{\mathcal{O}'}$  is in 1-1 correspondence with maps of operads  $\mathbf{lie} \rightarrow \mathcal{O}$ .

Assume that  $\mathcal{O}(1)$  is a nilpotent algebra ( $x^n = 0$  for any  $x \in \mathcal{O}(1)$ ). Let  $A$  be  $\mathcal{O}(1)$  with adjoined unit and let  $A^\times$  be the group of invertible elements.  $A^\times$  acts on  $\mathcal{O}$  by automorphisms. Therefore,  $A^\times$  acts on the set of maps  $\mathbf{lie} \rightarrow \mathcal{O}$ . The groupoid of this action is isomorphic to the Deligne groupoid of  $\mathfrak{g}_{\mathcal{O}'}$ .

**2.5.9.**

Similarly, let  $\mathcal{A}$  be a  $*$ -operad. Then formula 3 defines a Lie- $*$  algebra  $\mathfrak{g}_{\mathcal{A}}$ . We have natural action of a usual pro-Lie algebra  $\mathfrak{g}_{\mathcal{P}_{\mathbf{ch}}(M)}$  on a pro  $*$ -Lie algebra  $\mathfrak{g}_{U_M}$  by derivations. The chiral bracket  $b \in \mathfrak{g}_{\mathcal{P}_{\mathbf{ch}}(M)}^1$  satisfies  $[b, b] = 0$ . Therefore, the bracket with  $b$  defines a differential on  $\mathfrak{g}_{U_M}$ . Denote this differential graded  $*$ -Lie algebra by  $\mathfrak{d}_M$ .

**2.5.10.**

To avoid using derived functors, we will slightly modify  $\mathfrak{d}_M$ . Recall that  $M = \omega_X \oplus N$ , where  $N$  is free. Let

$$\mathcal{P}_{\mathbf{ch}}^{\mathbf{red}}(M, \dots, M; M \otimes \mathcal{D}_X) \subset \mathcal{P}_{\mathbf{ch}}(M, \dots, M; M \otimes \mathcal{D}_X)$$

be the subset of all operations vanishing under all restrictions

$$\mathcal{P}_{\mathbf{ch}}(M, \dots, M; M \otimes \mathcal{D}_X) \rightarrow \mathcal{P}_{\mathbf{ch}}(M, \dots, M, \omega_X, M, \dots, M; M \otimes \mathcal{D}_X).$$

Let  $\mathbf{def}_M \subset \mathfrak{d}_M$  be the submodule such that

$$\mathbf{def}_M = \oplus_n (\mathcal{P}_{\mathbf{ch}}^{\mathbf{red}}(M, \dots, M; M \otimes \mathcal{D}_X) \otimes \epsilon_n)^{S_n} [1 - n].$$

We see that  $\mathbf{def}_M$  is a  $*$ -Lie differential subalgebra of  $\mathfrak{d}_M$ .

**2.5.11.**

**Proposition 2.4** *The functors  $G_M$  and  $F_{\mathbf{def}_M}$  are canonically isomorphic.*

**2.6. Example**

Let  $K$  be a free left  $\mathcal{D}_X$ -module. Let  $T^i K = K^{\otimes_{\mathcal{D}_X} i}$ . The symmetric group  $S_i$  acts on the  $\mathcal{D}_X$ -module  $T^i K$ ; let  $S^i K = (T^i K)^{S_i}$  be the submodule of invariants and  $SK = \oplus_{i=0}^{\infty} S^i K$ .  $SK$  is naturally a free commutative  $\mathcal{D}_X$ -algebra and, hence,  $SK^r := SK \otimes \omega_X$  is a chiral algebra. We will compute the cohomology of the  $\mathcal{D}_X$ -module  $\mathbf{def}_{SK^r}$ . Let  $S_0 K = \oplus_{n=1}^{\infty} S^i K$ . We have:

$$\mathbf{def}_{SK^r} = \oplus_n (P_{\mathbf{ch}}(S_0 K^r[1], \dots, S_0 K^r[1]; SK^r \otimes \mathcal{D}_X)[1])^{S_n}.$$

On the other hand, denote by  $\Omega := SK \otimes K$ . Consider  $\Omega$  as an  $SK$ - $\mathcal{D}_X$ -module of differentials of  $SK$ . We have the de Rham differential  $D : S_0 K \rightarrow \Omega$ . We have a through map

$$\begin{aligned} c_n : P_{\mathbf{ch}}(K[1]^r, \dots, K[1]^r, SK[1]^r)^{S_n} &\cong P_{\mathbf{ch}}^{SK}(\Omega[1]^r, \dots, \Omega[1]^r, SK[1]^r)^{S_n} \\ &\xrightarrow{D} P_{\mathbf{ch}}(S_0 K[1]^r, \dots, S_0 K[1]^r, SK[1]^r)^{S_n}, \end{aligned}$$

where  $P_{\mathbf{ch}}^{SK}$  stands for  $SK$ -linear chiral operations. Denote by the same letter the induced map

$$c_n : P_{\mathbf{ch}}(K[1]^r, \dots, K[1]^r, SK[1]^r)^{S_n} \rightarrow \mathbf{def}_{SK}.$$



**Proposition 2.5** (1)  $dc_n = 0$ ;  
 (2)  $c_n$  induces an isomorphism

$$P_{\mathbf{ch}}(K[1]^r, \dots, K[1]^r, SK[1]^r)^{S_n} \rightarrow H^{n-1}(\mathbf{def}_{SK})[1-n].$$

### 2.6.1.

For a chiral algebra  $M$  denote by  $H_M$  the graded Lie algebra of cohomology of  $\mathbf{def}_M$ .

### 2.6.2.

Assume that  $K$  is finitely generated. Let

$$K^\vee = \text{hom}(K, \mathcal{D}_X) \otimes (\omega_X)^{-1}$$

be the dual module. Then

$$H_{SK^r} \cong \oplus_n (P_*(K^r, \dots, K^r; SK^r \otimes \mathcal{D}_X) \varepsilon_n)^{S_n} [1-n] = \oplus_n (\wedge^n K^\vee \otimes_{\mathcal{O}_X} SK)^r [1-n].$$

### 2.6.3.

We will postpone the calculation of the \*-Lie bracket on  $H_{SK^r}$  until we show in the next section that  $H_M$  has in fact a richer structure.

## 3. Algebraic structure on the cohomology of the deformation pro-\*-Lie algebra

We will keep the agreements and the notations from 2.2.1..

### 3.1. Cup product

We will define a chiral operation  $\cup \in P_{\mathbf{ch}} \mathbf{def}_M[-1](2)$  and then we will study the induced map on cohomology.

#### 3.1.1.

Recall that

$$\mathbf{def}_M[-1] \cong \oplus_n (a_n)^{S_n},$$

where

$$a_n = P_{\mathbf{ch}}(N[1], \dots, N[1]; M \otimes \mathcal{D}_X).$$

Let  $i_n : X \rightarrow X^n$  be the diagonal embedding and let  $p^i : X^n \rightarrow X$   $U_n \subset X^n$  be the complement to the union of all pairwise diagonals  $p^i x = p^j x$  and  $j_n : U_n \rightarrow X^n$  be the open embedding. Let  $U_{n,m} \subset X^{n+m}$  be the complement to the diagonals  $p^i x = p^j x$ , where  $1 \leq i \leq n$ ,  $n+1 \leq j \leq n+m$  and  $j_{nm} : U_{nm} \rightarrow X^{n+m}$  be the embedding

Compute

$$\begin{aligned}
j_{2*}j_2^*(a_n \boxtimes a_m) &\cong \text{hom}(j_{n*}j_n^*(N[1]^{\boxtimes n}) \boxtimes j_{m*}j_m^*(N[1]^{\boxtimes m}), \\
&\quad (i_n \times i_m)_*(j_{2*}j_2^*(M \otimes \mathcal{D}_X \boxtimes M \otimes \mathcal{D}_X))) \\
&\cong \text{hom}(j_{n*}j_n^*(N[1]^{\boxtimes n}) \boxtimes j_{m*}j_m^*(N[1]^{\boxtimes m}) \otimes j_*\mathcal{O}(U_{n,m}), \\
&\quad (i_n \times i_m)_*(j_{2*}j_2^*(M \otimes \mathcal{D}_X \boxtimes M \otimes \mathcal{D}_X) \otimes j_*\mathcal{O}(U_{n,m}))) \\
&\cong \text{hom}(j_{n+m*}j_{n+m}^*(N[1]^{\boxtimes n+m}), (i_n \times i_m)_*(j_{2*}j_2^*(M \boxtimes M)) \otimes \mathcal{D}_{X \times X}).
\end{aligned}$$

Taking the composition with the chiral operation on  $M$ , we obtain a chiral operation

$$j_{2*}j_2^*(a_n \boxtimes a_m) \rightarrow \text{hom}(j_{n+m*}j_{n+m}^*(N[1]^{\boxtimes n+m}), i_{n+m*}(M \otimes \mathcal{D}_X \otimes \mathcal{D}_X)) \cong i_{2*}a_{n+m},$$

which induces a chiral operation from  $P_{\mathbf{ch}}(a_n^{S_n}, a_m^{S_m}; a_{n+m}^{S_{n+m}})$  and, hence, an operation  $\cup \in (P_{\mathbf{ch}}(\mathbf{def}_M[-1], \mathbf{def}_M[-1]; \mathbf{def}_M[-1]))^{S^2}$ .

### 3.1.2.

To investigate the properties of this operation, consider the brace  $*$ -operation  $\cdot\{\} \in P_*(\mathbf{def}_M, \mathbf{def}_M; \mathbf{def}_M)$  defined by formula (2). Let

$$r : P_{\mathbf{ch}}(A_1, A_2; A_3) \rightarrow P_*(A_1, A_2; A_3)$$

be the natural map

**Proposition 3.1**  $d(\cdot\{\}) = r(\cup)$ .

Let  $\cup_h$  be the induced operation on  $H_M[-1]$ . The above proposition implies that  $r(\cup_h) = 0$ . In virtue of exact sequence

$$0 \rightarrow \text{hom}((A_1)^l \otimes (A_2)^l, (A_3)^l) \rightarrow P_{\mathbf{ch}}(A_1, A_2; A_3) \xrightarrow{r} P_*(A_1, A_2; A_3),$$

$\cup_h$  defines a  $\mathcal{D}_X$ -commutative product  $H_M[-1] \otimes H_M[-1] \rightarrow H_M[-1]$ , denoted by the same letter.

### 3.1.3.

**Proposition 3.2**  $\cup_h$  is associative.

### 3.1.4. Leibnitz rule

We are going to establish a relation between  $\cup$  and  $\cdot\{\}$ . This relation is similar to the one of coisson algebras. Our exposition will mimic the definition of coisson algebras from [1].

**3.1.5.**

Let  $A_i$  be right  $\mathcal{D}_X$ -modules. Write  $A_1 \overset{!}{\otimes} A_2 := (A_1^l \otimes A_2^l)^r$ ;  $P_!(A_1, A_2; A_3) := \text{hom}(A_1 \overset{!}{\otimes} A_2, A_3)$ . We have  $(A \overset{!}{\otimes} B) = i_2^*(B \boxtimes C)$ ;

$$i_{2*}(A \overset{!}{\otimes} B) \rightarrow i_{2*}A \otimes p_2^*(B^l).$$

We have a map

$$c : P_*(A_1, A_2; B) \otimes P_!(B, C; D) \rightarrow P_*(A_1, A_2 \overset{!}{\otimes} C; D)$$

defined as follows. Let  $u : A_1 \boxtimes A_2 \rightarrow i_{2*}B$  and  $m : B \overset{!}{\otimes} C \rightarrow D$ . Put

$$c(u, v) : A_1 \boxtimes (A_2 \overset{!}{\otimes} C) \cong (A_1 \boxtimes A_2) \otimes p_2^*(C^l) \rightarrow i_{2*}B \otimes p_2^*(C^l) \cong i_{2*}(B \overset{!}{\otimes} C) \rightarrow i_{2*}D.$$

**3.1.6.**

Denote

$$e = c(\square, \cup_h) \in P_*(H_M, H_M \overset{!}{\otimes} H_M; H_M).$$

Let  $T : H_M \overset{!}{\otimes} H_M \rightarrow H_M \overset{!}{\otimes} H_M$  be the action of symmetric group and let  $e^T$  be the composition with  $T$ . Let  $f \in P_*(H_M, H_M \overset{!}{\otimes} H_M; H_M)$  be defined by:

$$H_M \boxtimes (H_M \overset{!}{\otimes} H_M) \xrightarrow{\cup} H_M \boxtimes H_M \xrightarrow{\parallel} i_*H_M.$$

**Proposition 3.3** *We have  $f = e + e^T$ .*

In other words, the cup product and the bracket satisfy the Leibnitz identity.

**3.1.7.**

We see that  $H_M$  has a pro- $*$ -Lie bracket,  $(H_M)^l[1]$  has a commutative  $\mathcal{D}_X$ -algebra structure, and these structures satisfy the Leibnitz identity. Call this structure a *c-Gerstenhaber algebra* structure. Thus, our findings can be summarized as follows.

**Theorem 3.4** *The cohomology of the deformation pro- $*$ -Lie algebra of a chiral algebra is naturally a pro-c-Gerstenhaber algebra.*

**3.2. Example  $M = (SK)^r$** 

We come back to our example 2.6.. For simplicity assume  $K$  is finitely generated free  $\mathcal{D}_X$ -module. We have seen in this case that

$$(H_M)^l[-1] \cong \oplus_i \wedge^i K^\vee \otimes S^K[-i] \cong S(K^\vee[-1] \oplus K).$$

**Proposition 3.5** *The cup product on  $H_M$  coincides with the natural one on the symmetric power algebra.*

**3.2.1.**

To describe the bracket it suffices to define it on the submodule of generators  $G = (K^\vee[-1] \oplus K)^r$ . Define  $[\ ] \in P_*(G, G; H_M)$  to be zero when restricted onto  $K^r \boxtimes K^r$  and  $K^{\vee r}[-1] \boxtimes K^{\vee r}[-1]$ . Restriction onto  $K \boxtimes K^\vee[-1]$  takes values in  $\omega_X \subset H_M$  and is given by the canonic \*-pairing from [1]

$$(K^\vee \boxtimes K)^r \rightarrow i_{2*}\omega_X.$$

Recall the definition. We have  $K^{\vee r} = \text{hom}(K^r, \mathcal{D}_X \otimes \omega_X)$ . For open  $U, V \subset X$  we have the composition map

$$K(U) \otimes K^\vee(V) \rightarrow \mathcal{D}_X \otimes \omega_X(U \cap V) \cong i_{2*}\omega_X(U \times V)$$

which defines the pairing. This uniquely defines the \*-Lie bracket.

## 4. Formality Conjecture

Following the logic of Kontsevich's formality theorem, one can formulate a formality conjecture in this situation.

### 4.1. Quasi-isomorphisms

A map  $f : \mathfrak{g} \rightarrow \mathfrak{h}$  of differential graded pro-\*-Lie algebras is called quasi-isomorphism if it induces an isomorphism on cohomology. Call a pro-\*-Lie algebra *perfect* if such is its underlying complex of pro-vector spaces. The morphism  $f$  is called *perfect quasi-isomorphism* if it is a quasi-isomorphism and both  $\mathfrak{g}$  and  $\mathfrak{h}$  are perfect.

Two perfect pro-\*-lie algebras are called perfectly quasi-isomorphic if there exists a chain of perfect quasi-isomorphisms connecting  $\mathfrak{g}$  and  $\mathfrak{h}$ .

**Conjecture 4.1**  $\text{def}_{SK}$  and  $H_{SK}$  are perfectly quasi-isomorphic.

The importance of this conjecture can be seen from the following theorem:

**Theorem 4.2** Any chain of perfect quasi-isomorphisms between  $\text{def}_{SK^r}$  and  $H_{SK^r}$  establishes a bijection between the set of isomorphism classes of  $A$ -linear coisson brackets on  $SK^r \otimes A$  which vanish modulo the maximal ideal  $\mathcal{M}_A$  and the set of isomorphism classes of all deformations of the chiral algebra  $SK^r$  over  $A$ .

## References

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