MICROLOCAL CONDITION FOR NON-DISPLACEABILITY

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To Boris Tsygan on his 50-th birthday

ABSTRACT. We formulate a sufficient condition for non-displaceability (by Hamiltonian symplectomorphisms which are identity outside of a compact) of a pair of subsets in a cotangent bundle. This condition is based on micro-local analysis of sheaves on manifolds by Kashiwara-Schapira. This condition is used to prove that the real projective space and the Clifford torus inside the complex projective space are mutually non-displaceable

1. Introduction

Let M be a symplectic manifold and $A, B \subset M$ its compact subsets. A and B are called non-displaceable if $A \cap X(B) \neq \emptyset$, where X is any Hamiltonian symplectomorphism of M which is identity outside of a compact. Given such A and B, it is, in general, a non-trivial problem to decide, whether they are displaceable or not (see, for example, [3] and the literature therein). In non-trivial cases (when, say, A and B can be displaced by a diffeomorphism), all the methods known so far use different versions of Floer cohomology.

In this paper we introduce a sufficient condition for non-displaceability in the case when $M = T^*X$ with the standard symplectic structure. Our approach is based on Kashiwara-Shapira's microlocal theory of sheaves on manifolds and is independent of Floer's theory. We apply our condition in the following setting. Let our sympleectic manifold be $\mathbb{CP}^N$ with the standard symplectic structure and let our subsets be $\mathbb{RP}^N \subset \mathbb{CP}^N$ and $\mathbb{T}^N \subset \mathbb{CP}^N$, where $\mathbb{T}^{\mathbb{N}}$ is the Clifford torus consisting of all points $(z_0 : z_1 : \cdots : z_n)$ such that $|z_0| = |z_1| = \cdots = |z_n|$. Let A and B be arbitrarily chosen from the two subsets specified, we show that such A and B are non-displaceable. Same result has been proven in [3] using Hamiltonian Floer theory. Non-displaceability of Clifford torus has been proven in [4] via computing Floer cohomology.

Observe that our condition applies despite $\mathbb{CP}^N \neq T^*X$. We use a certain Lagrangian correspondence between $T^*\mathrm{SU}(N)$ and $\mathbb{CP}^N \times (\mathbb{CP}^N)^{\mathrm{opp}}$, where the symplectic form on $(\mathbb{CP}^N)^{\mathrm{opp}}$ equals the opposite to that on $\mathbb{CP}^N$, see Sec. 4.0.12. This way our original problem gets reduced to non-displaceablity of certain subsets in $T^*\mathrm{SU}(N)$.

Let us now get back to the non-displaceability condition for subsets in a symplectic manifold T^*X, where X is a smooth manifold. Fix a ground field $\mathbb{K}$. We start with a category $\mathcal{D}(X)$ which is defined as a full subcategory of the unbounded derived category of sheaves of $\mathbb{K}$-vector spaces on $X \times \mathbb{R}$, consisting of all objects $F \in D(X \times \mathbb{R})$ satisfying the following condition: for any open $U \subset X$ and any $c \in \mathbb{R} \cup \{\infty\}$, $R\Gamma_c(U \times (-\infty, c); F) = 0$. The category $\mathcal{D}(X)$ admits a microlocal definition. Let ∂_t be the vector field on $X \times \mathbb{R}$ corresponding to the infinitesimal shifts along $\mathbb{R}$. Let $\Omega_{\leq 0} \subset T^*(X \times \mathbb{R})$ be the subset consisting of all 1-forms η satisfying $i_{\partial_t}\eta \leq 0$. Let $\mathcal{C}_{\leq 0} \subset D(X \times \mathbb{R})$

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be the full subcategory consisting of all objects microsupported on $\Omega_{\leq 0}$. One can show that $\mathcal{D}(X)$ is the left orthogonal complement to $\mathcal{C}_{\leq 0}$.

One can show that the embedding $\mathcal{C}_{\leq 0} \subset D(X \times \mathbb{R})$ admits a left adjoint. Therefore, $\mathcal{D}(X)$ can be identified with a quotient $D(X \times \mathbb{R})/\mathcal{C}_{\leq 0}$. This motivates us to define microsupports of objects from $\mathcal{D}(X)$ as conic closed subsets of $\Omega_{>0} := T^*(X \times \mathbb{R}) \backslash \Omega_{\leq 0}$. Thus, we set $\mathrm{SS}_{\mathcal{D}}(F) := \mathrm{SS}(F) \cap \Omega_{>0}$ for any $F \in \mathcal{D}(X)$.

Let us identify $T^*(X \times \mathbb{R}) = T^*X \times T^*\mathbb{R}$. Let $A \subset T^*X$ be a subset. Define $\mathrm{Cone}(A) \subset \Omega_{>0}$ to consist of all points $(\eta, \alpha) \in T^*X \times T^*\mathbb{R}$ such that $i_{\partial_t}\alpha > 0$ (meaning that $(\eta, \alpha) \in \Omega_{>0}$) and

$$\frac{\eta}{i_{\partial_t}\alpha} \in A.$$

Let $\mathcal{D}_A(X) \subset \mathcal{D}(X)$ be the full subcategory consisting of all $F \in \mathcal{D}(X)$ such that $\mathrm{SS}_{\mathcal{D}}(F) \subset \mathrm{Cone}(A)$. This way we can link subsets of T^*X with the category $\mathcal{D}(X)$.

Let $c \in \mathbb{R}$, let $T_c : X \times \mathbb{R} \to X \times \mathbb{R}$ be the shift by c: $T_c(x, t) = (x, t + c)$. One sees that $T_c(\mathrm{Cone}(A)) = \mathrm{Cone}(A)$. Therefore, the endofunctor $T_{c*} : D(X \times \mathbb{R}) \to D(X \times \mathbb{R})$ preserves $\mathcal{D}_A(X)$ for all A. For any $c > 0$, one can construct a natural transformation $\tau_c : \mathrm{Id} \to T_{c*}$ of endofunctors on $\mathcal{D}_A(X)$ for any A, see Sec. 2.2.2.

We can now formulate the non-displaceablity condition (Theorem 3.1).

Let $A, B \subset T^*X$ be compact subsets. Suppose there exist $F_A \in \mathcal{D}_A(X)$; $F_B \in \mathcal{D}_B(X)$ such that for any $c \geq 0$, the natural map $R\hom(F_A; F_B) \to R\hom(F_A; T_{c*}F_B)$, induced by τ_c, does not vanish. Then A and B are non-displaceable.

Remark. For $c \in \mathbb{R}$ set $H_c(F_A, F_B) := H_c := R\hom(F_A; T_{c*}F_B)$. For any $d \geq 0$, the natural transformation τ_d induces a map $\tau_{c,c+d} : H_c \to H_{c+d}$.

Let $H(F_A, F_B) := H \subset \prod_{c \in \mathbb{R}} H_c$ be defined as a subset consisting of all collections $h_c \in H_c$ such that there exists a sequence $c_1 < c_2 < \cdots < c_n < \cdots$; $c_n \to \infty$ such that $h_c = 0$ for all $c \notin \{c_1, c_2, \ldots, c_n, \ldots\}$. The maps $\tau_{c,c+d}$ induce maps $\tau_d : H \to H$ for all $d \geq 0$. This way we get an action of the semigroup $\mathbb{R}_{\geq 0}$ on H. This implies that Novikov's ring, which is a group ring of $\mathbb{R}_{\geq 0}$, acts on H. There are indications that thus defined module over Novikov's ring H is related to Floer cohomology of the pair A, B. In this language, our nondisplaceability condition means that $H(F_A, F_B)$ has a non-trivial non-torsion part.

Remark It seems likely that under an appropriate version of Riemann-Hilbert correspondence our picture should become similar to the setting of [8]. This paper can be considered as an attempt to translate [8] into the language of constructible sheaves.

Remark There is some similarity between our theory and the approach from [7] where the authors identify the derived category of constructible sheaves on X with a certain version of the Fukaya category on T^*X. The authors use Lagrangian subsets of T^*X which are close to being conic, whereas we work with compact subsets of T^*X.

Let us now briefly describe the way our non-displaceability condition is applied to the above mentioned example $\mathbb{RP}^N, \mathbb{T}^N \subset \mathbb{CP}^N$. As was explained, the problem can be reduced to proving non-displaceability of certain subsets of $T^*\mathrm{SU}(N)$. Given such a subset, say A, it is, in general, a non-trivial problem to construct a non-zero object $F \in \mathcal{D}_A(\mathrm{SU}(N))$. Our major tool here is a certain object $\mathfrak{S} \in D(G \times \mathfrak{h})$ which is defined uniquely up-to a unique isomorphism by certain microlocal conditions to be now specified. Here $G = \mathrm{SU}(N)$ and $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}$, the Lie algebra of $\mathrm{SU}(N)$.

Let $C_+ \subset \mathfrak{h}$ be the positive Weyl chamber. For every $A \in \mathfrak{g}$ there exists a unique element $\|A\| \in \mathfrak{h}$ such that $\|A\|$ is conjugated with A. Let us identify $T^*(G\times\mathfrak{h}) = G\times\mathfrak{h}\times\mathfrak{g}^*\times\mathfrak{h}^*$ (via interpreting $\mathfrak{g}^*$ as the space of right-invariant 1-forms on G). Let us identify $\mathfrak{g}^* = \mathfrak{g}$, $\mathfrak{h}^* = \mathfrak{h}$ by means of Killing's form. Let $\Omega_{\mathfrak{S}} \subset G\times\mathfrak{h}\times\mathfrak{g}\times\mathfrak{h} = \Omega_{\mathfrak{S}} \subset G\times\mathfrak{h}\times\mathfrak{g}^*\times\mathfrak{h}^*$ consist of all points of the form (g, X, ω, η), where $\eta = \|\omega\|$. Let also $i_0 : G \to G\times\mathfrak{h}$ be the embedding $i_0(g) = (g, 0)$. We then define $\mathfrak{S}$ as an object of $D(G\times\mathfrak{h})$ such that $\mathrm{SS}(\mathfrak{S}) \subset \Omega_{\mathfrak{S}}$ and $i_0^{-1}\mathfrak{S} \cong \mathbb{K}_e$, where $\mathbb{K}_e$ is the skyscraper at the unit $e \in G$. One can show that this way $\mathfrak{S}$ is determined uniquely up-to a unique isomorphism. It turns out that the required objects $F_A \in \mathcal{D}_A(X), F_B \in \mathcal{D}_B(X), \ldots,$ can be easily expressed in terms of $\mathfrak{S}$.

Our next task is to compute the graded vector spaces $R\hom(F_A, T_{c*}F_B)$ and to make sure that the maps $\tau_c : R\hom(F_A, F_B) \to R\hom(F_A, T_{c*}F_B)$ are not zero for all $c \geq 0$. This problem gets gradually reduced to finding an explicit description of the restriction $i_e^{-1}\mathfrak{S} \in D(\mathfrak{h})$, where $i_e : \mathfrak{h} \to G\times\mathfrak{h}$, $i_e(X) = (e, X)$, and $e \in G$ is the unit.

Remark. Let $C_- := -C_+$, let $C_-^\circ \subset C_-$ be the interior. It turns out that the stalks of $(i_e^{-1}\mathfrak{S}|_{C_-})$ have a transparent topological meaning (however, this meaning won't be used in our proofs). Let $X \in C_-$; let $O(X) := \mathfrak{S}|_{e\times X}[-\dim\mathfrak{h}]$.

On the other hand, let us consider the smooth loop space ΩG. For $\gamma : [0,1] \to G$ being a smooth loop, we set $\|\gamma\| \in C_+$,

$$\|\gamma\| := \int_0^1 \|\gamma'(t)\| dt,$$

where $\gamma'(t) \in \mathfrak{g}$ is the t-derivative of γ. Let $\Omega_X \subset \Omega(G)$ be the subspace consisting of all loops γ such that $\|\gamma\| \leq -X$ (here $Y \leq -X$ means $< Y + X, C_+ > \leq 0$, where $<,>$ is the restriction of the positive definite invariant form on $\mathfrak{g}$ onto $\mathfrak{h}$). It can be shown that $\mathcal{O}(X) \cong H_\bullet(\Omega_X)$.

In regard with this setting, one can ask the following question (which will be probably discussed in a subsequent paper). We have an obvious concatenation map $\Omega_X \times \Omega_Y \to \Omega_{X+Y}$ whence a product $\mathcal{O}(X) \otimes \mathcal{O}(Y) \to \mathcal{O}(X+Y)$. One can show that this product is commutative so that the spaces $\mathcal{O}()$ form a filtered commutative algebra. It can be shown that this algebra can be obtained in the following algebro-geometric way. Let $\mathcal{FL}$ be the projective $\mathbb{K}$-variety of complete flags in $\mathbb{K}^N$. Fix a regular nilpotent operator $n : \mathbb{K}^N \to \mathbb{K}^N$ (that is, n consists of one Jordan block). Let $\mathbf{Pet} \subset \mathcal{FL}(N)$ be the closed subvariety consisting of all flags $0 = V_0 \subset V_1 \subset \cdots V_N = \mathbb{K}^N$ satisfying $nV_i \subset V_{i+1}$ for all $i < N$. This variety was discovered by Peterson, see e.g. [6].

Let $\mathbb{L} \in \mathfrak{h}$ be the lattice formed by all elements X such that e^X is in the center of G. Given $l \in \mathbb{L}$ we canonically have a line bunble L_l on $\mathcal{FL}$. It turns out that for all $l \in \mathbb{L} \cap C_+$ we have an isomorphism $O(l) = \Gamma(\mathbf{Pet}; L_l|_{\mathbf{Pet}})$, and this isomorphism is compatible with the natural product on both sides.

A related result is proven in [5], where, among other interesting results, the authors identify $H_\bullet(\Omega(G))$ with the algebra of functions on a certain affine open subset of $\mathbf{Pet}$.

Let us now go over the content of the paper. In Sec. 2-3 we formulate and prove the non-displaceability condition.

In Sec. 4 we start applying the non-displaceability condition to $\mathbb{RP}^N, \mathbb{T}^N \subset \mathbb{CP}^N$. Finally, the problem is reduced to the existence of an object $u_{\mathcal{O}} \in \mathcal{D}(G)$ satisfying certain properties (see Proposition 4.4).

In Sec. 5 the object $u_{\mathcal{O}}$ gets constructed out of $\mathfrak{S}$ (where we use certain properties of $\mathfrak{S}$ to be proven in the subsequent sections).

The rest of the paper is devoted to constructing and studying $\mathfrak{S}$. In Sec. 6 we construct an object $\mathfrak{S}$ and prove its uniqueness.

In Sec. 7 we compute an isomorphism type of $\mathfrak{S}|_{z\times C^\circ_-}$ where z is any element in the center of G. In essense, the computation is a version of Bott's computation of $H_\bullet(\Omega(G))$ using Morse theory.

The goal of Sec. 8 is to extend the result of the previous section to $z \times \mathfrak{h}$. This is done by means of establishing a certain periodicity propery of $\mathfrak{S}$ with respect to shifts along $\mathfrak{h}$ by elements of the lattice $\mathbb{L} = \{X \in \mathfrak{h} | e^X \in \mathbf{Z}\}$, where $\mathbf{Z} \subset G$ is the center. Namely, we show that $\mathfrak{S}$ is, what we call, *a strict B-sheaf.* (see Sec. 8.2). We show that any strict B-sheaf can be recovered from its restricton onto $\mathbf{Z} \times C^\circ_-$. By virtue of this statement we are able to identify the isomorphism type of $\mathfrak{S}|_{\mathbf{Z}\times G}$.

There are two appendices. In the first one we introduce the notation used when working with $\mathrm{SU}(N)$ and its Lie algebra. We also included a couple of useful Lemmas (which, most likely, can be found elsewhere in the literature). These Lemmas are mainly used when constructing and studying $\mathfrak{S}$. The notation is used systematically starting from Sec. 5.

In the second appendix we list, for the reader's convenience, the rules for computing the micro-support from [1]. These rules are used throughout the paper.

Strictly speaking, these rules are proved in [1] for the bounded derived category. However, one sees that they carry over directly to the unbounded derived category, in which case we use them.

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2. Generalities

2.1. Unbounded derived category.

2.1.1. Fix a ground field $\mathbb{K}$. Abelian category Sh_M of sheaves of $\mathbb{K}$-vector spaces on a smooth manifold M is of finite injective dimension. Therefore, one has a simple model of the unbounded derived category $D(M)$, namely one can take unbounded complexes of injective sheaves on M; given two such complexes, we define $\hom_{D(M)}(I_1, I_2) := H^0 \hom^\bullet(I_1, I_2)$. This definition is stable under quasi-isomorphisms precisely because of finite injective dimension of Sh_M. The main results of the formalism of 6 functors remain valid for $D(X)$ (excluding the Verdier duality).

2.1.2. We still have a notion of singular support of an object of $D(M)$ and it is defined in the same way as in [1] The results on functorial properties of singular support from Ch 5,6 of [1] are still valid for the unbounded derived category, and we will freely use them. For the convenience of the reader the results from [1] used in this paper are listed in Sec. 11

2.2. Sheaves on $X \times \mathbb{R}$. Let X be a smooth manifold. We will work with the manifold $X \times \mathbb{R}$. Let t be the coordinate on $\mathbb{R}$ and let $V = \partial/\partial t$ be the vector field corresponding to the infinitesimal shift along $\mathbb{R}$. Let $\Omega_{\leq 0} \subset T^*(X\times\mathbb{R})$ be the closed subset consisting of all 1-forms ω with $(\omega, V) \leq 0$. Let $\Omega_{>0} \subset T^*(X \times \mathbb{R})$ be the complement to $\Omega_{\leq 0}$, i.e. the set of all 1-forms ω such that $(\omega, V) > 0$.

Let $C_{\leq 0}(X) \subset D(X\times\mathbb{R})$ be the full subcategory of objects microsupported on $\Omega_{\leq 0}$. Let $\mathcal{D}(X) := D(X \times \mathbb{R})/C_{\leq 0}(X)$.

Proposition 2.1. *The embedding $C_{\le 0}(X) \to D(X \times \mathbb{R})$ has a left adjoint. Therefore, $\mathcal{D}(X)$ is equivalent to the left orthogonal complement to $C_{\le 0}(X)$ in $D(X \times \mathbb{R})$.*

Proof. Let $p_1 : X \times \mathbb{R} \times \mathbb{R} \to X \times \mathbb{R}$; $p_2 : X \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$; $a : X \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be given by $p_1(x, t_1, t_2) = (x, t_1)$; $p_2(x, t_1, t_2) = t_2$; $a(x, t_1, t_2) = t_1 + t_2$.. For $F \in D(X \times \mathbb{R})$ and $S \in D(\mathbb{R})$ set $F *_{\mathbb{R}} S := Ra_!(p_1^{-1} F \otimes p_2^{-1} S)$.

It is clear that $F *_{\mathbb{R}} \mathbb{K}_0 \cong F$ where $\mathbb{K}_0$ is the sky-scraper at $0 \in \mathbb{R}$.

We have a natural map $\mathbb{K}_{[0,\infty)} \to \mathbb{K}_0$ in $D(\mathbb{R})$.

For an $F \in D(X \times \mathbb{R})$, consider the induced map

$$F *_{\mathbb{R}} \mathbb{K}_{[0,\infty)} \to F *_{\mathbb{R}} \mathbb{K}_0 = F. \tag{1}$$

1) Let us show that $F *_{\mathbb{R}} \mathbb{K}_{[0,\infty)}$ *is in the left orthogonal complement to* $C_{\le 0}(X)$.

Indeed, let $G \in C_{\le 0}(X)$. Let $U \subset X$ be an open subset and let $(a, b) \subset \mathbb{R}$. Any object $F \in D(X \times \mathbb{R})$ can be produced from objects of the type $\mathbb{K}_{U \times (a,b)}$ for various U and (a, b) by taking direct limit. Therefore, without loss of generality, one can assume $F = \mathbb{K}_{U \times (a,b)}$. One then has

$$R\hom_{X \times \mathbb{R}}(F *_{\mathbb{R}} \mathbb{K}_{[0,\infty)}; G) =$$
$$= R\hom_{X \times \mathbb{R}}(\mathbb{K}_{U \times (a,b)} *_{\mathbb{R}} \mathbb{K}_{[0,\infty)}; G)$$
$$= R\hom_{X \times \mathbb{R}}(\mathbb{K}_{U \times [a,\infty)}[-1]; G)$$
$$= \mathrm{Cone}(R\Gamma(U \times \mathbb{R}; G) \xrightarrow{r} R\Gamma(U \times (-\infty; a); G)).$$

The map r is an isomorphism because $G \in C_{\le 0}$. Therefore, $\mathrm{Cone}(r) = 0$, whence the statement.

2) *Cone of the map (1) is in* $C_{\le 0}(X)$. Indeed, consider the cone of the map $\mathbb{K}_{[0,\infty)} \to \mathbb{K}_0$. This cone is isomorphic to $\mathbb{K}_{(0,\infty)}[1]$. One then has to check that $F *_{\mathbb{R}} \mathbb{K}_{(0,\infty)} \in C_{\le 0}(X)$. One can represent F as an inductive limit of compactly supported objects. Therefore, without loss of generality, one can assume F is compactly supported. One then can estimate the microsupport of $F *_{\mathbb{R}} \mathbb{K}_{[0,\infty)}$ using functorial properties of microsupport. Indeed, let us identify

$$T^*(X \times \mathbb{R} \times \mathbb{R}) = T^*X \times T^*(\mathbb{R} \times \mathbb{R}).$$

Let us also identify $T^*(\mathbb{R} \times \mathbb{R}) = \mathbb{R}^4$ so that a point $(t_1, t_2, k_1, k_2) \in \mathbb{R}^4$ corresponds to the 1-form $k_1 dt_1 + k_2 dt_2$ at the point $(t_1, t_2) \in \mathbb{R} \times \mathbb{R}$. We then have

$$p_1^{-1} F \otimes p_2^{-1} \mathbb{K}_{(0,\infty)} = F \boxtimes \mathbb{K}_{(0,\infty)};$$
$$\mathrm{SS}(F \boxtimes \mathbb{K}_{(0,\infty)})$$
$$\subset \{(\omega, t_1, t_2, k_1, k_2) \in T^*X \times \mathbb{R}^4 | (t_2, k_2) \in \mathrm{SS}(\mathbb{K}_{(0,\infty)})\}.$$

This means that either $t_2 = 0$ and $k_2 \le 0$ or $t_2 > 0$ and $k_2 = 0$.

As F is compactly supported, it follows that the map a is proper on the support of $F \boxtimes \mathbb{K}_{(0,\infty)}$.

Therefore, $\mathrm{SS}Ra_!(F \boxtimes \mathbb{K}_{(0,\infty)})$ is contained in the set of all points $(\omega, t, k) \in T^*X \times \mathbb{R}^2$ such that there exists a point $(\omega, t_1, t_2, k_1, k_2) \in \mathrm{SS}(F \boxtimes \mathbb{K}_{(0,\infty)})$ such that $t = t_1 + t_2$; $k_1 = k_2 = k$. This implies that $k \le 0$, therefore,

$$F *_{\mathbb{R}} \mathbb{K}_{(0,\infty)} = Ra_!(F \boxtimes \mathbb{K}_{(0,\infty)}) \in C_{\le 0}(X),$$

as was required.

The statements 1) and 2) imply that we have an exact triangle

$$\to F *_{\mathbb{R}} \mathbb{K}_{(0,\infty)} \to F *_{\mathbb{R}} \mathbb{K}_{[0,\infty)} \to F \to F *_{\mathbb{R}} \mathbb{K}_{(0,\infty)}[1] \to \cdots,$$

where $F *_{\mathbb{R}} \mathbb{K}_{(0,\infty)}[1]$ is in $C_{\leq 0}(X)$ and $F *_{\mathbb{R}} \mathbb{K}_{[0,\infty)}$ is in the left orthogonal complement to $C_{\leq 0}(X)$. Therefore, $F \mapsto F *_{\mathbb{R}} \mathbb{K}_{(0,\infty)}[1]$ is the left adjoint functor to the embedding $C_{\leq 0}(X) \to D(X \times \mathbb{R})$. $\square$

Thus, we have proven

Proposition 2.2. *An object $F \in D(X \times \mathbb{R})$ is in the left orthogonal complement to $C_{\leq 0}(X)$ iff the map (1) is an isomorphism.*

2.2.1. From now on we identify $\mathcal{D}(X)$ with a full subcategory of $D(X \times \mathbb{R})$ which is the left orthogonal complement to $C_{\leq 0}(X)$. Thus, the arrow (1) is an isomorphism for any $F \in \mathcal{D}(X) \subset D(X \times \mathbb{R})$ (and only for objects from $\mathcal{D}(X)$).

2.2.2. Let $T_c : X \times \mathbb{R} \to X \times \mathbb{R}$ be the shift along $\mathbb{R}$ by c: $T_c(x,t) = (x, t+c)$. We have $T_{c*}F = F *_{\mathbb{R}} \mathbb{K}_c$. If $F \in \mathcal{D}$, we have

$$T_{c*}F \cong F *_{\mathbb{R}} \mathbb{K}_{[0,\infty)} *_{\mathbb{R}} \mathbb{K}_c \cong F *_{\mathbb{R}} \mathbb{K}_{[c,\infty)}. \tag{2}$$

One can easily check that $T_{c*}F \in \mathcal{D}(X)$; for example, this follows from an isomorphism

$$F *_{\mathbb{R}} \mathbb{K}_{[c,\infty)} \cong T_{c*}F *_{\mathbb{R}} \mathbb{K}_{[0,\infty)},$$

which is the case for any $F \in D(X \times \mathbb{R})$.

For all $c \geq d$ we then have a natural map $T_{d*}F \to T_{c*}F$ which is induced by the embedding $[c,\infty) \subset [d,\infty)$ and we use the identification (2). This implies that we have natural transformations $\tau_{dc} : T_{d*} \to T_{c*}$ of endofunctors on $\mathcal{D}(X)$ for all $d \leq c$. It is clear that $\tau_{dc}\tau_{ed} = \tau_{ec}$ for all $e \leq d \leq c$.

2.2.3. Call an object $F \in \mathcal{D}(X)$ *a torsion object* if there exists $c > 0$ such that the natural map $\tau_{0c} : F \to T_{c*}F$ is zero in $\mathcal{D}(X)$.

2.2.4. Still thinking of $\mathcal{D}(X)$ as a quotient $D(X \times \mathbb{R})/C_{\leq 0}(X)$, the microsupport of an object $F \in \mathcal{D}(X)$ is naturally defined as a closed subset of $\Omega_{>0} \subset T^*(X \times \mathbb{R})$. Denote this microsupport by $\mathrm{SS}_{\mathcal{D}}(F) \subset \Omega_{>0}$.

Let us see what this means in terms of the identification of $\mathcal{D}(X)$ with a full subcategory of $\mathcal{D}(X)$ which is the left orthogonal complement to $C_{\leq}(X)$. Let $F \in \mathcal{D}(X) \subset D(X \times \mathbb{R})$. We then have $\mathrm{SS}_{\mathcal{D}}(F) = \mathrm{SS}(F) \cap \Omega_{>0}$, where $\mathrm{SS}(F)$ is the microsupport of F which is viewed as an object of $D(X \times \mathbb{R})$.

2.2.5. Let us identify $T^*\mathbb{R} = \mathbb{R} \times \mathbb{R}$ so that $(t_0, k) \in \mathbb{R} \times \mathbb{R}$ corresponds to the 1-form kdt at the point $t_0 \in \mathbb{R}$. We then have an induced identification $T^*(X \times \mathbb{R}) = T^*X \times \mathbb{R} \times \mathbb{R}$.

Let $A \subset T^*X$ be a subset. Define the conification $\mathrm{Cone}(A) \subset \Omega_{>0}$ to consist of all points $(\omega, t, k) \in T^*X \times \mathbb{R} \times \mathbb{R}$ such that $k > 0$, $(x, \omega/k) \in A$. Let $\mathcal{D}_A(X) \subset \mathcal{D}(X)$ be the full subcategory consisting of all objects $F \in \mathcal{D}(X)$ such that $\mathrm{SS}_{\mathcal{D}}(F) \subset \mathrm{Cone}(A)$.

3. Non-displaceability condition

Let X be a compact manifold. Let $L_1, L_2 \subset T^*X$ be compact subsets. Call L_1, L_2 mutually non-displaceable if for every Hamiltonian symplectomorphism Φ of T^*X which is identity outside of a compact, $\Phi(L_1) \cap L_2 \neq \emptyset$. Our goal is to prove

Theorem 3.1. *Suppose there exist objects $F_i \in \mathcal{D}_{L_i}(X)$, $i = 1, 2$ such that for all $c > 0$ the natural map*

$$\tau_c : R\hom(F_1, F_2) \to R\hom(F_1, T_c F_2)$$

is not zero. Then L_1 and L_2 are mutually non-displaceable.

The proof will occupy the whole section.

3.1. **Disjoint supports.** Our goal is to prove:

Theorem 3.2. *let $F_i \in \mathcal{D}_{A_i}(X)$, where $i = 1, 2$, $A_i \subset T^*X$ are compact sets and $A_1 \cap A_2 = \emptyset$. We then have $R\hom_{\mathcal{D}(X)}(F_1, F_2) = 0$.*

3.1.1. *Lemma.* Let M be a smooth manifold let E be a finite-dimensional real vector space of dimension ≥ 1. Let $p : M \times E \to M$ be the projection. Let $F \in D(M \times E)$. Let $\omega \in T^*M$, $\omega \neq 0$. Let $U \subset T^*M$ be a neighborhood of ω. Let $V \subset E^*$ be a neighborhood of 0 in the dual vector space. Let us identify $T^*(M \times E) = T^*M \times E \times E^*$.

Lemma 3.3. *Suppose that*

F is non-singular on the set

$$U \times E \times V \subset T^*M \times E \times E^* = T^*(M \times E)$$

Then $Rp_!F$ and Rp_*F are non-singular at ω.

Proof. We will only prove Lemma for $Rp_!F$; the proof for Rp_*F is similar.

Fix a Euclidean inner product $<,>$ on E. Without loss of generality one can assume that $V = B \subset E^*$ is an open unit ball.

Let $\theta : [0, \infty) \to [0, 1)$ be a function such that:

— $\theta'(x) > 0$ for all $x \geq 0$;

— there exists an $\varepsilon > 0$ such that for all $x \in [0, \varepsilon]$ we have $\theta(x) = x$.

— there exists an $M > 0$ such that for all $x > M$, $\theta(x) = 1 - 1/x$.

Let $B := \{v \in E | \ |v| < 1\}$. Let $Z : E \to B$ be the embedding given by

$$Z(v) = \frac{\theta(|v|)}{|v|} v.$$

It follows that Z is a diffeomorphism. Let $J : B \to E$ be the open embedding Let us split $p : M \times E \to M$ as

$$M \times E \overset{\mathrm{Id} \times Z}{\to} M \times B \overset{\mathrm{Id} \times j}{\to} M \times E \overset{p}{\to} X.$$

Denote $z := \mathrm{Id} \times Z$; $j := \mathrm{Id} \times J$. We have $Rp_!F = Rp_!j_!z_!F$. We then see that p is proper on the support of $j_!z_!F$. Let us estimate $\mathrm{SS}(z_!F)$.

Let $a(x) : [0, 1) \to [0, \infty)$ be the inverse function to θ. It follows that $a(x) = x$ for $x < \varepsilon$ and there exists $\delta > 0$ such that for all $x \in (1 - \delta; 1)$, $a(x) = 1/(1 - x)$. We then get $Z^{-1}v = (a(|v|)/|v|)v$. The condition of Lemma implies that for all $\omega \in U$ and for all $c \in B$ we have $(\omega, v, c) \notin \mathrm{SS}(F)$, where $v \in E$. Let

$$S_{UB} = \{(\omega, v, c) | \omega \in U; v \in E; c \in B\} \subset T^*(M \times E).$$

We then see that the set $(Z^{-1})^* S_{UB} \subset T^*(X \times B)$;

$$(Z^{-1})^* S_{UB} = \{(\omega, v, \sum_j c_j d((a(|v|)/|v|)v^j)\},$$

where $\omega \in U$, $v \in B$, and $|c| < 1$. Let us now estimate $\mathrm{SS}(j_! z_! F)$. According to 11.0.13, we have

$$\mathrm{SS}(j_! z_! F) \subset \mathrm{SS}(z_! F) \hat{+} N^*(X \times B)^a,$$

where on the RHS we have a Witney sum of the following conic subsets of $T^*(M \times E)$:

—we identify $\mathrm{SS}(z_! F)$ with a conic subset of $T^*(M \times E)$ as follows: $\mathrm{SS}(z_! F) \subset T^*(M \times B) \subset T^*(M \times E)$;

— $N^*(M \times B)^a$ is the exterior conormal cone to the boundary of $M \times B \subset M \times E$. We have

$$N^*(M \times B)^a = \{(\omega, b, tb) \in T^*M \times E \times E ||b| = 1; t \geq 0\},$$

where we identify $T^*M \times E \times E = T^*M \times E \times E^*$.

By definition one has:

$$\mathrm{SS}(z_! F) \hat{+} N^*(M \times B)^a = \mathrm{SS}(z_! F) \cup \Lambda,$$

where Λ consists of all points of the form $(\omega, b, \eta) \in T^*M \times E \times E$ where

—$\omega \in T^*_{x_0} M$; so let us choose a nieghborhood U_{x_0} of x_0 in M and identify $T^*U = U \times \mathbb{R}^{\dim M}$; let us denote points of T^*U by (x, ζ), $x \in U$; $\zeta \in \mathbb{R}^{\dim X}$;

—$b \in \partial B$ and there exists a sequence of points $(x_k, \omega_k, b_k, \eta_k) \in \mathrm{SS}(Z_! F) \cap T^*(U_{x_0} \times B)$; $(\beta_k; t_k) \in \partial B \times \mathbb{R}_{\geq 0}$ where $x_k \to x_0$; $b_k \to b$; $\beta_k \to b$; $\omega_k \to \omega$; $\eta_k + 2t_k \sum_j \beta_j dv_j \to \eta$; $t_k(|\beta_k - b_k| + |x_k - x_0|) \to 0$.

We will show that $(x_0, b, \omega, 0) \notin \Lambda$ for any $b \in \partial B$. Let us prove the statement by contradiction. Indeed, without loss of generality, one can assume that $(x_k, \omega_k) \in U$, therefore, $(b_k, \eta_k) \notin Z^{-1*}(E \times V)$. As $V \subset E^*$ is an open unit ball, this means that (b_k, η_k) is of the form

$$\eta_k = \sum_j c_k^j d(a(|b_k|) b_k^j / |b_k|)$$

and $|c_k| \geq 1$. as $b_k \to b$, $|b_k| \to 1$ and without loss of generality one can assume $|b_k| > 1 - \delta$ so that $a(|b_k|) = 1/(1 - |b_k|)$. Thus

$$\eta_k = \sum_j c_k^j d(b_k^j / (|b_k|(1 - |b_k|))$$

Let $R_k = |b_k|$. We then have

$$\eta_k = < c_k, db_k > /(R_k(1 - R_k)) + < c_k, b_k > \frac{2R_k - 1}{R_k^3 (1 - R_k)^2} < b_k, db_k >$$

so that

$$< \eta_k, \eta_k > = < c_k, c_k > /(R_k^2 (1 - R_k)^2) + < c_k, b_k >^2 \frac{(2R_k - 1)^2}{R_k^4 (1 - R_k)^4}$$
$$+ 2 < c_k, b_k >^2 \frac{2R_k - 1}{R_k^4 (1 - R_k)^3}$$
$$> < c_k, c_k > /(R_k^2 (1 - R_k)^2) > 1/(1 - R_k)^2$$

as long as $R_k > 1/2$ which is the case for all k large enough, without loss of generality we can assume that $R_k > 1/2$ for all k. Thus, $|\eta_k| > 1/(1 - R_k)$.

Therefore,

$$|\eta_k + 2t_k \sum_j \beta_k^j dv_k^j| \geq |\eta_k| - 2|t_k||\beta| > 1/(1 - R_k) - 2t_k$$

By assumption $|\eta_k + 2t_k \sum_j \beta_k^j dv_k^j| \to 0$, hence

$$1/(1 - R_k) - 2t_k \to 0$$

and $2t_k(1-R_k) \to 1$. On the other hand, we have

$$t_k(|b_k - \beta_k|) \geq t_k(1-R_k),$$

because $|\beta_k| = 1$ and $|b_k| = R_k$. Therefore, $t_k(1-R_k) \to 0$. We have a contradiction which shows that as long as $(x, \omega) \in U$, $(x, \omega, e, 0) \notin \mathrm{SS}(j_! Z_! F)$. Since the map $p : X \times E \to X$ is proper on the support of $j_! Z_! F$ (i.e. $X \times \overline{B}$) we know that $(x, \omega) \notin \mathrm{SS}(Rp_! j_! Z_! F)$ which proves Lemma $\square$

Corollary 3.4. *Let $F \in D(X \times E)$ and let $p : X \times E \to X$, $\kappa : T^*X \times E \times E^* \to T^*X \times E^*$ be the projections. Let $\mathcal{I} : T^*X \to T^*X \times E^*$ be the embedding given by $I(x, \omega) = (x, \omega, 0)$. We then have*

$$SS(Rp_! F), SS(Rp_* F) \subset \mathcal{I}^{-1}\overline{\kappa(SS(F))},$$

where the bar means the closure.

Proof. Clear $\square$

3.1.2. *Kernels and convolutions.* Let X_1, X_2, X_3 be manifolds. We are going to define a functor

$$D(X_1 \times X_2 \times \mathbb{R}) \times D(X_2 \times X_3 \times \mathbb{R}) \to D(X_1 \times X_3 \times \mathbb{R}).$$

Let

$$p_{ij} : X_1 \times X_2 \times X_3 \times \mathbb{R} \times \mathbb{R} \to X_i \times X_j \times \mathbb{R} \tag{3}$$

be the following maps

$$p_{12}(x_1, x_2, x_3, t_1, t_2) = (x_1, x_2, t_1);$$
$$p_{23}(x_1, x_2, x_3, t_1, t_2) = (x_2, x_3, t_2);$$

$$p_{13}(x_1, x_2, x_3, t_1, t_2) = (x_1, x_3, t_1 + t_2).$$

Let $A \in D(X_1 \times X_2 \times \mathbb{R})$ and $B \in D(X_2 \times X_3 \times \mathbb{R})$. Set

$$A \bullet_{X_2} B := Rp_{13!}(p_{12}^{-1} A \otimes p_{23}^{-1} B),$$

$A \bullet_{X_2} B \in D(X_1 \times X_3 \times \mathbb{R})$.

Let now X_k, $k = 1, 2, 3, 4$, are manifolds and let $A_k \in D(X_k \times X_{k+1} \times \mathbb{R})$, $k = 1, 2, 3$. We then have a natural isomorphism

$$(A_1 \bullet_{X_2} A_2) \bullet_{X_3} A_3 \cong A_1 \bullet_{X_2} (A_2 \bullet_{X_3} A_3).$$

Let $A \in D(X \times \mathbb{R})$ and $S \in D(\mathbb{R})$. Let **pt** be a point. We then have $A *_{\mathbb{R}} S \cong A \bullet_{\mathbf{pt}} S \cong S \bullet_{\mathbf{pt}} A$.

Let $A \in \mathcal{D}(X_1 \times X_2)$ and $B \in D(X_2 \times X_3 \times \mathbb{R})$. Then $A \bullet_{X_2} B \in \mathcal{D}(X_1 \times X_3 \times \mathbb{R})$. Indeed, according to Lemma 2.2, we need to check that the natural map

$$\mathbb{K}_{[0,\infty)} *_{\mathbb{R}} (A \bullet_{X_2} B) \to \mathbb{K}_0 *_{\mathbb{R}} (A \bullet_{X_2} B)$$

is an isomorphism.

It follows that this map is isomorphic to a map

$$(\mathbb{K}_{[0,\infty)} \bullet_{\mathbf{pt}} A) \bullet_{X_2} B \to (\mathbb{K}_0 \bullet_{\mathbf{pt}} A) \bullet_{X_2} B$$

which is, in turn, induced by the natural map

$$\mathbb{K}_{[0,\infty)} \bullet_{\mathbf{pt}} A \to \mathbb{K}_0 \bullet_{\mathbf{pt}} A$$

which is an isomorphism because $A \in \mathcal{D}(X_1 \times X_2)$.

In particular, it follows that

$$\bullet_{X_2} : \mathcal{D}(X_1 \times X_2) \times \mathcal{D}(X_2 \times X_3) \to \mathcal{D}(X_1 \times X_3).$$

3.1.3. *Fourier transform.* Let $E = \mathbb{R}^n$ be a real vector space and let E^* be the dual space. Let $G \subset E \times E^* \times \mathbb{R}$ be a closed subset $G = \{(X, P, t) | < X, P > +t \geq 0\}$, where $<,>: E \times E^* \to \mathbb{R}$ is the pairing. One sees that $\mathbb{K}_G \in \mathcal{D}(E \times E^*)$. Let $\Gamma \subset E^* \times E \times \mathbb{R}$ be a closed subset $G = \{(P, X, T)| - < P, X > +t \geq 0\}$. Again, we have $\mathbb{K}_\Gamma \in \mathcal{D}(E^* \times E \times \mathbb{R})$.

Define functors $F : \mathcal{D}(E) \to \mathcal{D}(E^*)$; $\Phi : \mathcal{D}(E^*) \to \mathcal{D}(E)$ as follows. Set

$$F(A) := A \bullet_E \mathbb{K}_G;$$

$$\Phi(B) := B \bullet_{E^*} \mathbb{K}_\Gamma.$$

F, Φ are called 'Fourier transform'.

Let us study the composition $\Phi \circ F : \mathcal{D}(E) \to \mathcal{D}(E)$. We have an isomorphism

$$\Phi \circ F(A) \cong A \bullet_E (\mathbb{K}_G \bullet_{E^*} \mathbb{K}_\Gamma).$$

Let us compute $\mathbb{K}_G \bullet_{E^*} \mathbb{K}_\Gamma$. Let

$$q : E \times E^* \times E \times \mathbb{R} \times \mathbb{R} \to E \times E \times \mathbb{R}$$

be given by $q(X_1, P, X_2, t_1, t_2) = (X_1, X_2, t_1 + t_2)$. By definition, we have

$$\mathbb{K}_G \bullet_{E^*} \mathbb{K}_\Gamma = Rq_! \mathbb{K}_K,$$

where

$$K = \{(X_1, P, X_2, t_1, t_2) | t_1 + < X_1, P > \geq 0; t_2 - < X_2, P > \geq 0\}$$

Let us decompose $q = q_1 q_2$, where

$$q_2 : E \times E^* \times E \times \mathbb{R} \times \mathbb{R} \to E \times E^* \times E \times \mathbb{R}$$

$q_2(X_1, P, X_2, t_1, t_2) = (X_1, P, X_2, t_1 + t_2)$; and

$$q_1 : E \times E^* \times E \times \mathbb{R} \to E \times E \times \mathbb{R},$$

$$q_1(X_1, P, X_2, t) = (X_1, X_2, t).$$

We see that $q_2(K) = L := \{(X_1, P, X_2, t) | t + < X_1 - X_2, P > \geq 0\}$. Furthermore, the map $q_2|_K : K \to L$ is proper; it is also a Serre fibration with a contractible fiber. Therefore, we have an isomorphism $Rq_{2!}\mathbb{K}_K \cong \mathbb{K}_L$.

Let us now compute $Rq_{1!}\mathbb{K}_L$. Let $\Delta \subset E \times E^* \times E \times \mathbb{R}$ be given by

$$\Delta = \{(X_1, P, X_2, t) | X_1 = X_2; t \geq 0\}.$$

We have $\Delta \subset L$ so that we have an induced map

$$\mathbb{K}_L \to \mathbb{K}_\Delta.$$

It is easy to check that the induced map

$$Rq_{1!}\mathbb{K}_L \to Rq_{1!}\mathbb{K}_\Delta$$

is an isomorphism.

We also have an isomorphism $Rq_{1!}\mathbb{K}_\Delta \cong \mathbb{K}_{\{(X_1,X_2,t)|X_1=X_2;t\geq 0\}}[-n]$.

Thus, we have an isomoprhism

$$Rq_! \mathbb{K}_K = \mathbb{K}_{\{(X_1,X_2,t)|X_1=X_2;t\geq 0\}}[-n]$$

For any $A \in D(E \times \mathbb{R})$, we have an isomorphism

$$A \bullet_E \mathbb{K}_{\{(X_1,X_2,t)|X_1=X_2;t\geq 0\}} \cong A *_{\mathbb{R}} \mathbb{K}_{[0,\infty)}.$$

Thus we have an isomorphism of functors

$$\Phi(F(\cdot)) \cong (\cdot) *_{\mathbb{R}} \mathbb{K}_{[0,\infty)}[-n]$$

The functor on the RHS acts on $\mathcal{D}(E)$ as the shift by $-n$. Thus we have established an isomorphism of functors $\Phi \circ F \cong \mathrm{Id}[-n]$. Analogously, we can prove $F \circ \Phi \cong \mathrm{Id}[-n]$. We have proven:

Theorem 3.5. *$\Phi[n]$ and F are mutually inverse equivalences of $\mathcal{D}(E)$ and $\mathcal{D}(E^*)$.*

3.1.4. Let us now study the effect of the Fourier transform on the microsupports. Let

$$a : T^*E = E \times E^* \to T^*E^* = E^* \times E$$

be given by $a(X, P) = (-P, X)$. It is clear that a is a symplectomorphism.

Theorem 3.6. *Let $A \subset T^*E$ be a closed subset and $S \in \mathcal{D}_A(E)$. Then $F(S) \in \mathcal{D}_{a(A)}(E^*)$.*

Let $B \in T^*E^*$ be a closed subset and $S \in \mathcal{D}_B(E^*)$. Then $\Phi(S) \in \mathcal{D}_{a^{-1}(B)}(E)$.

Proof. By definition, we have

$$F(S) = Rp_{13!}(p_{12}^{-1}S \otimes p_{23}^{-1}\mathbb{K}_G).$$

Here the maps p_{ij} are the same as in (3) for $X_1 = \mathbf{pt}$; $X_2 = E$; $X_3 = E^*$.

The condition $S \in \mathcal{D}_A(E)$ means that $\mathrm{SS}(S)$ is contained in the set Ω_0 of all points

$$(x, t, \omega, k) \in E \times \mathbb{R} \times E^* \times \mathbb{R} = T^*(E \times \mathbb{R}),$$

where either $k \leq 0$ or $k > 0$ and $(x, \omega/k) \in A$.

Therefore,

$$\mathrm{SS}(p_{12}^{-1}S) \subset \Omega_1 := \{(X, P, t_1, t_2, \omega, 0, k, 0)|(X, t, \omega, k) \in \Omega_0\}.$$

As $G \subset E \times E^* \times \mathbb{R}$ is defined by the equation $t+ < X, P > \geq 0$, we know that $\mathrm{SS}(\mathbb{K}_G)$ consists of all points of the form

$$(X, P, t, kP, kX, k) \in E \times E^* \times \mathbb{R} \times E^* \times E \times \mathbb{R}$$

where $t+ < X, P > \geq 0$, $k \geq 0$ and $k > 0$ implies $t+ < X, P >= 0$.

Therefore

$$\mathrm{SS}(p_{23}^{-1}\mathbb{K}_G) = \Omega_2 := \{(X, P, t_1, t_2, k_1P, k_1X, 0, k_1)|(X, P, t_2, k_1P, k_1X, k_1) \in \mathrm{SS}(\mathbb{K}_G)\}.$$

We see that $\Omega_1 \cap -\Omega_2$ is contained in the zero section of $T^*(E \times E^* \times \mathbb{R} \times \mathbb{R})$. Therefore,

$$\mathrm{SS}((p_{12}^{-1}S) \otimes (p_{23}^{-1}\mathbb{K}_G))$$

is contained in the set of all poits of the form $\omega_1 + \omega_2$ where $\omega_i \in \Omega_i$ and ω_1, ω_2 are in the same fiber of $T^*(E \times E^* \times \mathbb{R} \times \mathbb{R})$.

We have

$$\mathrm{SS}(p_{12}^{-1}S \otimes p_{23}^{-1}\mathbb{K}_G) \subset \Omega_3,$$

where Ω_3 consists of all points of the form

$$(X, P, t_1, t_2, \omega + k_1P, k_1X, k, k_1)$$

where:

— if $k > 0$, then $(X, \omega/k) \in A$;

— $t_2 + < X, P > \geq 0$;
— $k_1 \geq 0$;
— if $k_1 > 0$, then $t_2 + < X, P >= 0$.

Let $I : E \times E^* \times \mathbb{R} \times \mathbb{R} \to E \times E^* \times \mathbb{R} \times \mathbb{R}$ be given by

$$I(X, P, t_1, t_2) = (X, P, t_1 + t_2; t_2).$$

Let $\pi : E \times E^* \times \mathbb{R} \times \mathbb{R} \to E^* \times \mathbb{R}$ be given by $\pi(X, P, t_1, t_2) = (P, t_1)$. We then have $p_{13} = \pi I$;

$$Rp_{13!}(p_{12}^{-1} S \otimes p_{23}^{-1} \mathbb{K}_G) \cong R\pi_! I_! (p_{12}^{-1} S \otimes p_{23}^{-1} \mathbb{K}_G).$$

It is easy to see that

$$\mathrm{SS} I_! (p_{12}^{-1} S \otimes p_{23}^{-1} \mathbb{K}_G)$$

is contained in the set Ω_4 of of all points $(X, P, t_1+t_2, t_2, \omega, \eta, k, k_1-k)$ where $(X, P, t_1, t_2, \omega, \eta, k, k_1) \in \Omega_3$.

Suppose that a point $(P, \xi) \in E^* \times E = T^*E^*$ does not belong to $a(A)$, that is $(-\xi, P) \notin A$. We will prove that $R\pi_! I_! (p_{12}^{-1} S \otimes p_{23}^{-1} \mathbb{K}_G)$ is non-singular at any point of the form $(P, t, \xi, 1) \in E^* \times \mathbb{R} \times E \times \mathbb{R} = T^*(E^* \times \mathbb{R})$ (this means precisely that $R\pi_! I_! (p_{12}^{-1} S \otimes p_{23}^{-1} \mathbb{K}_G) \in \mathcal{D}_{a(A)}(E^*)$.)

According to Lemma 3.3, it suffices to find an $\varepsilon > 0$ such that any point of the form

$$(X, P', t_1, t_2, \omega, \eta, k', k_1')$$

with $|P' - P| < \varepsilon$; $|\omega| < \varepsilon$; $|\eta - \xi| < \varepsilon$; $|k' - 1| < \varepsilon$; $|k_1'| < \varepsilon$ is not in Ω_4. Assume it is, then there should exist a point $(X, P', t_1, t_2, \omega + k_1 P', k_1 X, k, k_1) \in \Omega_3$ such that $|P' - P| < \varepsilon$; $|\omega + k_1 P'| < \varepsilon$; $|k_1 X - \xi| < \varepsilon$; $|k - 1| < \varepsilon$; $|k' - k| < \varepsilon$. If ε is small enough, we have $k, k_1 > 0$ and $(X, \omega/k) \in A$. For any $\delta > 0$, there exists a $\varepsilon > 0$ such that these conditions imply:

$$|\omega + P| < \delta; |X - \xi| < \delta. \tag{4}$$

However,we know that $(\xi; -P) = a^{-1}(P, \xi) \notin A$. As A is closed, for δ small enough, there will be no points in A satisfying (4).

The proof of Part 2 is similar. $\square$

3.1.5. *Lemma.*

Lemma 3.7. *Let $S \in \mathcal{D}_A(X)$ where A is a compact. Then $SS(S) \cap \Omega_{\leq 0}(X) \subset T^*_{X \times \mathbb{R}}(X \times \mathbb{R})$. That is S is non-singular at every point of the form (x, t, ω, kdt), where either $k \leq 0$ and $\omega \neq 0$ or $k < 0$.*

Proof. Choose a point $x_0 \in X$, coordinates x^i near x_0 so that x_0 has zero coordinates and let U be a small neighborhood of x_0 given by $|x^i| < 1$ for all i. Consider the set $A \cap T^*U$. This set is contained in the set $B := \{(x, \sum a_i dx^i) | |a_i| \leq M\}$, for some $M > 0$ large enough. Let $\psi : \mathbb{R} \to (-1, 1)$ be an increasing surjective smooth function whose derivative is bounded (say $\psi(x) = (2/\pi)\arctan(x)$). Fix a constant $C > 0$ such that $0 < \psi'(x) \leq C$ for all x.

we then have a diffeomorphism $\Psi : E := \mathbb{R}^n \to U$, $\Psi(X^1, X^2, \ldots, X^n) = (\psi(X^1), \psi(X^2), \ldots, \psi(X^n))$. It then follows that the set $\Psi^{-1}B$ consists of all points $(X, \sum a_i d\psi(X^i))$, where $|a_i| < M$. But $\sum a_i d\psi(X_i) = \sum a_i \psi'(X_i) dX_i$. We know that $|a_i \psi'(X_i)| < CM =: M_1$. Let $V \subset E^*$ be given by $\{\sum_i b_i dX^i | |b_i| \leq M_1\}$ so that $\Psi^{-1}B$ is contained in the set $E \times V \subset E \times E^* = T^*E$.

Let $S \in \mathcal{D}_A(X)$. It follows that $G := \Psi^{-1}(S|_{U \times \mathbb{R}}) \in \mathcal{D}_{E \times V}(E)$. Our task now reduces to showing: *let $G \in \mathcal{D}_{E \times V}(E)$. Then G is nonsingular at a point $(X, t, \omega, kdt) \in E \times \mathbb{R} \times E^* \times \mathbb{R}$ if either $k < 0$ or $k = 0$ and $\omega \neq 0$.*

The statement will be proven using the Fourier transform.

First, we have an isomorphism $G = \Phi(F(G))[n]$. Next, Theorem 3.6 implies that $H := F(G) \in \mathcal{D}_{V\times E}(E^*)$. Let $W \subset E^*\backslash V$ be an open subset such that its closure is also a subset of $E^*\backslash V$. We then see that the restriction $H|_{W\times\mathbb{R}}$ is both in $C_{\leq 0}(U)$ (clear) and in the left orthogonal complemenent to $C_{\leq 0}(U)$ (follows from (2.2)). Therefore, $H|_{W\times\mathbb{R}} = 0$. Hence H is supported on $V \times \mathbb{R} \subset E^* \times \mathbb{R}$. Let us now study $\Phi(H)[n] = G$. We have

$$\Phi(H) = Rp_{13!}(p_{12}^{-1}H \otimes p_{23}^{-1}\mathbb{K}_{\{(P,X,t)|t-<X,P>\geq 0\}}),$$

where p_{ij} are the same as in (3) with $X_1 = \mathbf{pt}$; $X_2 = E^*$; $X_3 = E$. We need to show that *if* $(X, t, \omega, k) \in SS(\Phi(H))$ *and* $k \leq 0$*, then* $k = 0$ *and* $\omega = 0$.

We have

$$\mathrm{SS}(p_{12}^{-1}H) \subset \Omega_1 = \{(P, X, t_1, t_2, \pi, 0, k_1, 0)|P \in V\};$$

$$\mathrm{SS}(p_{23}^{-1}\mathbb{K}_{\{(P,X,t)|t-<X,P>\geq 0\}}) \subset \Omega_2 = \{(P, X, t_1, t_2, -kX, -kP, 0, k)|k \geq 0\}$$

Let $\omega_i \in \Omega_i$ belong to the fiber of $T^*(E^* \times E \times \mathbb{R} \times \mathbb{R})$ over a point (P, X, t_1, t_2). It is clear that $\omega_1 + \omega_2 = 0$ implies that $\omega_2 = \omega_1 = 0$. Therefore, we have

$$\mathrm{SS}(p_{12}^{-1}H \otimes p_{23}^{-1}\mathbb{K}_{\{(P,X,t)|t-<X,P>\geq 0\}})$$

$$\subset \Omega_3 = \{(P, X, t_1, t_2, \pi - kX, -kP, k_1, k)|k \geq 0; P \in V\}$$

Let us decompose $p_{13} = pI$, where $I : E^* \times E \times \mathbb{R} \times \mathbb{R} \to E^* \times E \times \mathbb{R} \times \mathbb{R}$ is given by $I(P, X, t_1, t_2) = (P, X, t_1 + t_2, t_2)$ and $p(P, X, T_1, T_2) = (P, T_1)$. We then see that

$$\mathrm{SS}(I_!(p_{12}^{-1}H \otimes p_{23}^{-1}\mathbb{K}_{\{(P,X,t)|t-<X,P>\geq 0\}})) \subset \Omega_4,$$

where Ω_4 consists of all points of the form $(P, X, t_1, t_2, \pi-kX, -kP, k_1, k-k_1)$, where $(P, X, t_1, t_2, \pi-kX, -kP, k_1, k) \in \Omega_3$.

Assume

$$(X', t, \omega, k') \in \mathrm{SS}(Rp_!I_!(p_{12}^{-1}H \otimes p_{23}^{-1}\mathbb{K}_{\{(P,X,t)|t-<X,P>\geq 0\}}))$$

and $k' \leq 0$. We are to show $k' = 0, \omega = 0$.

According to Lemma 3.3 for any $\varepsilon > 0$ there should exist a point $(P, X, t_1, t_2, \pi-kX, -kP, k_1, k-k_1) \in \Omega_4$ such that $|-kP - \omega| < \varepsilon$; $|k_1 - k'| < \varepsilon$; $|k - k_1| < \varepsilon$, $P \in V$, $k \geq 0$, $k' \leq 0$. Therefore, $-k' \leq k - k' = |k - k'| \leq |k - k_1| + |k_1 - k'| < 2\varepsilon$. Similarly, $k < 2\varepsilon$. Since ε can be made arbitrarily small, $k' = 0$. Next, $|\omega| < \varepsilon + |k||P|$. As V is bounded, there exists $D > 0$ such that $|P| < D$. Thus, $|\omega| < \varepsilon(1 + 2D)$ for any $\varepsilon > 0$. Therefore, $\omega = 0$. □

3.1.6. Choose F_1, F_2 in the left orthogonal complement to $C_{\leq 0}(X)$.

Consider the following sheaf on $X \times \mathbb{R}$:

$$H := Rp_{2*}R\underline{\mathrm{Hom}}(p_1^{-1}F_1; a^!F_2),$$

where $p_1, p_2, a : X \times \mathbb{R} \times \mathbb{R} \to X \times \mathbb{R}$ are given by: $p_i(x, t_1, t_2) = (x, t_i)$; $a(x, t_1, t_2) = (x, t_1 + t_2)$.

Let $q : X \times \mathbb{R} \to X$ be the projection.

Lemma 3.8. *One has 1)* $R\,\mathrm{hom}(F_1, F_2) = R\,\mathrm{hom}_{\mathbb{R}}(\mathbb{K}_0; Rq_*H)$*;*

2) $R\,\mathrm{hom}_{\mathbb{R}}(\mathbb{K}_{\mathbb{R}}; Rq_*H) = 0$*;*

3) Rq_*H *is locally constant along* $\mathbb{R}$.

Proof. Let $S \in D(\mathbb{R})$. We have

$$R\hom_{\mathbb{R}}(S; Rq_*H) = R\hom_{\mathbb{R}}(S; R\pi_*\underline{\mathrm{Hom}}(p_1^{-1}F_1; a^!F_2)),$$

where $\pi = qp_2 : X \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$; $\pi(x, t_1, t_2) = t_2$.

Next,

$$\begin{aligned} &R\hom_{\mathbb{R}}(S; R\pi_*\underline{\mathrm{Hom}}(p_1^{-1}F_1; a^!F_2)) \\ &\cong R\hom_{X\times\mathbb{R}}(Ra_!(\pi^{-1}S \otimes p_1^{-1}F_1); F_2) \\ &\cong R\hom_{X\times\mathbb{R}}(F_1 *_{\mathbb{R}} S; F_2). \end{aligned}$$

Thus,

$$R\hom(S; Rq_*H) \cong R\hom_{X\times\mathbb{R}}(F_1 *_{\mathbb{R}} S; F_2).$$

Let us now prove 1)

We have:

$$R\hom_{\mathbb{R}}(\mathbb{K}_0; Rq_*H) = R\hom_{X\times\mathbb{R}}(F_1 *_{\mathbb{R}} \mathbb{K}_0; F_2) = R\hom(F_1, F_2).$$

2) We have

$$R\hom_{\mathbb{R}}(\mathbb{K}_{\mathbb{R}}; Rq_*H) = R\hom_{X\times\mathbb{R}}(F_1 *_{\mathbb{R}} \mathbb{K}_{\mathbb{R}}; F_2)$$

As $F_1 \in \mathcal{D}(X)$, we have an isomorphism

$$F_1 *_{\mathbb{R}} \mathbb{K}_{[0,\infty)} *_{\mathbb{R}} \mathbb{K}_{\mathbb{R}} \to F_1 *_{\mathbb{R}} \mathbb{K}_{\mathbb{R}}.$$

However, one can easily check that $\mathbb{K}_{[0,\infty)} *_{\mathbb{R}} \mathbb{K}_{\mathbb{R}} = 0$.Therefore,

$$F_1 *_{\mathbb{R}} \mathbb{K}_{[0,\infty)} *_{\mathbb{R}} \mathbb{K}_{\mathbb{R}} = 0,$$

whence the statement.

3) Let us identify $T^*(X\times\mathbb{R}) = T^*X\times\mathbb{R}^2$; $T^*(X\times\mathbb{R}\times\mathbb{R}) = T^*X\times\mathbb{R}^4$ so that $(\omega, t, k) \in T^*X\times\mathbb{R}^2$ corresponds to a point $(\omega, \eta) \in T^*X \times T^*\mathbb{R}$, where η is a 1-form kdt at the point $t \in \mathbb{R}$; analogously, we let $(\omega, t_1, t_2, k_1, k_2)$ correspond to a point $(\omega, \zeta) \in T^*X \times T^*(\mathbb{R}\times\mathbb{R})$ where $\zeta = k_1dt_1 + k_2dt_2$ is a 1-form at the point $(t_1, t_2) \in \mathbb{R}^2$.

According to Lemma 3.7, We know that

$$\mathrm{SS}(F_1) \cap \{(\omega, t, k) | k \le 0\} \subset T^*_{X\times\mathbb{R}}(X \times \mathbb{R}).$$

Since $F_1 \in \mathcal{D}_{A_1}(X)$, we have

$$\mathrm{SS}(F_1) \cap \{(\omega, t, k) | k > 0\} \subset \{(\omega, t, k) | k > 0; (x, \omega/k) \in A_1\}.$$

Thus,

$$\mathrm{SS}(F_1) \subset \{(k\omega, t, k) | k \ge 0; \omega \in A_1\}$$

Analogously,

$$\mathrm{SS}(F_2) \subset \{(k\omega, t, k) | k \ge 0; \omega \in A_2\}.$$

Therefore,

$$\begin{aligned} \mathrm{SS}(p_1^{-1}F_1) &\subset \{(k_1\omega_1, t_1, t_2, k_1, 0) | k_1 \ge 0; \omega_1 \in A_1\}; \\ \mathrm{SS}(a^!F_2) &\subset \{(k_2\omega_2, t_1, t_2, k_2, k_2) | k_2 \ge 0; \omega_2 \in A_2\}. \end{aligned}$$

In order to estimate $\mathrm{SS}R\underline{\mathrm{Hom}}(p_1^{-1}F_1; a^!F_2)$ one should first check that $\mathrm{SS}(p_1^{-1}F_1) \cap \mathrm{SS}(a^!F_2) \subset T^*_{X\times\mathbb{R}\times\mathbb{R}}(X \times \mathbb{R} \times \mathbb{R})$. This is indeed so, because every point p in $\mathrm{SS}(p_1^{-1}F_1) \cap \mathrm{SS}(a^!F_2)$ is of the form

$$p = (k_1\omega_1, t_1, t_2, k_1, 0) = (k_2\omega_2, t_1, t_2, k_2, k_2).$$

which implies $k_1 = k_2 = 0$, hence $k_1\omega_1 = k_2\omega_2 = 0$. Therefore, one has

$$\mathrm{SS}R\underline{\mathrm{Hom}}(p_1^{-1}F_1; a^!F_2) \subset \{(k_2\omega_1 - k_1\omega_2, t_1, t_2; k_2 - k_1; k_2)|k_1, k_2 \geq 0; \omega_1 \in A_1; \omega_2 \in A_2\},$$

where it is also assumed that ω_1, ω_2 belong to the same fiber of T^*X. Let $q' : X \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ be the projection. Consider an object

$$G := Rq'_*R\underline{\mathrm{Hom}}(p_1^{-1}F_1; a^!F_2)$$

so that $Rq_*H = Rq_*Rp_{2*}R\underline{\mathrm{Hom}}(p_1^{-1}F_1; a^!F_2) = Rp'_{2*}G$, where $p'_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the projection along the first factor; $p'_1(t_1, t_2) = t_2$.

As the map q' is proper, the microsupport of G can be estimated as

$$\mathrm{SS}(G) \subset \{(t_1, t_2, k_2 - k_1, k_2)|k_1, k_2 \geq 0; \exists \omega_i \in A_i : k_1\omega_1 = k_2\omega_2\},$$

where again it is assumed that ω_i are in the same fiber of T^*X. Denote the set on the RHS by $\Gamma \subset \mathbb{R}^4 = T^*(\mathbb{R} \times \mathbb{R})$. Let us now estimate $\mathrm{SS}(Rq_*H) = Rp'_{2*}G$ using Lemma 3.4.

Let us first prove that $(t, 1) \notin \mathrm{SS}(q_*H)$, where we identify $T^*\mathbb{R} = \mathbb{R} \times \mathbb{R}$. Assuming the opposite implies that for any $\varepsilon > 0$ there should exist $(t_1, t_2, k_2 - k_1, k_2) \in \Gamma$ such that $|k_2 - k_1| < \varepsilon$; $|k_2 - 1| < \varepsilon$. As A_1, A_2 are compact and do not intersect, it is clear that for ε small enough we have $k_1A_1 \cap k_2A_2 = \emptyset$ which contradicts to $(t_1, t_2, k_2 - k_1, k_2) \in \Gamma$.

Let us now show that Rq_*H is non-singular at any point $(t, -1)$. Similar to above, assuming the contrary implies that for any $\varepsilon > 0$ there should exist $(t_1, t_2, k_2 - k_1, k_2) \in \Gamma$ such that $|k_2 + 1| < \varepsilon$. As $k_2 \geq 0$, this leads to contradiction. □

3.1.7. *Proof of Theorem 3.2.* It now follows that Rq_*H is a constant sheaf on $\mathbb{R}$ with $R\Gamma(\mathbb{R}, Rq_*H) = 0$, i.e. $Rq_*H = 0$. Hence $R\,\mathrm{hom}(F_1, F_2) = 0$ by Lemma 3.8 1).

This proves Theorem 3.2.

3.2. **Hamiltonian shifts.** Let $\Phi : T^*X \to T^*X$ be a Hamiltonian symplectomorphism which is equal to identity outside of a compact. Let $L \subset T^*X$ be a compact subset.

Theorem 3.9. *There exist:*

a collection of endofunctors $T_n : \mathcal{D}(X) \to \mathcal{D}(X)$,, $1 \leq n \leq N$ for some N, and a collection of transformations of functors $t_k : T_{2k} \to T_{2k+1}$ (for all k with $2k + 1 \leq N$). $s_k : T_{2k+2} \to T_{2k+1}$ (for all k with $2k + 2 \leq N$);

Such that

1) $T_N = Id$;

2) $T_1(\mathcal{D}_L(X)) \subset \mathcal{D}_{\Phi(L)}(X)$;

3) For all k and for all $F \in \mathcal{D}(X)$, we have $Cone(t_k(F))$ and $Cone(s_k(F))$ are torsion sheaves (see Sec. 2.2.3)

3.2.1. *Singular support of convolutions.* Let $A \in T^*X$ and $B \subset T^*(X \times Y) = T^*X \times T^*Y$ be compact subsets. Let $C \subset T^*Y$;

$$C := A \bullet B = \{p \in T^*Y|\exists q \in A : (-q, p) \in B\}.$$

3.2.2. *Lemma.*

Lemma 3.10. *Let $A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots \supset A$ be a collection of compact sets such that $\bigcap_i A_i = A$. Let $U \supset C$ be an open neighborhood. There exists an $N > 0$ such that for all $n > N$, $A_n \bullet B \subset U$.*

Proof. Assume not and pick points $b_n \in (A_n \bullet B)\backslash U$. One then has points $a_n \in A_n$ such that $(-a_n, b_n) \in B$. As B is compact, one can choose a convergent subsequence $a_{n_k} \to a$ and $b_{n_k} \to b$. It follows that $(-a, b) \in B$. We see that $a \in A_{n_k}$ for all k, hence $a \in A$. Therefore, $b \in C$. On the other hand, as $b_{n_k} \notin U$, $b \notin U$, we have a contradiction.

□

3.2.3. Let A, B, C are compact sets as above.

Proposition 3.11. *Let $F \in \mathcal{D}_A(X)$; $K \in \mathcal{D}_B(X \times Y)$. Then $F \bullet K \in \mathcal{D}_C(Y)$.*

Proof. It suffices to prove: *let $(y_0, \eta_0) \notin C$. Then $F \bullet K$ is nonsingular at $(y_0, t, \eta_0, 1)$ for all $t \in \mathbb{R}$.*

Let us identify of $T^*(X \times Y \times \mathbb{R} \times \mathbb{R}) = T^*X \times T^*Y \times T^*(\mathbb{R} \times \mathbb{R}) = T^*X \times T^*Y \times \mathbb{R}^4$, where we identify $T^*(\mathbb{R} \times \mathbb{R}) = \mathbb{R}^4$ in the same way as above: a point $(t_1, t_2, k_1, k_2) \in \mathbb{R}^4$ corresponds to a 1-form $k_1 dt_1 + k_2 dt_2$ at the point $(t_1, t_2) \in \mathbb{R}^2$.

Let us estimate the microsupport of $F \bullet K := p_{13!}(p_{12}^{-1}F \otimes p_{23}^{-1}K)$, where p_{ij} are the same as in (3) with $X_1 = \mathbf{pt}$; $X_2 = X$; $X_3 = Y$. We have $p_{12}^{-1}F$ is microsupported within the set S_F consisting of all points of the form

$$(k_1\omega_1, 0_y, t_1, t_2, k_1, 0),$$

where $0_y \in T^*_Y Y$, $(x, \omega_1) \in A$; $k_1 \geq 0$ (as follows from Lemma 3.7). Analogously, The sheaf $p_{23}^{-1}K$ is microsupported on the set S_K consisting of all points of the form

$$(k_2\omega_2, k_2\eta_2, t_1, t_2, 0, k_2),$$

where $k_2 \geq 0$, $(\omega_2, \eta_2) \in B$.

One sees that $S_K \cap -S_F \subset T^*_{X\times Y\times\mathbb{R}\times\mathbb{R}}(X\times Y\times\mathbb{R}\times\mathbb{R})$. Therefore, $p_{12}^{-1}F \otimes p_{23}^{-1}K$ is microsupported within the set of all points of the form

$$(k_1\omega_1 + k_2\omega_2, k_2\eta_2, t_1, t_2, k_1, k_2),$$

where $k_1, k_2 \geq 0$; $\omega_1 \in A$; $(\omega_2, \eta_2) \in B$.

Let $Q : X \times Y \times \mathbb{R} \times \mathbb{R} \to Y \times \mathbb{R} \times \mathbb{R}$, $a : Y \times \mathbb{R} \times \mathbb{R} \to Y \times \mathbb{R}$ be given by

$$Q(x, y, t_1, t_2) = (y, t_1, t_2);$$

$$a(y, t_1, t_2) = (y, t_1 + t_2)$$

so that $p_{13} = aQ$.

We see that the map Q is proper on the support of $p_{12}^{-1}F \otimes p_{23}^{-1}K$. It then follows that the sheaf $\Psi := RQ_!(p_{12}^{-1}F \otimes p_{23}^{-1}K)$ is microsupported on the set S_Q of all points

$$(k_2\eta_2, t_1, t_2, k_1, k_2)$$

such that $k_1, k_2 \geq 0$ and there exist $\omega_1 \in A$, $(\omega_2, \eta_2) \in B$ such that ω_1 and ω_2 are in the same fiber of T^*X and $k_1\omega_1 + k_2\omega_2 = 0$

Let us now estimate the mircosupport of $Ra_!\Psi$. We will use Lemma 3.4. Let us use an isomorphism $I : Y \times \mathbb{R} \times \mathbb{R} \to Y \times \mathbb{R} \times \mathbb{R}$, where

$$I(y, t_1, t_2) = (y, t_1 + t_2; t_2).$$

Let $p_2 : Y \times \mathbb{R} \times \mathbb{R} \to Y \times \mathbb{R}$ be given by $p_2(y, t_1, t_2) = (y, t_1)$ so that we have

$$a = p_2 I$$

and $Ra_!\Psi = Rp_{2!}I_!\Psi$. We see that the sheaf $I_!\Psi$ is microsupported on the set Γ_1 consisting of all points of the form

$$(k_2\eta_2, t_1, t_2, k_1, k_2 - k_1)$$

such that $k_1, k_2 \geq 0$ and there exist $\omega_1 \in A$, $(\omega_2, \eta_2) \in B$ such that ω_1 and ω_2 are in the same fiber of T^*X and $k_1\omega_1 + k_2\omega_2 = 0$. Let us now use Lemma 3.4 in order to estimate $\mathrm{SS}Rp_{2!}I_!\Psi$. Let $\eta \in T^*Y$; $\eta \notin C$. We need to show that $Rp_{2!}I_!\Psi$ is non-singular at any point of the form

$$(\eta, t, 1) \in T^*Y \times \mathbb{R} \times \mathbb{R} = T^*Y \times T^*\mathbb{R}.$$

Assuming the contrary, for any $\delta > 0$ there should exist a point $(k_2\eta_2, T_1, T_2, k_1, k_2 - k_1) \in \Gamma_1$ such that $|\eta - k_2\eta_2| < \delta$ and $|k_1 - 1|, |k_2 - k_1| < \delta$. Given $\varepsilon > 0$, one can choose $\delta > 0$ such that under the conditions specified, $|1 - k_1/k_2| < \varepsilon$. Let $A_\varepsilon = [1 - \varepsilon, 1 + \varepsilon].A$'. We then see that there should exist $\omega_2 \in T^*X$ such that $(\omega_2, \eta_2) \in B$ and $-\omega_2 \in A_\varepsilon$ (because $-\omega_2 = k_1/k_2\omega_1$ and $\omega_1 \in A$). Thus, $\eta_2 \in A_\varepsilon \bullet B$. We see that the sets $A_{1/n}$, $n = 1, 2, \ldots$ are compact and $\bigcap_n A_{1/n} = A$. Let $U \supset C$ be an open neighborhood.

By Lemma 3.10, there exists an N such that $A_{1/N} \bullet B \subset U$ i.e. for all $\varepsilon \leq 1/N$ we have $\eta_2 \in U$. Taking into account the inequality $|\eta - k_2\eta_2| < \delta$ and letting δ arbitrarily small, we see that $\eta \in U$. As U is any open neighborhood of C, we conclude $\eta \in C$. We get a contradiction. $\square$

3.2.4. If $\Phi = \Phi_1\Phi_2 \cdots \Phi_N$ and the statement of the Theorem is true for each Φ_k, it is true for Φ. In other words, if Z is the set of Hamiltonian symplectomorphisms of T^*X which are identity outside of a compact and if Z generates the whole group of Hamiltonian symplectomorphisms of T^*X which are idenitity outside a compact, then it suffices to prove Theorem for all $\Phi \in Z$.

Let us now choose an appopriate Z. Call a symplectomorphism $\Phi : X \to X$ *small* if

1) There exists a Darboux chart $U \subset T^*X$ with Darboux coordinates x, P, where x^i are local coordinates on x, $P^i = \partial/\partial x^i$, $|x^i| < 1$ and for some fixed

$$\pi \in \mathbb{R}^n, \tag{5}$$

$|P^i - \pi^i| < 1$ for all i. Let $p^i := P^i - \pi^i$. For $x \in \mathbb{R}^n$ we set $|x| := \max_i |x_i|$. We demand that Φ should be identity outside a subset $V \subset U$, $|x| < 1/2, |p| < 1/2$. 2) let $(x', p') = \Phi(x, p)$. Then (x, p') form a non-degenerate coordinate system on U so that (x, p') map U diffeomorphically onto a domain $W \subset \mathbb{R}^{2n}$.

It is well known that the set Z formed by small symplectomorphisms satisfies the conditions.

3.2.5. *Small symplectomorphisms in terms of generating functions.* The coordinates (x, p) define an embedding $U \subset \mathbb{R}^{2n}$. Let us extend $\Phi|_U$ to a map $\overline{\Phi} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ by setting $\overline{\Phi}(x, p_0) = (x, p_0)$ for all $(x, p_0) \notin U$. We see that $\overline{\Phi}$ is a diffeomorphism because it maps U diffeomorphically to itself, as well as the complement to U. Hence, $\overline{\Phi} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a symplectomorphism with respect to the standard symplectic structure.

As above let $\overline{\Phi}(x, p) = (x', p')$. Let $\Psi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ where $\Psi(x, p) = (x, p')$.

Lemma 3.12. *Ψ is a diffeomorphism.*

Proof. a) Ψ has a non-zero Jacobian everywhere. Indeed, if $|x| < 1, |p| < 1$ this is postulated by 2); otherwise $\Psi = \mathrm{Id}$ in a neighborhood of (x, p).

b) Φ is an injection. Suppose $\Psi(x_1, p_1) = \Psi(x_2, p_2)$. Then $x_1 = x_2 = x$ and $p'(x, p_1) = p'(x, p_2)$. Consider several cases:

1) $|x| \geq 1$, then $p'(x, p_1) = p_1$; $p'(x, p_2) = p_2$ and $p_1 = p_2$;

2) $|x| < 1$; $|p_1| < 1$. If $|p_2| < 1$, then $p_1 = p_2$ by Condition 2). If $|p_2| \geq 1$, then $p'(x, p_2) = p_2$; $|p'(x, p_2)| \geq 1$ and $|p'(x, p_1)| < 1$ because $\overline{\Phi}$ preserves U, so $p'(x_1, p_1) \neq p'(x_2, p_2)$;

3) $|x| < 1$ and $|p_2| < 1$— similar to 2);

4) $|x| < 1$ and $|p_1|, |p_2| = 1$. Then $p'(x, p_i) = p_i$, therefore $p_1 = p_2$.

c) Ψ is surjective. We know that $\Psi(x, p) = (x, p)$ if $|x| > 1$ or $|p| > 1$. Assume that, on the contrary, (x_0, p_0) does not belong to the image of Ψ. It follows that $|x_0| < 1$; $|p_0| < 1$. For $R > 0$ consider the sphere S_R given by the equation $\sum_i (x^i)^2 + \sum_i (p^i)^2 = R^2$. Choose R so large that $(x, p) \in S_R$ implies $|x| > 1$ or $|p| > 1$. We then have $\Psi|_{S_R} = \mathrm{Id}$. It also follows S_R cannot be homotopized to a point in $\mathbb{R}^{2n} \backslash (x_0, p_0)$ (because (x_0, p_0) is inside the open ball bounded by S_R). On the other hand it can: Let $\gamma : S_R \times [0, 1] \to \mathbb{R}^{2n}$ be any homotopy which contracts S_R to a point. Then $\Psi \circ \gamma$ is a required homotopy. This is a contradiction. □

Lemma 3.13. *There exists a smooth function $S(x, p')$ on $\mathbb{R}^{2n}$ such that*

1) $(x', p') = \overline{\Phi}(x, p)$ iff for all i:

$$p^i = (p')^i + \frac{\partial S}{\partial x^i};$$

$$(x')^i = x^i + \frac{\partial S}{\partial (p')^i};$$

2) $S = 0$ if $|x| \geq 1/2$ or $|p'| \geq 1/2$;

$$3) \qquad \max_{|x| \leq 1/2, |p'| \leq 1/2} |x^i + \partial S / \partial (p')^i| \leq 1/2$$

Proof. Consider the following 1-form on $\mathbb{R}^{2n}$: $\sum p^i dx^i + \sum (x')^i d(p')^i$. This form is closed, hence exact. So one can write

$$\sum p^i dx^i + \sum (x')^i d(p')^i = d(S(x, p') + < x, p' >)$$

by virtue of Lemma 3.12. This equation is equivalent to the part 1) of this Lemma.

We know that $\overline{\Phi} = \mathrm{Id}$ if $|x| \geq 1/2$ or $|p| \geq 1/2$. Therefore, $\overline{\Phi}$, being bijective, preserves the region $\{(x, p) | |x|, |p| < 1/2\}$. Therefore, if $|p'(x, p)| \geq 1/2$, then either $|x| \geq 1/2$ or $|p| \geq 1/2$, hence $p'(x, p) = p$; $x'(x, p) = x$ and $dS(x, p') = 0$ as soon as $|x| \geq 1/2$ or $|p'| \geq 1/2$. As the specified region is connected, S is a constant in this region, and one can choose S to be 0 as long as $|x| \geq 1/2$ or $|p'| \geq 1/2$. This proves 2).

It also follows that if $|x| \leq 1/2$ and $|p'(x, p)| \leq 1/2$ then $|p| \leq 1/2$, because otherwise $\Phi(x, p) = (x, p)$ and $p' = p$, which is a contradiction. This implies 3). □

3.2.6. Let J be the set of all smooth functions $S(x, p')$ on $\mathbb{R}^{2n}$ such that S is supported on the set $\{(x, p') | |x| \leq 1/2, |p'| \leq 1/2\}$ and the inequality 3) from Lemma 3.13 is satisfied. Our ultimate goal is: given such an S, we would like to construct certain kernels in $\mathcal{D}(\mathbb{R}^n \times \mathbb{R}^n)$ and then $\mathcal{D}(X \times X)$.

Let $\pi \in \mathbb{R}^n$ (this parameter has the same meaning as in (5). Let $S \in J$. We will start with constructing an appropriate object $\Lambda_{S,\pi} \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R}^n)$ and estimating its microsupport.

Let $\Sigma_\pi(x_1,x_2,p') := -S(x_1,p') - < x_1 - x_2, p' + \pi >$. We can decompose

$$d\Sigma_\pi = d_{x_1}\Sigma_\pi + d_{x_2}\Sigma_\pi + d_{p'}\Sigma_\pi.$$

Let $\Gamma_\pi(S) \subset T^*\mathbb{R}^n \times T^*\mathbb{R}^n$ consist of all points (x_1,p_1,x_2,p_2) satisfying: there exists p' such that $d_{p'}\Sigma_\pi(x_1,x_2,p') = 0$ and $p_1 = d_{x_1}\Sigma_\pi(x_1,x_2,p')$; $p_2 = d_{x_2}\Sigma_\pi(x_1,x_2,p')$.

Remark. Let us take S as in Lemma 3.13. The set $\Gamma_\pi(S)$ then consists of all points (x_1,P_1,x_2,P_2) such that $\overline{\Phi}(x_1;-P_1-\pi) = (x_2,P_2-\pi)$. That is, if $|P_1+\pi| < 1$, then $(x_2,P_2) = \Phi(x_1,-P_1)$; if $|P_1+\pi| \geq 1$, then $x_2 = x_1$, $P_2 = -P_1$, where we use notation from Sec 3.2.4.

We are now passing to constructing an object $\Lambda_{S,\pi} \in \mathcal{D}(\Gamma_\pi(S))$. Consider the following subset $C_{S,\pi} \subset \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$;

$$\{(x_1,x_2,p',t,)|t + \Sigma_\pi \geq 0\},$$

Let $q : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ be given by

$$q(x_1,x_2,p,t) = (x_1,x_2,t).$$

Set $\Lambda_{S,\pi} := Rq_!\mathbb{K}_{C_{S,\pi}}$.

Lemma 3.14. *Assume $S \in J$. Then $\Lambda_{S,\pi} \in \mathcal{D}_{\Gamma_\pi(S)}(\mathbb{R}^n \times \mathbb{R}^n)$.*

Proof. It is straightforward to check that $\Lambda_{S,\pi}$ is in the left orthogonal complement to $C_{\leq 0}(\mathbb{R}^n \times \mathbb{R}^n)$.

Let us now estimate the microsupport of $\Lambda_{S,\pi}$. Let us choose a large positive number C and consider objects

$$F_C := Rq_!\mathbb{K}_{\{(x_1,x_2,p',t)|t+\Sigma_\pi(x_1,x_2,p')\geq 0;|p'|<C\}}$$

so that $\Lambda_{S,\pi} = L\varinjlim_{C\to\infty} F_C$.

We will prove: *let $(x_1,x_2,t,\omega_1,\omega_2,k) \in T^*(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R})$ be a singular point of F_C. Then one of the following 3 statements is true:*

— *$k = \omega_1 = \omega_2 = 0$;*

— *$k > 0$ and $|\omega_i/k| \geq C - |\pi|$, $i = 1,2$.*

—$(x_1,x_2,\omega_1,\omega_2) \in \Gamma_\pi(S)$

This implies the Lemma as C can be chosen arbitrarily large.

Let us estimate the microsupport of the sheaf

$$\mathbb{K}_{\{(x_1,x_2,p',t)|t+\Sigma_\pi(x_1,x_2,p')\geq 0;|p'|<C\}}$$

we see that it is contained within the set of all points of the form

$$(x_1,x_2,p',t,kd\Sigma_\pi(x_1,x_2,p') + \sum a_i d(p')^i),$$

where $|p'| \leq C$ and $a_i \leq 0$ if $(p')^i = -C$; $a_i = 0$ if $|(p')^i| < C$, and $a_i \geq 0$ if $(p')^i = C$; also, $k \geq 0$ and if $k > 0$, then $t + \Sigma_\pi(x_1,x_2,p') = 0$.

Let us now estimate the singular support of the sheaf

$$Rq_!\mathbb{K}_{\{(x,x',p,t)|t+\Sigma_\pi(x_1,x_2,p')\geq 0;|p'|<C\}}.$$

As q is proper on the support of this sheaf, we see that

$$Rq_!\mathbb{K}_{\{(x,x',p,t)|t+\Sigma_\pi(x_1,x_2,p');|p'|<C\}}$$

is microsupported on the set of points

$$(x_1,x_2,t,\omega_1,\omega_2,k),$$

where there exists p', $|p'| \leq C$ such that

$$\omega_i = k d_{x_i} \Sigma_\pi(x_1, x_2, p') \tag{6}$$

where $k \geq 0$ and if $k > 0$ then there exists p' such that

$$\frac{\partial \Sigma_\pi}{\partial (p')^i}(x_1, x_2, p') \geq 0 \text{ if } (p')^i = -C;$$

$$\frac{\partial \Sigma_\pi}{\partial (p')^i}(x_1, x_2, p') = 0 \text{ if } |(p')^i| < C; \tag{7}$$

$$\frac{\partial \Sigma_\pi}{\partial (p')^i}(x_1, x_2, p') \leq 0 \text{ if } (p')^i = C.$$

Let us first consider the case $C > 1/2$, $|p'| = C$, and $k > 0$. Observe that if $|p'| > 1/2$, then $S(x, p') = 0$; $\Sigma_\pi = - < x_1 - x_2, p' + \pi >$. Eq. (6) then implies: If $C > 1/2$ and $|p'| = C$, then $\omega_1 = -k(p' + \pi)$; $\omega_2 = k(p' + \pi)$. Hence: if $k > 0$ and $|p| = C$, then $|\omega_1|/k, |\omega_2|/k \geq C - |\pi|$.

If $k > 0$ and $|p| < C$, then $(x_1, x_2, \omega_1, \omega_2) \in \Gamma_\pi(S)$ by (6) and (7).

If $k = 0$, then $\omega_1 = \omega_2 = 0$. Finally, k is always non-negative. This proves the statement. $\square$

3.2.7. Let $A, B \subset \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ be the following open subsets:

$$A = \{(x_1, x_2, t') | |x_1| > 1/2\}$$

$$B = \{(x_1, x_2, t') | |x_1| < 3/5; |x_2| > 4/5\}$$

Lemma 3.15. *For every $S \in J$ we have: 1) $\Lambda_{S,\pi}|_A \cong \mathbb{K}_{\{x_1 = x_2; t \geq 0\}}[-n]$;*

2) $\Lambda_{S,\pi}|_B = 0$.

Proof. 1) We have $S(x_1, p') = 0$ for all $x_1 > 1/2$. Therefore,

$$\Lambda_S|_A = R\pi_! \mathbb{K}_{\{t - < x_1 - x_2, p' + \pi > \geq 0\}} \cong \mathbb{K}_{\{x_1 = x_2; t \geq 0\}}[-n].$$

The last isomorhpism has been established in Sec. 3.1.3.

2) Let $|x_1| < 3/5$, $|x_2| > 4/5$, and consider the equation

$$\partial_{p'} \Sigma_\pi(x_1, x_2, p') = 0.$$

We have

$$\partial_{p'}(-S(x_1, p') - < x_1 - x_2, p' + \pi >) = -x_1 - \partial_{p'} S(x_1, p') + x_2 = x_2 - y,$$

where

$$y = x_1 + \partial_{p'} S(x_1, p')$$

if $|p'| \leq 1/2$ then $|y| \leq 1/2$ as $S \in J$. If $|p'| \geq 1/2$, then $y = x$ and $|y| < 3/5$. Thus, in any case $|y| < 3/5$, therefore,$x_2 - y \neq 0$ because $|x_2| > 4/5$.

Thus for all p',

$$\partial_{p'} \Sigma_\pi(x_1, x_2, p') \neq 0.$$

2)Fix $(x_1, x_2) \in B$. Set $G(p') := \Sigma_\pi(x_1, x_2, p')$. We know that $dG(p') \neq 0$ for all p'. For $|p'| > 1/2$, $G(p') = - < x_1 - x_2, p' + \pi > = < c, p' > + K$ for some constants $c \neq 0$ and K.

We need to show that for any such a function G, $Rq_! \mathbb{K}_{\{(t,p): t + G(p) \geq 0\}} = 0$ where $q : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is the projection.

Let $Y \subset \mathbb{R}^n$ be the hyperplane $< c, p > +K = -M$ for $M >> 0$. Let F_τ be the flow of the gradient vector field of G. We then get a map

$$\Gamma : Y \times \mathbb{R} \to \mathbb{R}^n,$$

$$\Gamma(y, \tau) = F_\tau(y)$$

The map Γ is clearly a diffeomorphism and $G(\Gamma(y, \tau)) = \tau - M$. Thus, under diffeomorphism Γ, the function $G(p')$ gets transformed into $\tau - M$. Therefore it suffices to show the statement for G being a linear function on $\mathbb{R}^n$, in which case the statement is clear. $\square$

3.2.8. Using the above Lemma we will now construct a kernel in $\mathcal{D}(X \times X)$ where X is as in Sec. 3.2.4. Observe that $A \cup B$ contains the set $C := \{(x, x', t) | |x| > 4/5 \text{ or } |x'| > 4/5\}$ and the above Lemma implies that $\Lambda_{S,\pi}|_C \cong \mathbb{K}_{\{x=x', t \geq 0\}}$.

Recall (Sec 3.2.4) that we have a Darboux chart $U \subset T^*X$. Let U_1 be the projection of U onto X. U_1 is identified with a cube $|x| < 1$ in $\mathbb{R}^n$. Let $V \subset U_1 \subset X$ be given by the equation $|x| < 1$ and $K \subset V$ by the equation $|x| < 4/5$. We then have a sheaf $\Lambda_{S,\pi}|_{V \times V \times \mathbb{R}}$ and a compact $K \subset V$ such that on $W := V \times V \times \mathbb{R} \backslash (K \times K \times \mathbb{R})$ we have an identification $\Lambda_{S,\pi}|_W = \mathbb{K}_{\{(x_1, x_2, t) \in W | x_1 = x_2; t \geq 0\}}[-n]$

One can now extend $\Lambda_{S,\pi}$ to a sheaf on $X \times \times X \times \mathbb{R}$ by setting $L_S|_{X \times \mathbb{R} \times X \times \mathbb{R} \backslash W} = \mathbb{K}_{\{(x, x', t) | x = x'; t \geq 0\}}$. Denote thus obtained sheaf by L_S. Let $\Gamma_\Phi = \{(-\omega, \Phi(\omega)\} \subset T^*X \times T^*X$.

Proposition 3.16. *We have $L_S \in \mathcal{D}_{\Gamma_\Phi}(X \times X)$.*

Proof. Follows easily from Lemma 3.14 and Remark before this Lemma. $\square$

3.2.9. Let $S_+(x, p)$ be a function on $\mathbb{R}^{2n}$ defined as follows: if $S(x, p) \leq 0$, then we set $S_+(x, p) = 0$; if $S(x, p) \geq 0$, then we set $S_+(x, p) = S(x, p)$.

Lemma 3.17. *For every $S \in J$ and any $\pi \in \mathbb{R}^n$ we have: 1) $\Lambda_{S_+,\pi}|_A \cong \mathbb{K}_{\{x_1 = x_2; t \geq 0\}}[-n]$;*
2) $\Lambda_{S_+,\pi}|_B = 0$.

Proof. There exists a sequence of smooth functions $g_n(x)$ on $\mathbb{R}$ with the following properties: 1) each function $g_n(x)$ is non-decreasing; furthermore, $0 \leq g_n'(x) \leq 1$ for all n and x;
2) for every x, the sequence $g_n(x)$ is non-decreasing;
3) for $x \leq 0$, $g_n(x) = 0$;
4) for $x \geq 1/n$, $g_n'(x) = 1$.
Fix such a sequence of functions.

For $S \in J$ consider functions $S_n(x, p) = g_n(S(x, p))$. Let us check that $S_n \in J$. Indeed, S_n are supported on the set $|x| \leq 1/2$, $|p| \leq 1/2$ because $g_n(0) = 0$. Next, we have $|x^i + \partial S/\partial p^i| \leq 1/2$ for all x with $|x| \leq 1/2$, i.e

$$\partial S/\partial p^i \in [-x_i - 1/2; -x_i + 1/2]$$

The interval on the RHS contains zero, therefore is closed under multiplication by any number $\lambda \in [0, 1]$.

We have

$$\partial S_n/\partial p^i = g_n'(S) \partial S/\partial p^i \in [-x_i - 1/2; -x_i + 1/2]$$

precisely because $0 \leq g_n' < 1$. Thus, $S_n \in J$.

Next, we see that $S_1(x) \leq S_2(x) \leq \cdots \leq S_n(x) \leq \cdots$ and that $S_n(x)$ converges uniformly to $S_+(x)$. It then follows that we have induced maps $\Lambda_{S_1,\pi} \to \Lambda_{S_2,\pi} \to \cdots \Lambda_{S_n,\pi} \to \cdots$ and we have an isomoprhism

$$L\varinjlim_n \Lambda_{S_n,\pi} \to \Lambda_{S_+,\pi}.$$

Since the sheaves $\Lambda_{S_n,\pi}$ satisfy the Lemma, so does $\Lambda_{S_+,\pi}$. □

This implies that in the same way as above, $\Lambda_{S_+,\pi}$ can be extended to $X \times X \times \mathbb{R}$ in the same way as $\Lambda_{S,\pi}$ and we denote thus obtained sheaf by $L_{S_+,\pi}$.

3.2.10. *Proof of the Theorem 3.9.* We will prove an equivalent statement as in Sec 3.2.4

Define a functor $T : \mathcal{D}(X \times \mathbb{R}) \to \mathcal{D}(X \times \mathbb{R})$ by setting $T(F) = F \bullet L_S$ (see Sec. 3.1.2). Because of Lemma 3.16 and Proposition 3.11 we see that if $F \in \mathcal{D}_L(X)$, then $TF \in \mathcal{D}_{\Phi(L)}(X)$.

Next, we have natural maps

$$L_{S,\pi} \xrightarrow{i} L_{S_+,\pi} \xleftarrow{j} L_{0,\pi}$$

Note that $L_{0,\pi} = \mathbb{K}_{\{(x_1,x_2,t)|x_1=x_2,t\geq 0\}}$. In order to finish the proof of the theorem, it suffices to show that the cones of the induced maps $F \bullet L_{S,\pi} \to F \bullet L_{S_+,\pi}$ and $F = F \bullet L_{0,\pi} \to F \bullet L_{S_+,\pi}$ are torsion sheaves for all $F \in \mathcal{D}(X)$. This easily follows from the fact that the cones of the maps $L_{S,\pi} \to L_{S_+,\pi}$ and $L_{0,\pi} \to L_{S_+,\pi}$ are torsion objects in $\mathcal{D}(X \times X \times \mathbb{R})$. This fact can be seen from the following: each of the cones in question is supported on the set $\{(x_1, x_2, t)|m \leq t \leq M\}$ where m is the minimum of S and M is the maximum of S. Any sheaf G with such a property is necessarily torsion, because the supports of G and $T_{c*}G$ are disjoint for $c >> 0$ and $R\hom(G, T_{c*}G) = 0$. This proves theorem 3.9.

3.2.11. *Proof of Theorem 3.1.* Let $F_1, F_2 \in \mathcal{D}(X)$ and let $f : F_1 \to F_2$. Call f *an isomorphism up-to torsion* if the cone of f is a torsion object. Call F_1 and F_2 *isomorphic up-to torsion* if they can be connected by a chain of isomorphisms up-to torsion.

It is easy to see that if F_1 and F_2 are isomorphic up-to torsion and for some $G \in \mathcal{D}(X)$, the natural map $R\hom(G, F_1) \to R\hom(G, T_{c*}F_1)$ is zero for some $c > 0$, then the map $R\hom(G, F_2) \to R\hom(G, T_{d*}F_2)$ is zero for some $d > 0$.

Suppose L_1 and L_2 are displaceable compact Lagrangians in T^*X, i.e. for some sympectomorphism Φ of T^*X such that Φ is identity outside of a compact, we have $L_1 \cap \Phi(L_2) = \emptyset$. Let $F_i \in \mathcal{D}_{L_i}(X)$. Theorem 3.1 is equivalent to the statement: *for some $c > 0$, the natural map $R\hom(F_1, F_2) \to R\hom(F_1, T_{c*}F_2)$ is zero.*

This statement can be proven as follows. By Theorem 3.9, there exists an object $F_3 \in \mathcal{D}_{\Phi(L_2)}(X)$ such that F_3 and F_2 are isomorphic up-to torsion. Therefore, it suffices to show that the natural map

$$R\hom(F_1, F_3) \to R\hom(F_1, T_{c*}F_3)$$

is zero for some $c > 0$. But Theorem 3.2 asserts that $R\hom(F_1, F_3) = R\hom(F_1, T_cF_3) = 0$, whence the statement.

4. Non-dispaceability of certain Lagrangian submanifolds in $\mathbb{CP}^n$

Consider $\mathbb{CP}^N$ with the standard symplectic structure. We have the following standard Lagrangian subvarieties in $\mathbb{CP}^N$: the Clifford torus $\mathbb{T} \subset \mathbb{CP}^N$ consisting of all points with homogeneous coordinates $(z_0 : z_1 : z_2 : \cdots : z_N)$ such that $|z_0| = |z_1| = \cdots = |z_N| > 0$. Another Lagrangian subvariety we will consider is $\mathbb{RP}^N \subset \mathbb{CP}^N$. Our main goal is to prove

Theorem 4.1. *1) $\mathbb{T}$ is non-displaceable from itself;*

2) $\mathbb{RP}^N$ is non-dispalceable from itself;

3) $\mathbb{T}$ and $\mathbb{RP}^N$ are non-displaceable from one another.

4.0.12. Let us first of all explain how Theorem 3.1 can be applied.

Let $G = \mathrm{SU}(N)$ Realize $\mathbb{CP}^N$ as a coadjoint orbit $\mathbb{CP}^N = \mathcal{O} \subset \mathfrak{g}^*$, where $\mathfrak{g} = \mathfrak{su}(N)$ is the Lie algebra of G. We identify $\mathfrak{g}$ with the real vector space of $N \times N$ skew-hermitian matrices. We have an invariant positive definite inner product on $\mathfrak{g}$ by the formula $< A, B >= -\mathrm{Tr}(AB)$. This way we get an identification $\mathfrak{g} \cong \mathfrak{g}^*$.

The orbit $\mathcal{O} \subset \mathfrak{g}^* \cong \mathfrak{g}$ is an orbit of the following diagonal skew-hermitian matrix

$$i\lambda(P_V - (1/N)I) \in \mathfrak{g}$$

where $\lambda \in \mathbb{R}$, $\lambda \neq 0$ is a fixed real number. For simplicity we will only work with $\lambda > 0$. However, the case $\lambda < 0$ is absolutly similar.

Consider T^*G. We have a diffeomorphism $I_R : T^*G \to G \times \mathfrak{g}^*$ where we identify $\mathfrak{g}^*$ with right-invariant forms on G. Any element $X \in \mathfrak{g}$ gives rise to a function on f_X on $\mathfrak{g}^*$. We have a standard Poisson structure on $\mathfrak{g}^*$ determined by the condition $\{f_X, f_Y\} = f_{[X,Y]}$. The canonical projection $p_R : T^*G \stackrel{I_R}{\to} G \times \mathfrak{g}^* \to \mathfrak{g}^*$ is then a Poisson map.

Let $\mathfrak{g}^{op}$ be the Lie algebra whose underlying vector space is $\mathfrak{g}$ but $[X,Y]_{\mathfrak{g}^{\mathrm{op}}} = -[X,Y]_{\mathfrak{g}}$, We then have an identification $I_L : T^*G \to G \times (\mathfrak{g}^{\mathrm{op}})^*$, where we identify $(\mathfrak{g}^{\mathrm{op}})^*$ with left-invariant forms on G. The composition $I_R I_L^{-1} : G \times (\mathfrak{g}^{\mathrm{op}})^* \to G \times \mathfrak{g}^*$ is as follows: $I_R I_L^{-1}(g, A) = (g, \mathrm{Ad}^*_{g^{-1}}(A))$.

Indeed, the conjugate map $(I_R I_L^{-1})^* : G \times \mathfrak{g} \to G \times \mathfrak{g}^{\mathrm{op}}$ is given by $(I_R I_L^{-1})^*(g, X) = (g, \mathrm{Ad}_{g^{-1}} X)$.

Respectively, $I_L I_R^{-1} : G \times \mathfrak{g}^* \to G \times (\mathfrak{g}^{\mathrm{op}})^*$ is given by $I_L I_R^{-1}(g, A) = (g, \mathrm{Ad}^*_g A)$.

One can easily check that the product $p_L \times p_R : T^*G \to (\mathfrak{g}^{\mathrm{op}})^* \times \mathfrak{g}^*$ is a Poisson map.

We know that $\mathcal{O}^{\mathrm{op}} \subset \mathfrak{g}^*$ is a symplectic leaf, hence a co-isotropic sub-variety. Therefore, so is $M := p_R^{-1}\mathcal{O} \subset T^*G$.

Let $\mathcal{O}^{\mathrm{op}} \subset (\mathfrak{g}^*)^{\mathrm{op}}$ be the image of $\mathcal{O} \subset \mathfrak{g}^*$ under the identification of vector spaces $\mathfrak{g}^* = (\mathfrak{g}^{\mathrm{op}})^*$.

We then see that $M = p_L^{-1}\mathcal{O}^{\mathrm{op}} = p_R^{-1}\mathcal{O}$. Indeed, we know that $I_L I_R^{-1}(g, A) = (g, \mathrm{Ad}_g A)$ and $A \in \mathcal{O}$ iff $\mathrm{Ad}_g A \in \mathcal{O}$.

Hence, we have $M = (p_L \times p_R)^{-1}(\mathcal{O}^{\mathrm{op}} \times \mathcal{O})$. Given any Poisson fibration $f : X \to Y$ and a coisotropic subvariety $N \subset Y$, the subvariety $f^{-1}N \subset X$ is also co-isotropic. Let $n \in f^{-1}N$ and let $V \in T_n f^{-1}N$ be a co-isotropic vector (i.e $V = X_H$ where H is a function in a neighborhood of n and $H|_{f^{-1}N} = 0$), we then see that $f_*V \in T_{f(n)}N$ is also a co-isotropic vector.

Let us apply this observation to our case. We see that $\mathcal{O}^{\mathrm{op}} \times \mathcal{O}$ has only zero co-isotropic vectors. Therefore, all co-isotropic vectors in TM are tangent to fibers of the map $p_L \times p_R : M \to \mathcal{O}^{\mathrm{op}} \times \mathcal{O}$. Comparison of dimensions shows that the inverse is also true: co-isotropic vectors in TM are precisely those tangent to the fibers of the map $p_L \times p_R$. Thus, co-isotropic foliation to M is the tangent foliation to $p_L \times p_R$. We know that this implies an induced symplectic structire on $\mathcal{O}^{\mathrm{op}} \times \mathcal{O}$. As the map $p_L \times p_R$ is Poisson, it follows that the induced Poisson structure coincides with that induced by the inclusion $\mathcal{O}^{\mathrm{op}} \times \mathcal{O} \hookrightarrow \mathfrak{g}^{\mathrm{op}} \times \mathfrak{g}$. The corresponding symplectic 2 form is equal to $(-\omega; \omega)$ where ω is Kirillov's symplectic form on $\mathcal{O}$ and we use the identification of manifolds $\mathcal{O}^{\mathrm{op}} = \mathcal{O}$.

Let $I : M \to T^*G$ be the inclusion and $P = p_L \times p_R : M \to \mathcal{O}^{\mathrm{op}} \times \mathcal{O}$. It then follows that $I^*\omega_{T^*G} = P^*\omega_{\mathcal{O}^{\mathrm{op}} \times \mathcal{O}}$.

It follows that if $L \subset \mathcal{O}^{\mathrm{op}} \times \mathcal{O}$ is a Lagrangian manifold, then so is $IP^{-1}L \subset T^*G$.

Another important observation: let H be a function on $\mathcal{O}^{\mathrm{op}} \times \mathcal{O}$ and let H' be a function on T^*G such that $H'|_M = P^{-1}H$.

1) Then the Hamiltonian vector field $X_{H'}$ is tangent to M;

2) given any function f on $\mathcal{O}^{\mathrm{op}} \times \mathcal{O}$ we have

$$X_{H'} P^{-1} f = P^{-1} X_H f.$$

Let $e^{tX_{H'}}$ be the Hamiltonian flow of H' and e^{tX_H} the Hamiltonian flow of H. It then follows that for any point $m \in M$, $Pe^{tX_{H'}}(m) = e^{tX_H}(P(m))$.

These observations imply:

Proposition 4.2. *Let $L_1, L_2 \subset \mathcal{O}^{op} \times \mathcal{O}$ be subsets such that $IP^{-1}L_1, IP^{-1}L_2 \subset T^*G$ are non-displaceable. Then so are L_1, L_2.*

Proof. Suppose L_1 and L_2 are displaceable. Then there exist functions $H_1, \ldots, H_k$ on $\mathcal{O}^{\mathrm{op}} \times \mathcal{O}$ such that $e^{X_{H_1}} \cdots e^{X_{H_k}} L_1 \cap L_2 = \emptyset$. Choose compactly supported functions $H'_1, \ldots, H'_k$ on T^*G such that $H'_i|_M = P^{-1}H_i$. One then has

$$Pe^{X_{H'_1}} \cdots e^{X_{H'_k}} m = e^{X_{H_1}} \cdots e^{X_{H_k}} Pm$$

for every $m \in M$. Therefore,

$$IP^{-1}L_1 \cap e^{X_{H'_1}} \cdots e^{X_{H'_m}} P^{-1}L_2 = \emptyset,$$

i.e the Lagrangians $IP^{-1}L_1$ and $IP^{-1}L_2$ are displaceble, whence the statement $\square$

let $\Delta \subset \mathcal{O}^{\mathrm{op}} \times \mathcal{O}$ be the diagonal. Δ is clearly Lagranginan.

It then follows that Theorem 4.1 follows from the following one:

Theorem 4.3. *1) $IP^{-1}\Delta$ and $IP^{-1}\mathbb{T} \times \mathbb{T}$ are non-displaceble;*

2) $IP^{-1}\Delta$ and $IP^{-1}\mathbb{RP}^N \times \mathbb{RP}^N$ are non-displaceable;

3) $IP^{-1}\mathbb{RP}^N \times \mathbb{RP}^N$ and $IP^{-1}\mathbb{T} \times \mathbb{T}$ are non-displaceable

4.0.13. We will prove Theorem 4.3 using Theorem 3.1.

Our main tool will be a certain object $u_{\mathcal{O}} \in \mathcal{D}(G)$ which will be now introduced.

We need a notation. Let $S \in \mathcal{D}(G)$. Let $F \in D(G)$. Let $m : G \times G \times \mathbb{R} \to G \times \mathbb{R}$ be the map induced by the product on G. Set $F *_G S := Rm_!(F \boxtimes S)$ (this is nothing else but a convolution). One can easily check that $F *_G S \in \mathcal{D}(G)$ (use Proposition 2.2).

Proposition 4.4. *There exists an object $u_{\mathcal{O}} \in \mathcal{D}_{IP^{-1}\Delta}(G)$ with the following properties:*

1) there exists a neigborhood of the unit $U \subset G$; $e \in U$ with the following property:

for every $g \in G$ and every object $F \in D(G)$ such that F is supported on gU and $R\Gamma(G, F) = 0$, the object $F *_G u_{\mathcal{O}}$ is a torsion object;

2) The object $u_{\mathcal{O}}$ is not a torsion object.

The proof of this Proposition is rather long, so we will first show how this Proposition (along with Theorem 3.1) implies Theorem 4.3.

4.0.14.

Lemma 4.5. *Let $\mathfrak{h} \subset \mathfrak{g}$ be the standard Cartan subalgebra consisting of the diagonal traceless skew-hermitian matrices. Let $\mathfrak{k} := \mathfrak{so}(N) \subset \mathfrak{su}(N)$. We then have $\mathbb{T} = (\mathfrak{g}/\mathfrak{h})^* \cap \mathcal{O}$; $\mathbb{RP}^N = (\mathfrak{g}/\mathfrak{k})^* \cap \mathcal{O}$.*

Proof. The symplectomorphism $f : \mathbb{CP}^N \to \mathcal{O}$ is as follows. Given a line $l \in \mathbb{C}^N$ we set $f(l) := i(\lambda P_L - \lambda/NI)$, where $\lambda > 0$ is a fixed positive real number. Let $v = (v_1, v_2, \ldots, v_N) \in l$; $v \neq 0$. We then have

$$f(l)_{pq} = (i\lambda/|v|^2)v_p\overline{v_q} - i\lambda/N\delta_{pq},$$

where δ_{pq} is the Kronecker symbol.

Thus, $f(l) \in \mathcal{O} \cap (\mathfrak{g}/\mathfrak{h})^*$ iff $f(l)_{pp} = 0$ for all p, i.e. $|v_p|^2/|v|^2 = 1/N$, i.e. $|v_1|^2 = |v_2|^2 = \cdots |v_N|^2$, i.e. $l \in \mathbb{T}$.

Analogously, $f(l) \in (\mathfrak{g}/\mathfrak{k})^*$ iff $f(l)_{pq} \in i\mathbb{R}$ for all p, q, i.e $v_p\overline{v_q} \in \mathbb{R}$ for all p, q. Let $v_{p_0} \neq 0$. Then $v_q = t_q/\overline{v_{p_0}}$for some $t_q \in \mathbb{R}$ and for all q. Let $t = (t_1, t_2, \ldots, t_N)$ then $v = t/\overline{v_{p_0}}$ and $l \in \mathbb{RP}^N \subset \mathbb{CP}^N$. The inverse can be easily checked as well. $\square$

Proposition 4.6. *Let $T \subset SU(N)$ be the subgroup of diagonal matrices and let $SO(N) \subset SU(N)$ be the subgroup of special orthogonal matrices.*

We then have

1) $\mathbb{K}_T *_G u_{\mathcal{O}} \in \mathcal{D}_{IP^{-1}(\mathbb{T}\times\mathbb{T})}(G)$;

2) $\mathbb{K}_{SO(N)} *_G u_{\mathcal{O}} \in \mathcal{D}_{IP^{-1}(\mathbb{RP}^N\times\mathbb{RP}^N)}(G)$.

Proof. Let us prove 1). First of all, one can easily check that $\mathbb{K}_T *_G u_{\mathcal{O}} \in \mathcal{D}(G)$ using Proposition 2.2. It only remains to show that $\mathbb{K}_T *_G u_{\mathcal{O}}$ *is microsupported on the set* $\{(g, t, k\omega, k)|k \geq 0; \omega \in IP^{-1}\mathbb{T} \times \mathbb{T}\}$. We have $\mathbb{K}_T *_G u_{\mathcal{O}} = Rm_!(\mathbb{K}_T \boxtimes u_{\mathcal{O}})$, where $m : G \times G \times \mathbb{R} \to G \times \mathbb{R}$ is induced by the product on G. Let also $M : G \times G \to G$ be the product on G Let $g_1, g_2 \in G$. We then have an induced map

$$M_{g_1,g_2*} : T_{(g_1,g_2)}G \times G \to T_{g_1g_2}G$$

Let $(g_1, g_2, X_1, X_2) \in G\times G\times\mathfrak{g}\times\mathfrak{g} = T(G\times G)$. One then has $M_{g_1,g_2*}(g_1, g_2, X_1, X_2) = X_1+\mathrm{Ad}_{g_1}X_2$. The dual map

$$M^*_{g_1,g_2} : T^*_{g_1g_2}G \to T^*_{(g_1,g_2)}G \times G$$

is as follows

$$M^*_{g_1,g_2}(g_1g_2, \omega) = (g_1, g_2, \omega; \mathrm{Ad}^*_{g_1}\omega).$$

Finally, the map

$$m^*_{g_1,g_2,t} : T^*_{(g_1g_2,t)}(G \times \mathbb{R}) \to T^*_{(g_1,g_2,t)}(G \times G \times \mathbb{R})$$

is given by

$$m^*_{g_1,g_2,t}(g_1g_2, t, \omega, k) = (g_1, g_2, t, \omega, \mathrm{Ad}^*_{g_1}\omega, k). \tag{8}$$

The map m being proper, we know that the object $Rm_!(\mathbb{K}_T \boxtimes u_{\mathcal{O}})$ is microsupported on the set of all points of the form

$$(g_1g_2, t, \omega, k) \tag{9}$$

where

$$m^*_{g_1,g_2,t}(g_1g_2, t, \omega, k) \in \mathrm{SS}(\mathbb{K}_T \boxtimes u_{\mathcal{O}}),$$

i.e

$$(g_1, \omega) \in \mathrm{SS}(\mathbb{K}_T); \tag{10}$$

$$(g_2, t, \mathrm{Ad}^*_{g_1}\omega, k) \in \mathrm{SS}(u_{\mathcal{O}}). \tag{11}$$

We have,

$$\mathrm{SS}(\mathbb{K}_T) \subset \{(g, \omega_1) \in G \times \mathfrak{g}^*|g \in T; \omega_1 \in (\mathfrak{g}/\mathfrak{t})^*\}; \tag{12}$$

$$\mathrm{SS}(u_{\mathcal{O}}) \subset \{(g, t, k\omega_2, k)|k \geq 0; (g, \omega_2) \in IP^{-1}\Delta\}, \tag{13}$$

as follows from Lemma 3.7. The condition $(g, \omega_2) \in IP^{-1}\Delta$ means that $\omega_2 \in \mathcal{O}$ and $P_L\omega_2 = P_R\omega_2$, i.e $\omega_2 = \mathrm{Ad}^*_g\omega_2$.

Therefore

$$(g_1g_2, t, \omega, k) \in \mathrm{SS}Rm_!(\mathbb{K}_T \boxtimes u_{\mathcal{O}})$$

only if (compare (11) and (13)):

$$k \geq 0 \tag{14}$$

$$\mathrm{Ad}^*_{g_1}\omega = k\omega_2, \tag{15}$$

where

$$\omega_2 \in \mathcal{O} \tag{16}$$

and

$$\mathrm{Ad}^*_{g_2}\omega_2 = \omega_2. \tag{17}$$

We should also have (compare (10) and (12)):

$$g_1 \in T \tag{18}$$

and

$$\omega \in (\mathfrak{g}/\mathfrak{h})^*. \tag{19}$$

Let us now show that (g_1g_2, ω, k) is of the form $(g_1g_2, k\omega^1, k)$, where $k \geq 0$ and $(g_1g_2, \omega^1) \in IP^{-1}(\mathbb{T} \times \mathbb{T})$. The latter means that $(P_L \times P_R)(g_1g_2, \omega^1) \in \mathcal{O}^{\mathrm{op}} \times \mathcal{O}$ i.e. *both* ω^1 *and* $Ad^*_{g_1g_2}\omega^1$ *belong to* $\mathbb{T} = \mathcal{O} \cap (\mathfrak{g}/\mathfrak{h})^*$. We have $k \geq 0$ (see (14). If $k = 0$, then $\omega = \mathrm{Ad}^*_{g_1^{-1}}k\omega_2 = 0$ and $(g_1g_2, \omega, k) = (g_1g_2, 0, 0)$, the condition is fulfilled.

Let now $k > 0$. We have $\omega = k\mathrm{Ad}^*_{g_1^{-1}}\omega_2$ (see (15)) so that $\omega^1 = \mathrm{Ad}^*_{g_1^{-1}}\omega_2$.

As $\omega_2 \in \mathcal{O}$ (see (16)), it follows that $\omega^1 = \mathrm{Ad}^*_{g_1^{-1}}\omega_2 \in \mathcal{O}$. We also have $\omega_2 = \omega/k \in (\mathfrak{g}/\mathfrak{h})^*$ (see (19).

Next, let us consider

$$\begin{aligned}\mathrm{Ad}^*_{g_1g_2}\omega^1 &= \mathrm{Ad}^*_{g_1g_2}\mathrm{Ad}^*_{g_1^{-1}}\omega_2\\ &= \mathrm{Ad}^*_{g_2}\omega_2 = \omega_2\end{aligned}$$

(the latter equality comes from (17), and we have already shown that $\omega_2 \in \mathcal{O} \cap (\mathfrak{g}/\mathfrak{t})^*$.

This proves the statement 1). The statement 2) can be proven in precisely the same way. $\square$

4.1. Our goal is to prove the following statements

Proposition 4.7. *The object* $\mathbb{K}_T *_G u_{\mathcal{O}} \in \mathcal{D}(G)$ *is isomorphic up to torsion to the object* $u_{\mathcal{O}} \otimes_{\mathbb{K}} H^*(T, \mathbb{K})$.

Proposition 4.8. *Suppose that* $\mathbb{K}$ *is a field of characteristic 2. The object* $\mathbb{K}_{SO(N)} *_G u_{\mathcal{O}} \in \mathcal{D}(G)$ *is isomorphic up-to torsion to* $u_{\mathcal{O}} \otimes_{\mathbb{K}} H^*(SO(N), \mathbb{K})$.

4.1.1. These Propositions imply Theorem 4.3. Let $\mathbb{K}$ have characteristic 2 and let each of objects F_1 and F_2 be either $\mathbb{K}_T *_G u_{\mathcal{O}}$ or $\mathbb{K}_{\mathrm{SO}(N)} *_G u_{\mathcal{O}}$. Taking into account Proposition 4.6 and Theorem 3.1, it suffices to show that for any $c > 0$, the induced map $R\hom(F_1, F_2) \to R\hom(F_1; T_{c*}F_2)$ does not vanish (for all choices of F_1 and F_2). By virtue of the just formulated Propositions, this follows from $u_{\mathcal{O}}$ being non-torsion which is promised in Proposition 4.4. Thus, Theorem 4.3 is now reduced to Propositions 4.4, 4.7, and 4.8. We will first deduced the last two Propositions from the first one, and, finally, we will prove Proposition 4.4.

4.1.2. In order to prove Propositions 4.7 and 4.8 we need to develop corollaries from Proposition 4.4 1).

Let C_U be the full subcategory of $D(G)$ generated by all objects F as in Proposition 4.4 1) and their finite extensions.

Lemma 4.9. *let $Q := [0,1]^M$, $M \geq 0$. Let $\pi : Q \to G$ be any continuous map. Let $F \in D(Q)$, $R\Gamma(Q, F) = 0$. Then $R\pi_! F \in C_U$.*

Proof. The case $M = 0$ is obvious. Let $M > 0$. Let $Q_0 := [0, 1/2] \times [0,1]^{M-1}$ and let $Q_1 = [1/2, 1] \times [0,1]^{M-1}$.

1) We will first prove that F *can be obtained by a finite number of extensions from objects $X_1, X_2, \ldots, X_m$, where each X_i is supported on either Q_0 or Q_1 and $R\Gamma(Q, X_i) = 0$.* Call such objects and their extensions *admissible*. Thus, we are to show that F is admissible.

Let $I := Q_1 \cap Q_2$. Let $i_k : Q_k \to Q$ and $i : I \to Q$ be inclusions. Realize F as a complex of soft sheaves on Q Let $F_k := F|_{G_k}$ and $F_I := F|_I$. Each of these objects is also a complex of soft sheaves.

We then have an isomorphism

$$F \to \mathrm{Cone}(i_{1*}F_1 \oplus i_{2*}F_2 \to i_* F_I)$$

Let $p_k : Q_k \to \mathbf{pt}$ and $p_I : I \to \mathbf{pt}$ be the natural projections. Let $V_k := p_{k*}F_k = p_{k!}F_k$; let $V_I := p_{I*}F_I = p_{I!}F_I$. V_k and V_I are just complexes of $\mathbb{K}$-vector spaces. We then have maps

$$a_k : p_k^{-1}V_k \to F_k; a_I : \ p_I^{-1}V_I \to F_I;$$

$$b_k : i_{k*}p_k^{-1}V_k \to i_{I*}p_I^{-1}V_I$$

We then have the following commutative diagram of complexes of sheaves

$$\begin{tikzcd}
i_{1*}F_1 \oplus i_{2*}F_2 \arrow[r] & i_{I*}F_I \\
i_{1*}p_1^{-1}V_1 \oplus i_{2*}p_2^{-1}V_2 \arrow[u, "a_1 \oplus a_2"] \arrow[r, "b_1 \oplus b_2"] & i_{I*}p_I^{-1}V_I \arrow[u]
\end{tikzcd}$$

Let Φ be the total complex of this diagram. Φ can be obtained by successive extensions from the following objects

$$\mathrm{Cone}(i_{k*}p_k^{-1}V_k \to i_{k*}F_k);$$

$$\mathrm{Cone}(i_{I*}p_I^{-1}V_I \to i_{I*}F_I);$$

each of these objects is admissible. Hence Φ is admissible.

Next, we have a natural map

$$\Phi \to \mathrm{Cone}(i_{1*}p_1^{-1}V_1 \oplus i_{2*}p_2^{-1}V_2 \to i_{I*}p_I^{-1}V_I)$$

The cone of this map is quasi-isomrophic to F. Thus, in order to show that F is admissible, it suffices to show that

$$\mathrm{Cone}(i_{1*}p_1^{-1}V_1 \oplus i_{2*}p_2^{-1}V_2 \to i_{I*}p_I^{-1}V_I)$$

is admissible.

Let us study the arrow $i_{1*}p_1^{-1}V_1 \oplus i_{2*}p_2^{-1}V_2 \to i_{I*}p_I^{-1}V_I$. This arrow is induced by the natural maps $V_1 \to V_I$ and $V_2 \to V_I$. The cone of the induced map $f : V_1 \oplus V_2 \to V_I$ is quasi-isomorphic to $R\Gamma(Q, F) = 0$. Therefore, f is a quasi-isomorphism and we have an induced quasi-isomorphism

$$\mathrm{Cone}(i_{1*}p_1^{-1}V_1 \oplus i_{2*}p_2^{-1}V_2 \to i_{I*}p_I^{-1}(V_1 \oplus V_2)) \to \mathrm{Cone}(i_{1*}p_1^{-1}V_1 \oplus i_{2*}p_2^{-1}V_2 \to i_{I*}p_I^{-1}V_I).$$

The object on the left hand side is isomorphic to

$$\mathrm{Cone}(i_{1*}p_1^{-1}V_1 \to i_{I*}p_I^{-1}V_1) \oplus \mathrm{Cone}(i_{2*}p_2^{-1}V_2 \to i_{I*}p_I^{-1}V_1).$$

We see that this object is a direct sum of admissible objects, hence is itself admissible, therefore the object

$$\mathrm{Cone}(i_{1*}p_1^{-1}V_1 \oplus i_{2*}p_2^{-1}V_2 \to i_{I*}p_I^{-1}V_I)$$

is also admissible, whence the statement.

2)Choose a positive integer M and subdivide Q into 2^M small cubes, denote these small cubes by $\mathbf{q}_i$, $i = 1, \ldots 2^M$. Call an object $X \in D(Q)$ *M-admissible* if either

a) X is supported on one of $\mathbf{q}_i$ and $R\Gamma(Q, X) = 0$ or b) X can be obtained from objects as in a) by a finite number of extensions.

By repeatedly applying the statement from 1) we see that *every object $F \in D(Q)$ such that $R\Gamma(Q, F) = 0$ is M-admissible.*

3) For M large enough one has: for every i there exists $g_i \in G$ such that $\pi(\mathbf{q}_i) \subset g_iU$. This implies that given any object $X \in D(Q)$ supported on $\mathbf{q}_i$ and satisfying $R\Gamma(Q, X) = 0$, one has $R\pi_!X \in C_U$. Therefore, every M-admissible object is in C_U, including F. ☐

Corollary 4.10. *Let U be a neighborhood of unit in G such that U is diffeomorphic to an open ball. Then $C_U = C$, where $C \subset D(G)$ is the full subcategory formed by finite extensions of objects of the form $R\pi_!X$, where $\pi : Q \to G$, $X \in D(Q)$, $R\Gamma(Q, X) = 0$.*

Corollary 4.11. *Let $F \in C$ and $X \in D(G)$. One then has $F *_G X \in C$; $X *_G F \in C$.*

Proof. Choose a small open ball $U \in G$, $e \in U$, small means that there exists another open ball $V \subset G$ such that $U \cdot U \subset V$. It is not hard to see that any $X \in D(G)$ can be realized as a finite extension of objects X_i, where each X_i is supported on g_iU for some U. Without loss of generality, one then can assume that $X = X_i$. Therefore, $X *_G F$ is supported on $g_iU^2 \subset g_iV$. One also sees that $R\Gamma(G, X *_G F) = R\Gamma(G, X) \otimes R\Gamma(G, F) = 0$. Thus, $X *_G F \in C_V$.

The case of $F *_G X$ can be proven in a similar way. ☐

4.1.3. Call a map $f : F \to H$ in $D(G)$ a C-isomorphism if the cone of f is in C. Call two objects $F, H \in D(G)$ *C-isomorphic* if they can be joined by a chain of C-isomorphisms.

Corollary 4.12. *if F_1 and F_2 are C-isomorphic and H_1 and H_2 are C-isomorphic, then $F_1 *_G H_1$ and $F_2 *_G H_2$ are C-isomorphic*

4.1.4. We have

Claim 4.13. *If F and H are C-isomorphic, then $F *_G u_{\mathcal{O}}$ and $H *_G u_{\mathcal{O}}$ are isomoprhic up-to torsion*

Proof. Indeed, $C = C_U$, where U is the same as in Proposition 4.4. The statement follows immediately from part 1) of this Proposition. $\square$

4.2. **Proof of Proposition 4.7.** Let $S_k \subset \mathrm{SU}(N)$ be the one-parametric subgroup consisting of all matrices of the form $\mathrm{diag}(1, 1, \ldots, e^{i\phi}; e^{-i\phi}, 1, \ldots, 1)$, where $e^{i\phi}$ is at the k-th position. We then have $T = S_1 S_2 \cdots S_{N-1}$; $\mathbb{K}_T = \mathbb{K}_{S_1} *_G \mathbb{K}_{S_2} *_G \cdots *_G \mathbb{K}_{S_{N-1}}$.

It is clear that the statement of Proposition follows from

Lemma 4.14. *For any k, $\mathbb{K}_{S_k}$ is C-isomorphic to $\mathbb{K}_e \oplus \mathbb{K}_e[-1]$*

Indeed, Corollary 4.12 will then imply that $\mathbb{K}_T$ is C-isomorphic to $(\mathbb{K}_e \oplus \mathbb{K}_e[-1])^{*N-1} = \mathbb{K}_e \otimes_{\mathbb{K}} H^\bullet(T, \mathbb{K})$. Therefore, by Claim 4.13, the objects $\mathbb{K}_T * u_{\mathcal{O}}$ and $(\mathbb{K}_e \otimes_{\mathbb{K}} H^\bullet(T, \mathbb{K}))) * u_{\mathcal{O}} = u_{\mathcal{O}} \otimes_{\mathbb{K}} H^\bullet(T, \mathbb{K})$ are isomorphic up-to torsion.

It now remains to prove Lemma

4.2.1. *Proof of Lemma 4.14.* As all subgroups S_k are conjugated in G, it suffices to prove Lemma for S_1. One then has $S_1 \subset \mathrm{SU}(2) \subset \mathrm{SU}(N)$, where the embedding $\mathrm{SU}(2) \subset \mathrm{SU}(N)$ is induced by the standard decomposition $\mathbb{C}^N = \mathbb{C}^2 \oplus \mathbb{C}^{N-2}$. Let U be an open neighborhood of unit in $\mathrm{SU}(N)$ and let $U' := U \cap \mathrm{SU}(2)$. Let $\iota : \mathrm{SU}(2) \subset \mathrm{SU}(N)$ be the inclusion. It is clear that $i_* C_{U'} \subset C_U$, hence if two objects $F_1, F_2 \in D(\mathrm{SU}(2))$ are C-isomorphic, then so are $i_* F_1$ and $i_* F_2$. Therefore, in order to prove Lemma, it suffices to show that $\mathbb{K}_{S_1}$ *and $\mathbb{K}_e \oplus \mathbb{K}_e[-1]$ viewed as objects of $D(SU(2))$ are C-isomorphic.*

Let $B \subset \mathfrak{su}(2)$ consist of all matrices of the form iM, where M is a Hermitian matrix whose eigenvalues has absolute value of at most π. Let $B_\pi \subset B$ be the subset of all matrices iM, where the eigenvalues of M are precisely π and $-\pi$. It is clear the B is diffeomorphic to a 3-dimensional closed ball and $B_\pi \subset B$ is the boundary 2-sphere.

Let $I : [-\pi; \pi] \to B$ be given by $I(\phi) = i\mathrm{diag}(\phi; -\phi)$.

We then have a diagram

$$\begin{tikzcd}
[-\pi;\pi] \arrow[r, "i_1"] & B \arrow[r, "i_2"] & \mathrm{SU}(2) \\
\{-\pi,\pi\} \arrow[u, "a_1"] \arrow[r, "k_1"] & B_\pi \arrow[u, "a_2"] \arrow[r, "k_3"] & \{-I\} \arrow[u, "a_3"]
\end{tikzcd}$$

where i_2 is induced by the exponential map $\mathfrak{su}(2) \to \mathrm{SU}(2)$; a_1, k_1, a_2, a_3 are obvious inclusions; k_3 is the projection. We then have

$$\mathbb{K}_{S_1} = \mathrm{Cone}(R(i_2 i_1)_! \mathbb{K}_{[-\pi;-\pi]} \oplus a_{3!} \mathbb{K}_{-I} \to R(i_2 i_1 a_1)_! \mathbb{K}_{\{-\pi,\pi\}}). \tag{20}$$

The arrow in this equation is induced by natural maps

$$\alpha : R(i_2 i_1)_! \mathbb{K}_{[-\pi;-\pi]} \to R(i_2 i_1 a_1)_! \mathbb{K}_{\{-\pi,\pi\}})$$

and

$$\beta : a_{3!} \mathbb{K}_{-I} \to R(i_2 i_1 a_1)_! \mathbb{K}_{\{-\pi,\pi\}} = R(a_3 k_3 k_1)_! \mathbb{K}_{\{-\pi,\pi\}}$$

where α is induced by the natural map

$$\mathbb{K}_{[-\pi;-\pi]} \to a_{1!}\mathbb{K}_{\{-\pi,\pi\}}$$

induced by the embedding $\{-\pi,\pi\} \subset [-\pi,\pi]$.

The map β is induced by the natural map

$$\mathbb{K}_{-I} \to (k_3k_1)_!\mathbb{K}_{\{-\pi,\pi\}} = (k_3k_1)_*(k_3k_1)^{-1}\mathbb{K}_{\{-\pi,\pi\}}.$$

We have a C-isomorphism

$$\gamma : Ri_{2!}\mathbb{K}_B \to R(i_2i_1a_1)_!\mathbb{K}_{[-\pi,\pi]}$$

Therefore the object in (20) is C-isomorhic to

$$\text{Cone}(Ri_{2!}\mathbb{K}_B \oplus \mathbb{K}_{-I} \overset{\alpha_1\oplus\beta}{\to} R(i_2i_1a_1)_!\mathbb{K}_{\{-\pi,\pi\}}) \tag{21}$$

where $\alpha_1 = \alpha\gamma$.

The map $\alpha_1 : Ri_{2!}\mathbb{K}_B \to R(i_2i_1a_1)_!\mathbb{K}_{\{-\pi,\pi\}}$ can be factored as

$$Ri_{2!}\mathbb{K}_B \to R(i_2a_2)_!\mathbb{K}_{B_\pi} \to R(i_2i_1a_1)_!\mathbb{K}_{\{-\pi,\pi\}}.$$

Observe that $B_\pi = \mathbb{CP}^1$ and that $R(i_2a_2)_!\mathbb{K}_{B_\pi} \cong H^*(B_\pi,\mathbb{K}) \otimes_{\mathbb{K}} \mathbb{K}_{-I}$. Next $R(i_2i_1a_1)_!\mathbb{K}_{\{-\pi,\pi\}} = \mathbb{K}_{-I} \oplus \mathbb{K}_{-I}$. The map $R(i_2a_2)_!\mathbb{K}_{B_\pi} \to R(i_2i_1a_1)_!\mathbb{K}_{\{-\pi,\pi\}}$ factors as

$$R(i_2a_2)_!\mathbb{K}_{B_\pi} = a_{3!}\mathbb{K}_{-I} \otimes_{\mathbb{K}} H^*(\mathbb{CP}^1) \to a_{3!}\mathbb{K}_{-I} \overset{\beta}{\to} a_{3!}(\mathbb{K}_{-I} \oplus \mathbb{K}_{-I}) = R(i_2i_1a_1)_!\mathbb{K}_{\{-\pi,\pi\}}$$

. Thus we see that α_1 factors as $\alpha_1 = \beta u$. It is well known that in this case we have a quasi-isomorphism

$$\text{Cone}(\alpha_1 \oplus \beta) \cong \text{Cone}(0 \oplus \beta).$$

meaning that the object in (21) is isomorphic to $Ri_{2!}\mathbb{K}_B \oplus \mathbb{K}_{-I}[-1]$ (because $\text{Cone}(\beta) \cong \mathbb{K}_{-I}[-1]$).

Let $\varepsilon : 0 \in B$ be the zero matrix. one then has a C-isomorphism $Ri_{2!}\mathbb{K}_B \to Ri_{2!}\varepsilon_!\mathbb{K}_0 = \mathbb{K}_e$. Analogously, by choosing a point $0' \in B_\pi$, one gets a C-isomorphism $Ri_{2!}\mathbb{K}_B \to \mathbb{K}_{-I}$. Therefore, the object in (21) is C-isomorhic to $\mathbb{K}_e \oplus \mathbb{K}_{-I}[-1]$ and $\mathbb{K}_{-I}$ is C-isomorphic with $\mathbb{K}_e$ (via $Ri_{2!}\mathbb{K}_B$). Thus, the object in (21), hence $\mathbb{K}_{S_1}$ is C-isomorphic to $\mathbb{K}_e \oplus \mathbb{K}_e[-1]$. Lemma is proven.

4.3. **Proof of Proposition 4.8.** . In this subsection we fix char $\mathbb{K} = 2$.

We have standard embeddings

$$\text{SO}(2) \subset \text{SO}(3) \subset \cdots \subset \cdots \text{SO}(N) \subset \text{SU}(N)$$

where the embedding $\text{SO}(k) \subset \text{SO}(N)$ is induced by the embedding $\mathbb{R}^k \hookrightarrow \mathbb{R}^N$; $(x_1, x_2, \ldots, x_k) \mapsto (x_1, x_2, \ldots, x_k, 0, \ldots, 0)$.

We will prove the following statement.

Lemma 4.15. *The sheaf $\mathbb{K}_{SO(k)} \in D(SU(N))$ is C-isomorphic to $\mathbb{K}_{SO(k-1)} \oplus \mathbb{K}_{SO(k-1)}[1-k]$, for all $k \geq 2$.*

It is clear that this Lemma implies the Proposition. Let us now prove Lemma

4.3.1. We have an embedding $\text{SO}(k) \subset \text{SU}(k) \subset \text{SU}(N)$ and, in the same way as in the proof of Lemma 4.14, it suffices to prove that $\mathbb{K}_{\text{SO}(k)}$ is C-isomorphic to $\mathbb{K}_{\text{SO}(k-1)} \oplus \mathbb{K}_{\text{SO}(k-1)}[1-k]$ in $D(\text{SU}(k))$.

4.3.2. Let $M := \mathrm{SU}(k)/\mathrm{SO}(k-1)$, let $\Pi : \mathrm{SU}(k) \to M$ be the canonical projection.

For any smooth manifold Y Let $C(Y) \subset D(Y)$ be the full subcategory formed by finite extensions of objects of the form $Rp_!X$ where $p : Q \to Y$ is a continuos map, $Q = [0,1]^M$, $M \geq 0$, $X \in D(Q)$; $R\Gamma(Q, X) = 0$.

Lemma 4.16. *If $F \in C(M)$, then $\Pi^{-1}F \in C(SU(k))$.*

Proof. Let $p : Q \to M$ be a continuous map. Π is a locally trivial fibration with fiber $\mathrm{SO}(k-1)$, let $\Pi_Q : \mathrm{SU}(k) \times_M Q \to Q$ be the pull-back of this fibration with respect to the map $p : Q \to M$. The fibration Π_Q is trivial, hence we have a homeomorphism

$$\mathrm{SO}(k-1) \times Q \cong \mathrm{SU}(k) \times_M Q.$$

We then have natural maps

$$\pi : \mathrm{SO}(k-1) \times Q \cong \mathrm{SU}(k) \times_M Q \to \mathrm{SU}(k);$$

Let $q' : \mathrm{SO}(k-1) \times Q \to \mathrm{SO}(k-1)$, $q : \mathrm{SO}(k-1) \times Q \to Q$, be projections. Let $X \in D(Q)$, $R\Gamma(Q, X) = 0$. We then have $\Pi^{-1}Rp_!X = R\pi_!q^{-1}X$.

Let us cover

$$\mathrm{SO}(k-1) = \bigcup_{i=1}^{n} Q_i,$$

where each $Q_i \subset \mathrm{SO}(k-1)$ is a closed subset homeomorphic to a cube. One then can represent the sheaf $\mathbb{K}_{\mathrm{SU}(k-1)}$ (actually any object of $D(\mathrm{SU}(k-1))$ as a finite extension formed by objects $Y_i \in D(\mathrm{SU}(k-1))$ such that each Y_i is supported on Q_{l_i} for some l_i. Let $Z_i \in D(Q_{l_i})$, $Z_i = Y_i|_{Q_{l_i}}$. The object $q^{-1}X$ is then a finite extension of objects of the form

$$q^{-1}X \otimes (q')^{-1}Y_i$$

Let $\pi_i : Q_{l_i} \times Q \to \mathrm{SO}(k-1) \times Q \to \mathrm{SU}(k)$ be the through map. Let $q_i : Q_{l_i} \times Q \to Q$, $p_i : Q_{l_i} \times Q \to Q_{l_i}$ be projections.

We then have $R\pi_!q^{-1}X$ is a finite extension formed by objects

$$R\pi_!(q^{-1}X \otimes (q')^{-1}Y_i) \cong R\pi_{i!}(q_i^{-1}X \otimes p_i^{-1}Z_i) \in C(\mathrm{SU}(k)).$$

Therefore, $\Pi^{-1}R\pi_!X \in C(\mathrm{SU}(k))$, whence the statement. □

4.3.3. We have an identification $\mathrm{SO}(k)/\mathrm{SO}(k-1) = S^k$. We have the natural map $S^k = \mathrm{SO}(k)/\mathrm{SO}(k-1) \to \mathrm{SU}(k)/\mathrm{SO}(k-1) = M$.

This map is an embedding; denote the image of this embedding $S \subset M$. Let $\overline{e} \in S^{k-1}$ be the image of the unit of $\mathrm{SO}(k)$. Fix the standard basis $(e^1, e^2, \ldots, e^k)$ in $\mathbb{R}^k$. Then S^k gets identified with the unit sphere in $\mathbb{R}^k$ and $\overline{e} = e^k$. The point $\overline{e}$ determines a point on S, to be also denoted by $\overline{e}$.

Lemma 4.16 implies that Lemma 4.15 follows from the following statement:

Lemma 4.17. *The object $\mathbb{K}_S$ is $C(M)$-equivalent to $\mathbb{K}_{\overline{e}} \oplus \mathbb{K}_{\overline{e}}[1-k]$.*

Proof. As was explained above, S is identified with the unit sphere in $\mathbb{R}^k$. Let $V \subset \mathbb{R}^k$ be an orthogonal complement to e_k. Let us denote $e := e_k$ and $\varepsilon = -e$. Let $B \subset V$ be the ball of radius π. We have a surjective map $P : B \to S$: let $f = \phi n \in B$, where $0 \leq \phi \leq \pi$ and $n \in B$. Set $P(\phi n) = \cos(\phi)e + \sin(\phi)n$. It follows that P is 1-to 1 on the interior of B and that P takes the boundary of B to the point $\varepsilon \in S$. Let $c : B \xrightarrow{P} S \to M$ be the through map Let $\partial B \subset B$ be the boundary. We have a commutative diagram

(22)
$$\begin{array}{ccc} B & \xrightarrow{c} & M \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle \iota} \\ \partial B & \xrightarrow{p} & \varepsilon \end{array}$$

One has

(23)
$$\mathbb{K}_S \cong \mathrm{Cone}(Rc_!\mathbb{K}_B \oplus \iota_!\mathbb{K}_\varepsilon \xrightarrow{f_0} \iota_! Rp_!\mathbb{K}_{\partial B}),$$

where $f_0 = \alpha \oplus \beta$; the map $\alpha : Rc_!\mathbb{K}_B \to \iota_! Rp_!\mathbb{K}_{\partial B} = Rc_! i_*\mathbb{K}_{\partial B}$ is induced by the canonical map

$$\mathbb{K}_B \to i_*\mathbb{K}_{\partial B},$$

and the map

$$\beta : \iota_!\mathbb{K}_e \to \iota_! Rp_!\mathbb{K}_{\partial B}$$

is induced by the canonical map

$$\mathbb{K}_\varepsilon \to Rp_*\mathbb{K}_{\partial B} = Rp_!\mathbb{K}_{\partial B}.$$

Let $M : B \to \mathrm{SO}(k)$ as follows:

—$M(\phi n)$ is identity on any vector which is orthogonal to both n and e;

—$M(\phi n)e = \cos(\phi)e + \sin(\phi)n$;

—$M(\phi n)n = -\sin(\phi)e + \cos(\phi)n$.

One then sees that the composition

$$B \xrightarrow{M} \mathrm{SO}(k) \xrightarrow{\Pi} S$$

equals $P : B \to S$. Thus, $P = \Pi M$. One can also rewrite:

$$M(\phi n) = I + (e^{i\phi} - 1)\mathbf{pr}_{(e+in)/\sqrt{2}} + (e^{-i\phi} - 1)\mathbf{pr}_{(e-in)/\sqrt{2}},$$

where **pr** is the orthognal projector.

For $0 \le \alpha \le \pi/4$, set

$$\mu(\alpha, \phi n) = I + (e^{i\phi} - 1)P_{(\cos\alpha e + i\sin\alpha n)} + (e^{-i\phi} - 1)P_{(\sin\alpha e - \cos\alpha i n)}$$

One sees that:

$\mu : [0, \pi/4] \times B \to \mathrm{SU}(k)$;

$$\mu(\alpha, 0) = I;$$
$$\mu(\alpha, \pi n) \in \mathrm{SO}(k);$$
$$\mu(\pi/4, \phi n) = M(\phi n);$$
$$\mu(\alpha, \pi n)e = -e.$$

Let $\nu : [0; \pi/4] \times B \xrightarrow{\mu} \mathrm{SU}(k) \to M$ be the through map. It then follows that $\nu(\alpha, \pi n) = \varepsilon$.

We have a commutative diagram

(24)
$$\begin{array}{ccccc} B & \xrightarrow{i} & [0; \pi/4] \times B & \xrightarrow{\nu} & M \\ {\scriptstyle k_0}\uparrow & & {\scriptstyle k_1}\uparrow & & \uparrow{\scriptstyle \iota} \\ \partial B & \xrightarrow{i_0} & [0, \pi/4] \times \partial B & \xrightarrow{\pi} & \varepsilon \end{array}$$

Here $i(b) = (\pi/4, b)$ for all $b \in B$; $i_0(b) = (\pi/4, b)$ for all $b \in \partial B$.

We have $c = \nu i$; $\pi i_0 = p$ (where p is as in diagram (22)).

In a way similar to above we can construct a map

$$f : R\nu_! \mathbb{K}_{[0;\pi/4]\times B} \oplus i_! \mathbb{K}_\varepsilon \to \iota_! R\pi_! \mathbb{K}_{[0;\pi/4]\times\partial B}$$

The diagram (24) gives rise to a commutative diagram in $D(M)$:

(25)
$$\begin{tikzcd} Rc_!\mathbb{K}_B \oplus \iota_!\mathbb{K}_\varepsilon \arrow[r, "f_0"] & \iota_! Rp_! \mathbb{K}_{\partial B} \\ R\nu_!\mathbb{K}_{[0;\pi/4]\times B} \oplus \iota_!\mathbb{K}_\varepsilon \arrow[u, "a"] \arrow[r, "f"] & \iota_! R\pi_! \mathbb{K}_{[0;\pi/4]\times\partial B} \arrow[u] \end{tikzcd}$$

in which the right vertical arrow is an isomorhism; the left vertical arrow is a direct sum of the identity arrow $\iota_!\mathbb{K}_e$ and the natural arrow

$$a : R\nu_!\mathbb{K}_{[0;\pi/4]\times B} \to R\nu_! Ri_! \mathbb{K}_B = Rc_!\mathbb{K}_B.$$

This diagram defines uniquely a map $A : \mathrm{Cone}(f) \to \mathrm{Cone}(f_0)$ (because the rightmost arrow in diagram (25) is an isomorphism) the cone of this map is isomorphic to the cone of the map a. It easily follows that $\mathrm{Cone}(a) \in C(M)$, therefore, A is a $C(M)$-isomorphism.

Consider now the diagram (25) where all ingredients are the same except that the map $i : B \to [0;\pi/4] \times B$ gets replaced with the map $i_1 : B \to [0;\pi/4] \times B$, where $i_1(b) = (0, b)$. Let us compute $c_1 := \nu i_1 : B \to M$. We have

(26)
$$\mu(0, \phi n) = I + (1 - e^{i\phi})\mathbf{pr}_e + (1 - e^{-i\phi})\mathbf{pr}_n;$$

(27)
$$c_1(\phi n) = P\mu(0, \phi n).$$

We then have a commutative diagram obtained from diagram (22) by replacement c with c_1. Hence we have a map

(28)
$$\mathbb{K}_S \cong \mathrm{Cone}(Rc_{1!}\mathbb{K}_B \oplus \iota_!\mathbb{K}_e \stackrel{f_1}{\to} \iota_! Rp_! \mathbb{K}_{\partial B}),$$

constructed in the same way as the map f_0 in (23).

In the same way as above one can show that $\mathrm{Cone}(f_1)$ is $C(M)$-isomorphic to $\mathrm{Cone}(f)$, hence to $\mathrm{Cone}(f_0)$, hence to $\mathbb{K}_S$.

Let us now work with $\mathrm{Cone}(f_1)$.

1) Eq. (26) and (27) imply that $c_1(rn) = c_1(-rn)$ for any $rn \in B$. Let $B/2$ be the quotient of B in which $b \in B$ gets identified with $-b$. Let $\delta : B \to B/2$ be the projection. We then have a unique map $c_2 : B/2 \to M$ such that $c_1 = \delta c_2$. Let $\partial B/2$ is the image of ∂B in $B/2$. Of course, $\partial B \cong S^{k-2}$ and $\partial B/2 \cong \mathbb{RP}^{k-2}$. We have a natural quotient map $\delta_1 : \partial B \to \partial B/2$. These maps fit into the following commutative diagram:

$$\begin{tikzcd} B \arrow[r, "\delta"] & B/2 \arrow[r, "c_2"] & M \\ \partial B \arrow[u, "i"] \arrow[r, "\delta_1"] & \partial B/2 \arrow[u, "i_1"] \arrow[r, "p_1"] & \varepsilon \arrow[u, "\iota"] \end{tikzcd}$$

One then can construct an arrow

$$f_2 : Rc_{2!}\mathbb{K}_{B/2} \oplus \mathbb{K}_\varepsilon \to \iota_!(p_1\delta_1)_!\mathbb{K}_{\partial B}$$

in the same way as above. Similar to above, there exists a natural map

$$\mathrm{Cone}(f_2) \to \mathrm{Cone}(f_1)$$

whose cone is isomorphic to the cone of the natural map

$$Rc_{2!}\mathbb{K}_{B/2} \to Rc_{2!}R\delta_!\mathbb{K}_B. \tag{29}$$

Let us show that *the cone of this map is in $C(M)$.*

Indeed, choose a covering $\partial B = \bigcup_{k=1}^m C_k$ where C_k, and all non-empty intresections of these sets are closed sets homeomorphic to the closed disk of the same dimension as dimension of ∂B and $C_k \cap -C_k = \emptyset$.

Consider the set of all multiple non-empty intersections of the sets C_k and denote elements of this set by $C'_1, C'_2, \dots, C'_M$. Each of these sets is homeomorphic to a closed disk of the same dimension as dimension of ∂B and for each i, $C'_i \cap -C'_i = \emptyset$.

Let $B_k \subset B$ be the cones of C'_k:

$$B_k = \{rn | 0 \le r \le \pi; n \in C'_k\}.$$

It is clear that B_k cover B and that $B_k \cap -B_k = \{0\}$.

Let $B_k/2$ be the images of B_k in $B/2$. The map $\delta|_{B_k} : B_k \to B_k/2$ is a homeomorophism. It follows that $\mathbb{K}_{B/2}$ is a finite extension of objects, each of them being of the form $\mathbb{K}_{B_k/2}$. It then suffices to show that the cone of the natural map

$$c_{2!}\delta_!\mathbb{K}_{B_k \cup -B_k} = c_{2!}\delta_!\delta^{-1}\mathbb{K}_{B_k/2} \to c_{2!}\mathbb{K}_{B_k/2} \in C(M)$$

We have

$$\begin{aligned} \delta_!\mathbb{K}_{B_k\cup -B_k} &= \delta_!(\mathrm{Cone}(\mathbb{K}_{B_k} \oplus \mathbb{K}_{B_k} \to \mathbb{K}_0)) \\ &= \mathrm{Cone}(\mathbb{K}_{B_k/2} \oplus \mathbb{K}_{B_k/2} \to \mathbb{K}_0) \end{aligned}$$

The natural map $\delta_!\mathbb{K}_{B_k \cup -B_k} \to \mathbb{K}_{B_k/2}$ is given by the natural map

$$\mathrm{Cone}(\mathbb{K}_{B_k/2} \oplus \mathbb{K}_{B_k/2} \to \mathbb{K}_0) \to \mathbb{K}_{B_k/2} \tag{30}$$

induced by

$$\mathrm{Id} \oplus \mathrm{Id} : \mathbb{K}_{B_k/2} \oplus \mathbb{K}_{B_k/2} \to \mathbb{K}_{B_k/2}$$

Therefore, the cone of the map (30) is isomorphic to the cone of the natural map

$$\mathbb{K}_{B_k/2} \to \mathbb{K}_0$$

Denote this cone by F' and let $F := F'|_{B_k/2}$. It follows that $R\Gamma(B_k/2, F) = 0$. Let $P : B_k/2 \to B/2 \to M$ be the trough map. Our task is now reduced to showing that $RP_!F \in C(M)$. This follows from the fact that $B_k/2$ is homeomorphic to a unit cube.

Thus, the cone of the map (29) is in $C(M)$, therefore $\mathrm{Cone}(f_1)$ and $\mathrm{Cone}(f_2)$ are $C(M)$ isomorphic.

Let us now study $\mathrm{Cone}(f_2)$. The map f_2 is a direct sum of two maps: one of them is the natural map $g : Rc_{2!}\mathbb{K}_{B/2} \to \iota_!(p_1\delta_1)_!\mathbb{K}_{\partial B} = Rc_{2!}i_{1!}\delta_{1!}\mathbb{K}_{\partial B}$ and the other is the natural map

$$h : \mathbb{K}_\varepsilon \to \iota_!(p_1\delta_1)_!\mathbb{K}_{\partial B} \tag{31}$$

The map g factors as

$$Rc_{2!}\mathbb{K}_{B/2} \xrightarrow{g_1} Rc_{2!}i_{1!}\mathbb{K}_{\partial B/2} \xrightarrow{l} Rc_{2!}i_{1!}\delta_{1!}\mathbb{K}_{\partial B} \tag{32}$$

We have $Rc_{2!}\mathbb{K}_{\partial B/2} = \iota_! Rp_{1!}\mathbb{K}_{\partial B/2} \cong H^*(\partial B/2, \mathbb{K}) \otimes_{\mathbb{K}} \iota_!\mathbb{K}_\varepsilon$;

$$Rc_{2!}i_{1!}\delta_!\mathbb{K}_{\partial B} = \iota_! Rp_!\mathbb{K}_{\partial B} = H^*(\partial B; \mathbb{K}) \otimes_{\mathbb{K}} \iota_!\mathbb{K}_\varepsilon.$$

The map l in (32) is induced by the map

$$\delta_1^* : H^*(\partial B/2; \mathbb{K}) \to H^*(\partial B; \mathbb{K}).$$

Recall that $\partial B \cong S^{k-2}$; $\partial B/2 \cong \mathbb{RP}^{k-2}$ and δ_1 is the quotient map. As char $\mathbb{K} = 2$, it follows that the map δ_1^* factors as

$$H^*(\partial B/2; \mathbb{K}) \xrightarrow{n_1} \mathbb{K} \xrightarrow{n_2} H^*(\partial B; \mathbb{K}),$$

where the arrow n_1 is induced by any embedding $\mathrm{pt} \to \partial B/2$ and the arrow n_2 is induced by the projection $\partial B \to \mathrm{pt}$. This means that $l = l_2 l_1$, where

$$l_1 : Rc_{2!}\mathbb{K}_{\partial B/2} \to \iota_!\mathbb{K}_\varepsilon$$

is induced by n_1, and

$$l_2 : \mathbb{K}_\varepsilon \to Rc_{2!}i_{1!}\delta_{1!}\mathbb{K}_{\partial B}$$

is induced by n_2. Let us now consider the map h in (31). As was explained above, $\iota_!(p_1\delta_1)_!\mathbb{K}_{\partial B} \cong H^*(\partial B; \mathbb{K}) \otimes_{\mathbb{K}} \iota_!\mathbb{K}_\varepsilon$ and the map h is induced by the map $\mathbb{K} \to H^*(\partial B; \mathbb{K})$ induced by the projection $\partial B \to \mathbf{pt}$. That is $h = l_2$

These observations show that the map $g = l_2 l_1 g_1 = h l_1 g_1$ factors through h. This implies that

$$\mathrm{Cone}(f_2) = \mathrm{Cone}(g \oplus h) = \mathrm{Cone}(0 \oplus h) = Rc_{2!}\mathbb{K}_{B/2} \oplus \mathrm{Cone}(h) = Rc_{2!}\mathbb{K}_{B/2} \oplus \iota_!\mathbb{K}_\varepsilon[1-k]$$

As was explained above, $Rc_{2!}\mathbb{K}_{B/2}$ is C-isomorphic to $Rc_!\mathbb{K}_B$. Let $x \in \partial B$. We then have natural C-isomorphisms

$$Rc_!\mathbb{K}_B \to Rc_!\mathbb{K}_0 = \mathbb{K}_e$$

and

$$Rc_!\mathbb{K}_B \to Rc_!\mathbb{K}_{0'} = \mathbb{K}_\varepsilon$$

hence, $Rc_{2!}\mathbb{K}_{B/2}$ is C-isomorphic with both $\mathbb{K}_e$ and $\mathbb{K}_\varepsilon$, as well as with $Rc_{2!}\mathbb{K}_{B/2}$.

Thus,

$$Rc_{2!}\mathbb{K}_{B/2} \oplus \iota_!\mathbb{K}_\varepsilon[1-k]$$

is C-isomorphic with $\mathbb{K}_e \oplus \mathbb{K}_e[1-k]$, hence so is $\mathrm{Cone}(f_2)$. This proves Lemma □

5. Proof of Proposition 4.4: constructing $u_{\mathcal{O}}$

The rest of this paper will be devoted to proving Proposition 4.4. In this section we will construct the object $u_{\mathcal{O}}$. In the subsequent sections we will check it satisfies all the required properties.

5.1. **Constructing $u_{\mathcal{O}}$.** Our construction is based on a certain object $\mathfrak{S} \in D(G \times \mathfrak{h})$. This object is introduced and studied in the subsequent Sec. 6 It is defined as any object satisfying the conditions in Theorem 6.1.

5.1.1. *Convolution on* $\mathfrak{h}$. Let X, Y are manifolds. Let $a : X \times \mathfrak{h} \times Y \times \mathfrak{h} \to X \times Y \times \mathfrak{h}$ be given by $a(x, A_1, y, A_2) = (x, y, A_1 + A_2)$. Let $F \in D(X \times \mathfrak{h})$ and $G \times D(Y \times \mathfrak{h})$. Set $F *_{\mathfrak{h}} G := Ra_!(F \times G)$.

5.1.2. Let $L := \mathcal{O} \cap C_+$. We have $L = \lambda e_1$, where $\lambda > 0$.

Let $\gamma_L \in D(\mathfrak{h} \times \mathbb{R})$ be given by $\gamma_L = \mathbb{K}_{\{(A,t)|t+<A,L>\geq 0\}}$. Let $I_0 : G \times \mathbb{R} \to G \times \mathfrak{h} \times \mathbb{R}$ be given by $I_0(g,t) = (g, 0, t)$. Set

$$u_{\mathcal{O}} = I_0^{-1}(\mathfrak{S} *_{\mathfrak{h}} \gamma_L).$$

Let us first of all prove that $u_{\mathcal{O}} \in \mathcal{D}_{IP^{-1}\Delta}(G)$. Using proposition 2.2 it is easy to show that $u_{\mathcal{O}}$ is in the left orthogonal complement to $C_{\leq 0}(G)$. Let us now estimate $\mathrm{SS}(u_{\mathcal{O}})$.

Let $p_3 : G \times \mathfrak{h} \times \mathbb{R} \to G \times \mathfrak{h}$; $p_1 : G \times \mathfrak{h} \times \mathbb{R} \to \mathfrak{h} \times \mathbb{R}$; $p_2 : G \times \mathfrak{h} \times \mathbb{R} \to G \times \mathbb{R}$ be the projections.

One can show that

$$u_{\mathcal{O}} = Rp_{2!}(p_1^{-1}\mathbb{K}_{\{(A,t)|t\geq<A,L>\}} \otimes p_3^{-1}\mathfrak{S})$$

As usual let us identify

$$T^*(G \times \mathfrak{h} \times \mathbb{R}) = G \times \mathfrak{h} \times \mathbb{R} \times \mathfrak{g}^* \times \mathfrak{h}^* \times \mathbb{R}$$

We see that $p_1^{-1}\mathbb{K}_{\{(A,t)|t\geq<A,L>\}}$ is microsupported on the set

$$\Omega_1 := \{(g, A, t, 0, -kL; k)|k \geq 0\}.$$

The object $p_2^{-1}\mathfrak{S}$ is microsupported on the set

$$\Omega_2 := \{(g, A, t, \omega, \eta, 0)\},$$

where $(g, A, \omega, \eta) \in \Omega_{\mathfrak{S}}$ (See Sec. 51 for the definition of$\Omega_{\mathfrak{S}}$).

One sees that if $\zeta_j \in \Omega_j \cap T^*_{(g,A,t)}(G \times \mathfrak{h} \times \mathbb{R})$ and $\zeta_1 + \zeta_2 = 0$, then $k = 0$ and $\zeta_1 = 0$, hence $\zeta_2 = 0$. Therefore, the object

$$\Psi := p_1^{-1}\mathbb{K}_{\{(A,t)|t\geq<A,L>\}} \otimes p_3^{-1}\mathfrak{S}$$

is microsupported on the set

$$\Omega_3 := \{(g, A, t, \omega_1 + \omega_2; \eta_1 + \omega_2; k_1 + k_2)|(g, A, t, \omega_j; \eta_j; k_j) \in \Omega_j\}$$

We have

$$\Omega_3 = \{(g, A, t, \omega; \eta - kL; k)|k \geq 0; (g, A, \omega, \eta) \in \Omega\}$$

Let us now apply Corollary 3.4 to the projection p_2 (so that $E = \mathfrak{h}$).

Let

$$\pi : G \times \mathfrak{h} \times \mathbb{R} \times \mathfrak{g}^* \times \mathfrak{h}^* \times \mathbb{R} \to G \times \mathbb{R} \times \mathfrak{g}^* \times \mathfrak{h}^* \times \mathbb{R}.$$

Let us find $\pi(\Omega_3)$ We see that

$$\pi(\Omega_3) \subset \{(g, t, \omega, \eta - kL; k)|k \geq 0, \mathrm{Ad}_g\omega = \omega; \eta = |\omega|\} =: \Omega_4.$$

The set Ω_4 is closed. Therefore, $\mathrm{SS}(Rp_{2!}\Psi)$ is confined within the set of all points of the form $\{(g, t, \omega, k)|(g, t, \omega, 0, k) \in \Omega_4\}$ Thus $\|\omega\| = kL$,$\mathrm{Ad}_g\omega = \omega$ and $k \geq 0$. If $k = 0$. then $\omega = 0$ and we have $(g, t, 0, 0) \in \mathrm{SS}(Rp_{2!}\Psi)$. If $k > 0$, then set $\omega = k\zeta$. We then have $|\zeta| = L$ (which means that $\zeta \in \mathcal{O}$) and $\mathrm{Ad}_g\zeta = \zeta$. This is the same as to say $(g, \zeta) \in IP^{-1}\mathcal{O}$. This proves the statement

5.2. **Proof of Proposition 4.4 1).**

5.2.1. *The map* $\tau_c : u_{\mathcal{O}} \to T_{c*}u_{\mathcal{O}}$. We will rewrite this map in a way more convenient to us.

Let $c > 0$. We then have an obvious map $\tau_c^\gamma : \gamma_L \to T_{c*}\gamma_L$;

$$T_{c*}u_{\mathcal{O}} = I_0^{-1}(\mathfrak{S} *_{\mathfrak{h}} T_{c*}\gamma_L)$$

The natural map $\tau_c : u_{\mathcal{O}} \to T_{c*}u_{\mathcal{O}}$ (coming from the fact that $u_{\mathcal{O}} \in \mathcal{D}(G)$), in terms of the above identifications, is given by the map

$$I_0^{-1}(\mathfrak{S} *_{\mathfrak{h}} \gamma_L) \to I_0^{-1}(\mathfrak{S} *_{\mathfrak{h}} T_{c*}\gamma_L)$$

which is induced by the map τ_L^γ.

Let $A_1, A_2 \in \mathfrak{h}$. For $A \in \mathfrak{h}$ set $U_A = \{A_1 \in \mathfrak{h} | A_1 << A\}$. Set $V_A := \mathbb{K}_{U_A}[\dim \mathfrak{h}]$. We then have a natural map

$$e_A : \mathbb{K}_A \to V_A. \tag{33}$$

It is defined as follows. Let us identify $\mathbb{R}^{N-1} = \mathfrak{h}$, where

$$(x_1, x_2, \ldots, x_{N-1}) \mapsto \sum x_k f_k.$$

Let $A = \sum t_k f_k$ Upon this identification, $U_A = \{(x_1, x_2, \ldots, x_{N-1}) | x_k < t_k\}$ and $V_A = \boxtimes_k(\mathbb{K}_{(-\infty,t_k)}[1])$; $\mathbb{K}_A = \boxtimes_k \mathbb{K}_{t_k}$. The map e_A is defined as a product of maps $\varepsilon_k : \mathbb{K}_{t_k} \to \mathbb{K}_{(-\infty;t_k)}[1]$ which represents the class of the extension

$$0 \to \mathbb{K}_{(-\infty;t_k)} \to \mathbb{K}_{(-\infty,t_k]} \to \mathbb{K}_{t_k} \to 0.$$

Let $A \in C_+$, we then have $c =< A, L >\geq 0$ because $A, L \in C_+$.

Lemma 5.1. *Let $A \in \mathfrak{h}$ be such that $< A, L >= c$*

The natural map

$$\mathbb{K}_A *_{\mathfrak{h}} \gamma_L \stackrel{e_A}{\to} V_A *_{\mathfrak{h}} \gamma_L$$

is an isomorphism.

Proof. Clear ☐

Let now $A \in C_+$. Since $A, L \in C_+$, it follows that $c =< A, L >\geq 0$. We also have a natural isomorphism

$$\mathbb{K}_A *_{\mathfrak{h}} \gamma_L \cong T_{c*}\gamma_L.$$

Let us combine this isomorphism with that of the Lemma, we will get an isomorhpism

$$V_A * \gamma_L \cong T_{c*}\gamma_L$$

By substituting $A = 0$, we get an isomorphism

$$V_0 * \gamma_L \cong \gamma_L.$$

Upon these identifications, the map τ_c^γ corresponds to a map

$$\tau_A^V : V_0 \to V_A$$

induced by the inclusion $U_0 \subset U_A$.

Thus, the map $\tau_c : u_{\mathcal{O}} \to T_c u_{\mathcal{O}}$ is isomorphic to the map

$$I_0^{-1}(\mathfrak{S} *_{\mathfrak{h}} V_0 *_{\mathfrak{h}} \gamma_L) \to I_0^{-1}(\mathfrak{S} *_{\mathfrak{h}} V_A *_{\mathfrak{h}} \gamma_L)$$

induced by the natural map $\tau_A^V : V_0 \to V_A$. As $\mathfrak{h}$ is an abelian Lie group, we can rewrite the above map as

$$I_0^{-1}(V_0 *_{\mathfrak{h}} \mathfrak{S} *_{\mathfrak{h}} \gamma_L) \to I_0^{-1}(V_A *_{\mathfrak{h}} \mathfrak{S} *_{\mathfrak{h}} \gamma_L). \tag{34}$$

5.2.2. Let $*_{G\times\mathfrak{h}}$ denote the convolution on $D(G\times\mathfrak{h})$.

Taking into account the expression (34) for τ_c, the Proposition 4.4 1) can be deduced from the following Proposition:

Proposition 5.2. *Let U and $F\in D(G)$ be as in Proposition 4.4 1). Then there exists $A\in C_+$ such that the natural map*

$$(F\boxtimes V_0)*_{G\times\mathfrak{h}}\mathfrak{S}\to(F\boxtimes V_A)*_{G\times\mathfrak{h}}\mathfrak{S} \tag{35}$$

induced by the map $\tau^V_A:V_0\to V_A$, is zero in $D(G\times\mathfrak{h})$

Thus, Proposition 4.4 1) is now reduced to Proposition 5.2

5.3. **Proof of Proposition 5.2.** Let H be any sheaf on $\mathfrak{h}$. Let $\alpha:\mathfrak{h}\to\mathfrak{h}$ be the antipode map. We then have $H*_{\mathfrak{h}}\mathfrak{S}=Rp_{2!}(p_1^{-1}\alpha_*H\otimes a^{-1}\mathfrak{S})$, where as usual $p_1:G\times\mathfrak{h}\times\mathfrak{h}\to\mathfrak{h}$ is given by

$$p_1(g,A_1,A_2)=A_1;$$

and $p_2:G\times\mathfrak{h}\times\mathfrak{h}\to G\times\mathfrak{h}$ is given by $p_2(g,A_1,A_2)=(g,A_2)$. Set $H^\alpha:=\alpha_*H$. We then have

$$Rp_{2!}(p_1^{-1}H^\alpha\otimes a^{-1}\mathfrak{S})=Rp_{2!}((p_1^{-1}H^\alpha)\otimes(\mathfrak{S}*_G\mathfrak{S})),$$

where we have used the isomorphism (64). Next,

$$Rp_{2!}((p_1^{-1}H^\alpha)\otimes(\mathfrak{S}*_G\mathfrak{S}))\cong[Rp_!(\pi^{-1}H^\alpha\otimes\mathfrak{S})]*_G\mathfrak{S},$$

where $\pi:G\times\mathfrak{h}\to\mathfrak{h}$; $p:G\times\mathfrak{h}\to G$ are projections.

One then has

$$Rp_!(\pi^{-1}H^\alpha\otimes\mathfrak{S})=I_0^{-1}(H*_{\mathfrak{h}}\mathfrak{S}).$$

Let $S_A:=I_0^{-1}(V_A*_{\mathfrak{h}}\mathfrak{S})$. We then have a natural map $\tau^S_A:S_0\to S_A$.

We have $V_A*_{\mathfrak{h}}\mathfrak{S}\cong S_A*_G\mathfrak{S}$ and

$$(F\boxtimes V_A)*_{G\times\mathfrak{h}}\mathfrak{S}\cong F*_G(S_A*_G\mathfrak{S})=(F*_GS_A)*_G\mathfrak{S}$$

The map 35 is then induced by the map τ^S_A.

Thus, Proposition 5.2 is now reduced to

Proposition 5.3. *There exist: a neighborhood $U\subset G$ of the unit $e\in G$ and $A\in C_+$ such that the natural map*

$$F*_GS_0\to F*_GS_A$$

induced by τ^S_A is zero for any $F\in D(G)$ which is supported on gU for some $g\in G$ and satisfies $R\Gamma(G,F)=0$.

Proof. We have a natural map $\mathbb{K}_A\to V_A$, as in (33). Hence, we have an induced map

$$I_0^{-1}(\mathbb{K}_A*_{\mathfrak{h}}\mathfrak{S})\to I_0^{-1}(V_A*_{\mathfrak{h}}\mathfrak{S})=:S_A. \tag{36}$$

One sees that *this map is actually an isomorphism.* Indeed, one can easily show that for any object $F\in D(G\times\mathfrak{h})$ such that $\mathrm{SS}(F)\subset T^*G\times\mathfrak{h}\times C_+\subset T^*G\times T^*\mathfrak{h}$, the map

$$I_0^{-1}(\mathbb{K}_A*_{\mathfrak{h}}F)\to I_0^{-1}(V_A*_{\mathfrak{h}}F)$$

induced by the map (33), is an isomorphism, and $\mathfrak{S}$ is of this type by virtue of Theorem 6.1.

One also sees that $I_0^{-1}(\mathbb{K}_A *_{\mathfrak{h}} \mathfrak{S}) = I_{-A}^{-1}\mathfrak{S}$, where $I_{-A} : G \to G \times \mathfrak{h}$; $I_{-A}g = (G, -A)$. Taking into account (36), we obtain an isomorphism

$$S_A \cong I_{-A}^{-1}\mathfrak{S}.$$

Let us choose a small A, $A >> 0$.

As was shown in the course of proving Theorem 6.1, for $0 << A << b$ we have

$$S_A = I_{-A}^{-1}\mathfrak{S} \cong \mathbb{K}_{\mathcal{U}_A}.$$

where $\mathcal{U}_A = \{e^X | \|X\| << A\} \subset G$. We also know that $S_0 = \mathbb{K}_e$.

Without loss of generality one can assume that for some $A \in C_+$; $A << b$, $U \subset \mathcal{U}_{A/10}$. Let $h \in U$ so that $h = e^X$, where $\|X\| << A/10$. We have $(F *_G S_A)|_{gh} = R\Gamma_c(\{ghr^{-1}|r \in \mathcal{U}_A\}; F)[\dim G]$. It follows that $gU \subset \{ghr^{-1}|r \in \mathcal{U}_A\}$ (Indeed, let $gh' \in gU$ so that $h' = e^{X'}, \|X'\| << A/10$. We have $h' = hr^{-1}$, $r = (h')^{-1}h$. By Lemma 10.4, $r = e^Z$, where $\|Z\| \leq \| - X'\| + \|X\| << A$. So $r \in \mathcal{U}_A$). Therefore, $(F *_G S_A)|_{gh} = R\Gamma(gU, F)[\dim G] = 0$. Thus, $F *_G S_A$ is supported away from gU. But $F *_G S_0 = F$ is supported on gU. Therefore, $R\hom(F *_G S_0; F *_G S_A) = 0$ which proves the statement. $\square$

Thus, we have proven Proposition 4.4 1). The rest of the paper is devoted to proving the second part of the Proposition.

5.4. Recall that we have a sheaf $\gamma_L := \mathbb{K}_{\{(A,t)|t+<A,L>\geq 0\}}$ on $\mathfrak{h} \times \mathbb{R}$. Let $\iota : \mathbb{R} \to \mathfrak{h} \times \mathbb{R}$ be given by $\iota(t) = (0, t)$. We have a natural isomorphism

$$\mathbb{K}_{[0,\infty]}[-\dim \mathfrak{h}] = \iota^! \gamma_L$$

hence a natural map

$$\mathbb{K}_{0\times[0,\infty)}[-\dim \mathfrak{h}] \to \gamma_L. \tag{37}$$

This map induces a map

$$I_0^{-1}(\mathfrak{S} *_{\mathfrak{h}} \mathbb{K}_{0\times[0,\infty)})[-\dim \mathfrak{h}] \to \mathbb{K}_{I_0^{-1}(\mathfrak{S}*_{\mathfrak{h}}\gamma_L)} = u_{\mathcal{O}} \tag{38}$$

where $I_0 : G \times \mathbb{R} \to G \times \mathfrak{h} \times \mathbb{R}$, $I_0(g, t) = (g, 0, t)$ Next, one has

$$I_0^{-1}(\mathfrak{S} *_{\mathfrak{h}} \mathbb{K}_{0\times[0,\infty)}) = i_0^{-1}\mathfrak{S} \boxtimes \mathbb{K}_{[0,\infty)}$$

where $i_0 : G \to G \times \mathfrak{h}$, $i_0(g) = (g, 0)$. We know that $i_0^{-1}\mathfrak{S} = \mathbb{K}_e$, thus we have an isomoprhism

$$I_0^{-1}(\mathfrak{S} *_{\mathfrak{h}} \mathbb{K}_{0\times[0,\infty)}) = \mathbb{K}_{e\times[0,\infty)}$$

The map (38) then can be rewritten as:

$$\mathbb{K}_{e\times[0,\infty]}[-\dim \mathfrak{h}] \to u_{\mathcal{O}} \tag{39}$$

Proposition 5.4. *Let $\Phi \in \mathcal{D}_{G\times\mathcal{O}}(G)$ The natural map*

$$\hom_{G\times\mathbb{R}}(u_{\mathcal{O}}; \Phi) \to R\hom_{G\times\mathbb{R}}(\mathbb{K}_{(e,0)}[-\dim \mathfrak{h}]; \Phi)$$

induced by the map (39) is an isomorphism.

Proof. We have

$$u_{\mathcal{O}} = Rp_{2!}(p_3^{-1}\mathfrak{S} \otimes p_1^{-1}\mathbb{K}_{\{(A,t)|t\geq (A,L)\}});$$
$$\mathbb{K}_{e\times[0,\infty)} = I_0^{-1}(\mathfrak{S} *_{\mathfrak{h}} \mathbb{K}_{0\times[0,\infty)})$$
$$= Rp_{2!}(p_3^{-1}\mathfrak{S} \otimes p_1^{-1}\mathbb{K}_{0\times[0,\infty)})$$

where $p_1 : G\times\mathfrak{h}\times\mathbb{R} \to \mathfrak{h}\times\mathbb{R}$; $p_2 : G\times\mathfrak{h}\times\mathbb{R} \to G\times\mathbb{R}$; $p_3 : G\times\mathfrak{h}\times\mathbb{R} \to G\times\mathfrak{h}$ are projections.

Let $X \in D(\mathfrak{h}\times\mathbb{R})$. We then have

$$R\hom(Rp_{2!}(p_3^{-1}\mathfrak{S} \otimes p_1^{-1}X);\Phi)$$
$$= R\hom(p_3^{-1}\mathfrak{S} \otimes p_1^{-1}X; p_2^!\Phi)$$
$$= R\hom(p_1^{-1}X; R\underline{\mathrm{Hom}}(p_3^{-1}\mathfrak{S}; p_2^!\Phi)$$

(40)
$$= R\hom_{\mathfrak{h}\times\mathbb{R}}(X; Rp_{1*}R\underline{\mathrm{Hom}}(p_3^{-1}\mathfrak{S}; p_2^!\Phi)).$$

Let us estimate the microsupport of the sheaf

$$Rp_{1*}R\underline{\mathrm{Hom}}(p_3^{-1}\mathfrak{S}; p_2^!\Phi).$$

We know that $\mathrm{SS}(\mathfrak{S}) \subset \{(g, A, \omega, |\omega|) \in G\times\mathfrak{h}\times\mathfrak{g}^*\times\mathfrak{h}^*\}$. Therefore,

$$\mathrm{SS}(p_3^{-1}\mathfrak{S}) \subset \Omega_1 := \{(g, A, t, \omega_1, |\omega_1|, 0)\} \subset G\times\mathfrak{h}\times\mathbb{R}\times\mathfrak{g}^*\times\mathfrak{h}^*\times\mathbb{R}.$$

Analogously,

$$\mathrm{SS}(p_2^!\Phi) \subset \Omega_2 = \{(g, A, t, k\omega, 0, k)|k \geq 0, \omega \in \mathcal{O}\}$$

One sees that if $(g, A, t, \omega_i, \eta_i, k_i) \in \Omega_i$ and $\omega_2 = \omega_1, \eta_2 = \eta_1, k_1 = k_2$, then $0 = k_1 = k_2$, hence $\omega_2 = \omega_1 = 0$; also $0 = \eta_2 = \eta_1$. Therefore,

$$\mathrm{SS}(R\underline{\mathrm{Hom}}(p_3^{-1}\mathfrak{S}; p_2^!\Phi)) \subset \Omega_3 := \{(g, A, t, \omega_2 - \omega_1; \eta_2 - \eta_1; k_2 - k_1)|(g, A, t, \omega_i, \eta_i, k_i) \in \Omega_i\}$$
$$= \{(g, A, t, k\omega - \omega_1; -|\omega_1|; k)|k \geq 0; \omega \in \mathcal{O}\}$$

As the map p_1 is proper, one has

$$\mathrm{SS}(Rp_{1*}R\underline{\mathrm{Hom}}(p_3^{-1}\mathfrak{S}; p_2^!\Phi)) \subset \Omega_4 := \{(A, t, \eta, k)|\exists g \in G : (g, A, t, 0, \eta, k) \in \Omega_3\}.$$

We see that

$$\Omega_4 = \{(A, t, -k|\omega|, k)\} = \{(A, t, -kL, k)\}.$$

Let $\pi : \mathfrak{h}\times\mathbb{R} \to \mathbb{R}$; $\pi(A, t) = t - < A, L >$. It then follows that $Rp_{1*}R\underline{\mathrm{Hom}}(p_3^{-1}\mathfrak{S}; p_2^!\Phi)$ is locally constant along the fibers of π i.e. there exists a sheaf Γ on $\mathbb{R}$ such that

$$Rp_{1*}R\underline{\mathrm{Hom}}(p_3^{-1}\mathfrak{S}; p_2^!\Phi) = \pi^!\Gamma$$

Taking into account (40) the statement is reduced to showing that the natural map

$$R\hom_{\mathfrak{h}\times\mathbb{R}}(\mathbb{K}_{\{(A,t)|t\geq <A,L>\}}; \pi^!\Gamma) \to R\hom_{\mathfrak{h}\times\mathbb{R}}(\mathbb{K}_{0\times[0,\infty)}; \pi^!\Gamma)$$

is an isomorphism for any sheaf $\Gamma \in D(\mathbb{R})$. This is equivalent to showing that the map

$$R\pi_!\mathbb{K}_{0\times[0,\infty)}[-\dim\mathfrak{h}] \to R\pi_!\mathbb{K}_{\{(A,t)|t\geq <A,L>\}}$$

induced by the map (37) is an isomorphism, which is easy. ∎

It then follows that for all $c \in \mathbb{R}$, we have an isomorphism

$$R\hom(u_{\mathcal{O}}; T_{c*}u_{\mathcal{O}}) \cong R\hom(\mathbb{K}_{e\times[0,\infty)}[-\dim \mathfrak{h}]; T_{c*}u_{\mathcal{O}})$$

Let $i : \mathbb{R} \to G \times \mathbb{R}$; $i(t) = (e, t)$. We then have

$$R\hom(\mathbb{K}_{e\times[0,\infty)}; T_{c*}u_{\mathcal{O}}) = R\hom(\mathbb{K}_{[0,\infty)}; i^! T_{c*}u_{\mathcal{O}}).$$

One sees that the submanifold $i(\mathbb{R}) \subset G\times\mathbb{R}$ is non-characteristic for $T_{c*}u_{\mathcal{O}}$ (because $\mathrm{SS}(T_{c*}u_{\mathcal{O}}) \subset \{(g, t, k\omega, k), k \geq 0; \omega \in \mathcal{O}\}$). Therefore, according to Proposition 11.0.11, we have an isomorphism

$$i^{-1}T_{c*}u_{\mathcal{O}}[-\dim G] \cong i^! T_{c*}u_{\mathcal{O}}.$$

Thus, we have an isomorphism

$$\rho : R\hom(u_{\mathcal{O}}; T_{c*}u_{\mathcal{O}}) \cong R\hom(\mathbb{K}_{[0,\infty)}[-\dim \mathfrak{h}]; i^{-1}T_{c*}u_{\mathcal{O}}[-\dim G])$$

For $c > 0$ the natual maps

$$R\hom(u_{\mathcal{O}}; u_{\mathcal{O}}) \to R\hom(u_{\mathcal{O}}; T_{c*}u_{\mathcal{O}})$$

and

$$R\hom(\mathbb{K}_{[0,\infty)}[-\dim \mathfrak{h}]; i^{-1}u_{\mathcal{O}}[-\dim G]) \to R\hom(\mathbb{K}_{[0,\infty)}[-\dim \mathfrak{h}]; i^{-1}T_{c*}u_{\mathcal{O}}[-\dim G])$$

commute with our isomorphism.

Proposition (4.4) 2) reduces to

Proposition 5.5. *For any $c > 0$, the natural map*

$$R\hom(\mathbb{K}_{[0,\infty)}[-\dim \mathfrak{h}]; i^{-1}u_{\mathcal{O}}[-\dim G]) \to R\hom(\mathbb{K}_{[0,\infty)}[-\dim \mathfrak{h}]; i^{-1}T_c u_{\mathcal{O}}[-\dim G]) \tag{41}$$

is non-zero

5.4.1. Let $I : \mathfrak{h} \to G \times \mathfrak{h}$ be given by $I(A) = (e, A)$. Let $\mathcal{S}_e := I^!\mathfrak{S} = I^{-1}\mathfrak{S}[-\dim G]$. We then have

$$i^{-1}u_{\mathcal{O}} = I_0^{-1}(\mathcal{S}_e *_{\mathfrak{h}} \gamma_L)[\dim G] \tag{42}$$

This equation dictates us to find an explicit expression for $\mathcal{S}_e$. It turns out to be more convenient to work with a slightly different object. Namely, let $\mathbf{Z} \subset G$ be the center of G. Let $I_{\mathbf{Z}} : \mathbf{Z}\times\mathfrak{h} \to G\times\mathfrak{h}$ be the obvious embedding. Set $\mathcal{S} := I_{\mathbf{Z}}^!\mathfrak{S} = I_{\mathbf{Z}}^{-1}\mathfrak{S}[-\dim G]$. We will identify this object up-to an isomorphism.

5.5. **Identifying $\mathcal{S}$.** We will now give an explicit description of the object $\mathcal{S}$ up-to isomorphism. The proof of this result will be given in the subsequent sections of the paper.

5.5.1. *Object $\mathcal{Y}$.* We first define an object $\mathcal{Y} \in D(\mathbf{Z} \times \mathfrak{h})$ as follows. Let $\mathbb{L} \subset \mathfrak{h}$ be the lattice consisting of all $A \in \mathfrak{h}$ such that $e^A \in \mathbf{Z}$.

For a subset $J \subset \{1, 2, \ldots, N-1\}$ and $l \in \mathbb{L}$ let $K(J, l) \subset e^l \times \mathfrak{h} \subset \mathbf{Z} \times h$ be defined as follows:

$$K(J, l) := \{(e^l, x) \in \mathbf{Z} \times \mathfrak{h} | \forall j \in J :< x - l, e_j >\geq 0\}.$$

Let $V(J, l) := \mathbb{K}_{K(J,l)}[D(l)]$, where $D(l)$ is an integer defined in (7.5.5). That is, decompose $l = \sum l_k e_k$, where $e^1, e^2, \ldots, e^n$ is a basis in $\mathfrak{h}$ as in (96). Then $D(l) = \sum l_k D_k$, where $D_k = k(N-k)$ and $N = \dim \mathfrak{h} + 1$. Let $\mathbb{L}_J = \{l \in \mathbb{L} | \forall i \notin J :< l, f_j >\leq 0\}$ Let $\Psi^J := \bigoplus_{l\in\mathbb{L}_J} V(J, l)$.

Let $J_1 \subset J_2$. Construct a map

$$I_{J_1J_2} : \Psi^{J_1} \to \Psi^{J_2}.$$

It is defined as the direct sum of the natural maps

$$V(J_1,l)\to V(J_2,l)$$

for all $l\in\mathbb{L}_{J_1}\subset\mathbb{L}_{J_2}$. These maps come from the closed embeddings $K(J_2,l)\subset K(J_1,l)$.

Let Subsets be the poset (hence the category) of all subsets of $\{1,2,\ldots,N-1\}$. We then see that Ψ is a functor from Subsets to the category of sheaves on $\mathbf{Z}\times\mathfrak{h}$. We then construct the standard complex $\mathcal{Y}^\bullet$ such that

$$\mathcal{Y}^k:=\bigoplus_{I,|I|=k}\Psi^I \tag{43}$$

and the differential $d_k:\mathcal{Y}^k\to\mathcal{Y}^{k+1}$ is given by

$$d_k=\sum(-1)^{\sigma(J_1,J_2)}I_{J_1J_2}, \tag{44}$$

where the sum is taken over all pairs $J_1\subset J_2$ such that $|J_1|=k$ and $|J_2|=k+1$. The set $J_2\backslash J_1$ then consists of a single element e and $\sigma(J_1J_2)$ is defined as the number of elements in J_2 which are less than e.

5.5.2. *Object $\mathcal{S}$.* Let $I\subset\{1,2,\ldots,N-1\}$ be a subset. Denote $e_I:=\sum\limits_{i\in I}e_i\in\mathfrak{h}$. Let also $G(I)$ be a graded vector space as in Lemma 8.2.

For any $l\in\mathfrak{h}$, let $T_l:\mathbf{Z}\times\mathfrak{h}\to\mathbf{Z}\times\mathfrak{h}$ be the shift by l: $T_l(z,A):=(z,A+l)$

Theorem 5.6. *We have an isomoprhism*

$$\mathcal{S}\cong\bigoplus_I G_I[D(-2\pi e_I)]\otimes T_{-2\pi e_I*}\mathcal{Y}, \tag{45}$$

Proof of this theorem is obtained as a result of a study of the obect $\mathcal{S}$ in Sec. 6- 9.

Given this description of $\mathcal{S}$, we can now compute $i^{-1}u_{\mathcal{O}}$.

5.6. **Computing $i^{-1}u_{\mathcal{O}}$.** Let $\mathcal{O}$ be the orbit of $L\in\mathfrak{g}^*$, where $L=\lambda e_1$, $\lambda>0$. For each $z\in\mathbf{Z}$, let us define objects $\mathcal{V}_z\in D(\mathbb{R})$ by the formula:

$$\mathcal{V}_z:=\bigoplus_{l\in\mathbb{L}^z;\forall j\neq 1:<l,f_j>\leq 0}\mathbb{K}_{[<l,L>;\infty)}[D(l)-\dim\mathfrak{h}], \tag{46}$$

where $\mathbb{L}^z:=\{l\in\mathbb{L}|e^l=z\}$. For every $d>0$ we have natural maps $\tau_d:\mathcal{V}_z\to T_{d*}\mathcal{V}_z$, where T_d is the shift by d. The map τ_d is induced by the obvious maps

$$\mathbb{K}_{[<l,L>;\infty)}\to\mathbb{K}_{[<l,L>+d;\infty)}=T_{d*}\mathbb{K}_{[<l,L>,\infty)}.$$

Theorem 5.7. *1) We have an isomrophism*

$$i^{-1}u_{\mathcal{O}}\cong\bigoplus_I G_I[D(-2\pi e_I)]\otimes T_{<-2\pi e_I,L>*}\mathcal{V}_{e^{2\pi e_I}}[\dim G] \tag{47}$$

2) The natural map $i^{-1}u_{\mathcal{O}}\to i^{-1}T_{d*}u_{\mathcal{O}}$ is induced by the maps τ_d.

Proof. Let $\mathbb{L}^c=\{l\in L;e^l=c\}$. Let $\mathbb{L}^c_J=\mathbb{L}^c\cap\mathbb{L}_J$. Let $\mathcal{Y}_c\in D(\mathfrak{h})$; $\mathcal{Y}_c=\mathcal{Y}|_{c\times\mathfrak{h}}$.

It follows from (45) and (42) that we have an isomorphism

$$i^{-1}u_{\mathcal{O}}\cong\bigoplus_I G_I[D(-2\pi e_I)]\otimes I_0^{-1}(T_{-2\pi e_I*}\mathcal{Y}|_{e\times\mathfrak{h}}*_{\mathfrak{h}}\gamma_L)[\dim G].$$

Let $\mathcal{U}_z := I_0^{-1}\mathcal{Y}|_{z\times\mathfrak{h}} *_{\mathfrak{h}} \gamma_L$. We then have

$$\begin{aligned} I_0^{-1}[T_{-2\pi e_I *}\mathcal{Y}|_{e\times\mathfrak{h}} *_{\mathfrak{h}} \gamma_l] &= I_0^{-1}[\mathcal{Y}_{e^{2\pi e_I}} *_{\mathfrak{h}} T_{-2\pi e_I *}\gamma_l] \\ &= I_0^{-1}[\mathcal{Y}_{e^{2\pi e_I}} *_{\mathfrak{h}} T_{<-2\pi e_I, L>*}\gamma_L] \\ &= T_{<-2\pi e_I, L>*}\mathcal{U}_{e^{2\pi e_I}}, \end{aligned}$$

where for a real number t, we define a map $T_t : G\times\mathbb{R} \to G\times\mathbb{R}$ to be the shift along $\mathbb{R}$ by t, whereas for $A\in\mathfrak{h}$, T_A is the shift by A along $\mathfrak{h}$ in $G\times\mathfrak{h}$.

We then have an isomorphism

$$i^{-1}u_{\mathcal{O}} \cong \bigoplus_I G_I[D(-2\pi e_I)] \otimes T_{<-2\pi e_I, L>*}\mathcal{U}_{e^{2\pi e_I}}[\dim G] \tag{48}$$

One also sees that the natural map

$$i^{-1}u_{\mathcal{O}} \to i^{-1}T_{d*}u_{\mathcal{O}}$$

for $d > 0$ corresponds under this isomorphism to the natural map induced by the maps

$$\tau_d : \mathcal{U}_c \to T_{d*}\mathcal{U}_c, \tag{49}$$

in turn induced by the natural map $\gamma_L \to T_{d*}\gamma_L$ coming from the embedding

$$\{(t,A)|t \geq -<A,L>+d\} \subset \{(t,A)|t \geq -<A,L>\}$$

(we have $\gamma_L = \mathbb{K}_{\{t\geq -<A,L>\}}$ and $T_{d*}\gamma_L = \mathbb{K}_{\{(t,A)|t\geq -<A,L>+d\}})$).

Let us compute $\mathcal{U}_z$ for $z\in\mathbf{Z}$. We will actually see that $\mathcal{U}_z \cong \mathcal{V}_z$.

Lemma 5.8. *We have $I_0^{-1}((V(J,l)|_{e^l\times\mathfrak{h}}) *_{\mathfrak{h}} \gamma_L) = 0$ for all $J \neq \{1\}$.*

Proof. Let $V'(J,l) := V(J,l)|_{e^l\times\mathfrak{h}}$.

We have $\gamma_L = \mathbb{K}_{\{(A,t)|t+<A,L>\geq 0\}}$. The inequality $t+<A,L>\geq 0$ is equivalent to $t/\lambda + <A,e_1>\geq 0$. Set $T = t/\lambda$. Then our statement can be reformulated as:

$$V'(J,l) *_{\mathfrak{h}} \mathbb{K}_{\{(A,T)|T+<A,e_1>\geq 0\}} = RP_!((V'(J,l)\boxtimes\mathbb{K}_{\mathbb{R}})\otimes\mathbb{K}_{\{(A,T)|T\geq <A,e_1>\}}) = 0,$$

where $P : \mathfrak{h}\times\mathbb{R}\to\mathbb{R}$ is the projection. This is equivalent to showing that for any $T\in\mathbb{R}$,

$$R\Gamma_c(\mathfrak{h}; V'(J,l)\otimes\mathbb{K}_{\{A\in\mathfrak{h}|T\geq(A,e_1)\}}) = 0.$$

Let $x_j : \mathfrak{h}\to\mathbb{R}$; $x_j = <A,e_j>$. We then have

$$V'(J,l)\otimes\mathbb{K}_{\{A\in\mathfrak{h}|T\geq<A,e_1>\}} = \mathbb{K}_S[D(l)],$$

where $S = \{A\in\mathfrak{h}|x_1(A)\leq T; \forall j\in J : x_j(A)\geq x_j(l)\}$.

Suppose there exists $j\in J$, $j\neq 1$. Decompose $\mathfrak{h} = \mathbb{R}.f_j\times E$, where E is the span of all f_i, $i\neq j$ (recall that f_j form the basis dual to $e_1, e_2, \ldots, e_{N-1}$). Thus, $\mathfrak{h} = \mathbb{R}\times E$. Then $\mathbb{K}_S[D(l)] = \mathbb{K}_{[0,\infty)}\boxtimes A$ for some $A\in D(E)$. Let $\pi : \mathfrak{h}\to E$ be the projection. Then $R\pi_!\mathbb{K}_S[D(l)] = 0$ because $R\Gamma_c(\mathbb{R}, \mathbb{K}_{[0,\infty)}) = 0$. If $J = \emptyset$, then $S = \{A\in\mathfrak{h}|x_1(A)\leq T\}$. It is easy to see that $R\Gamma_c(\mathfrak{h}, \mathbb{K}_S[D(l)]) = 0$. This exhausts all subsets $J\neq\{1\}$. $\square$

It now follows that $I_0^{-1}(\Psi^J *_{\mathfrak{h}} \gamma_L) = 0$ for all $J\neq\{1\}$ Therefore, we have an isomorphism

$$\mathcal{U}_z = I_0^{-1}(\Phi_z *_{\mathfrak{h}} \gamma_L)[\dim G] \cong I_0^{-1}(\Psi_z^{\{1\}} *_{\mathfrak{h}} \gamma_L)[-1][\dim G]$$

$$\cong \bigoplus_{l\in\mathbb{L}^z_{\{1\}}} I_0^{-1}[V(\{1\};l))_z *_{\mathfrak{h}} \gamma_L][-1][\dim G],$$

where the subscript z hear and below means the restriction onto $z \times \mathfrak{h} \subset \mathbf{Z} \times \mathfrak{h}$. Let us compute

$$I_0^{-1}[V(\{1\};l)_z *_{\mathfrak{h}} \gamma_L] = RP_!(\mathbb{K}_{(A,t);x_1(A)\geq x_1(l)} \otimes \mathbb{K}_{\{(A,t)|\lambda x_1(A)\leq t\}})[D(l)],$$

where $P : \mathfrak{h} \times \mathbb{R} \to \mathbf{Z} \times \mathbb{R}$ is the projection. We have

$$RP_!(\mathbb{K}_{\{(A,t);x_1(A)\geq x_1(l)\}} \otimes \mathbb{K}_{\{(A,t)|\lambda x_1(A)\leq t\}})$$

$$= RP_!(\mathbb{K}_{\{(A,t);x_1(l)\leq x_1(A)\leq t/\lambda\}}) = \mathbb{K}_{[\lambda x_1(l),\infty)}[1 - \dim \mathfrak{h}]$$

Thus,

$$I_0^{-1}[V(\{1\};l)_c *_{\mathfrak{h}} \gamma_L] \cong \mathbb{K}_{[\lambda x_1(l),\infty)}[1 - \dim \mathfrak{h}][D(l)]$$

Let $d \geq 0$. We need to compute the map

$$\tau_d : I_0^{-1}[V(\{1\};l)_c *_{\mathfrak{h}} \gamma_L] \to T_{d*}I_0^{-1}[V(\{1\};l)_c *_{\mathfrak{h}} \gamma_L]$$

induced by the natural map

$$\gamma_L \to T_{d*}\gamma_L.$$

It is easy to see that the map τ_d is isomorphic to the natural map

$$\mathbb{K}_{[\lambda x_1(l),\infty)}[1 - \dim \mathfrak{h}] \to T_{d*}\mathbb{K}_{[\lambda x_1(l),\infty)}[1 - \dim \mathfrak{h}]$$

$$= \mathbb{K}_{[\lambda x_1(l)+d,\infty)}[1 - \dim \mathfrak{h}],$$

induced by the embedding

$$[\lambda x_1(l) + d, \infty) \subset [\lambda x_1(l), \infty).$$

Thus, we have,

$$\mathcal{U}_z = \bigoplus_{l\in\mathbb{L}^z_{\{1\}}} \mathbb{K}_{[\lambda x_1(l),\infty)\}}[D(l)][-\dim \mathfrak{h}]$$

$$= \bigoplus_{l\in\mathbb{L}^z;\forall j\neq 1:<l,f_j>\leq 0} \mathbb{K}_{[\lambda<l,e_1>,\infty)}[D(l) - \dim \mathfrak{h}].$$

Thus, we see that $\mathcal{U}_z \cong \mathcal{V}_z$. It is now straightforward to check that the maps τ_d on both sides do match □

Let us substitute (46) into (47). We will get

$$i^{-1}u_{\mathcal{O}} \cong \bigoplus_I G(I) \otimes \upsilon(I)[-\dim \mathfrak{h} + \dim G],$$

where

$$\upsilon(I) = \bigoplus_{l\in\mathbb{L}^{e^{2\pi e_I}};\forall j\neq 1:<l,f_j>\leq 0} \mathbb{K}_{[<l-2\pi e_I,L>;\infty)}[D(l - 2\pi e_I)].$$

Let us replace l with $l + 2\pi e_I$. We will get an ultimate formula

(50) $$\upsilon_I = \bigoplus_{l\in\mathbb{L}^0;\forall j\neq 1:<l+2\pi e_I,f_j>\leq 0} \mathbb{K}_{[<l,L>;\infty)}[D(l)].$$

The map $\tau_d : i^{-1}u_{\mathcal{O}} \to T_{d*}i^{-1}u_{\mathcal{O}}$, $d \leq 0$ is induced by natural maps $\tau_d : \upsilon_I \to T_{d*}\upsilon_I$ which are produced by the embeddings $T_d[< l, L >; \infty)) \subset [< l, L >; \infty)$.

5.6.1. *Proof of Proposition 5.5.* We have

$$R\hom(\mathbb{K}_{[0,\infty)}[-\dim\mathfrak{h}];T_{d*}i^{-1}u_{\mathcal{O}}[-\dim G]) = \bigoplus_I G(I)\otimes H_I(d),$$

where

$$H_I(d) := R\hom(\mathbb{K}_{[0,\infty)};T_{d*}\upsilon_I) \cong \bigoplus_{l\in S_I(d)} \mathbb{K}[D(l)],$$

and

$$S_I(d) := \{l\in L^0|\forall j\neq 1 :< l+2\pi e_I, f_j >\leq 0; < l, L > +d\geq 0\}.$$

The map (41) is induced by maps $\tau_d : H_I(0)\to H_i(d)$, which are in turn induced by the maps $\tau_d : \upsilon\to T_{d*}\upsilon$. It is not hard to see that the map $\tau_d : H_I(0)\to H_I(d)$ is induced by the inclusion $S_I(0)\subset S_I(d)$. As $S_I(0)$ is not empty, the maps $\tau_d : H_I(0)\to H_I(d)$ do not vanish for any $d\geq 0$, which proves the Proposition.

6. An object $\mathfrak{S}$

We will freely use notations from Sec. 10.

The object $\mathfrak{S}$ will be characterized microlocally. Let us first define a subset

$$\Omega_{\mathfrak{S}} \in T^*(G\times\mathfrak{h}) \tag{51}$$

which will serve as a microsupport of $\mathfrak{S}$. Define $\Omega_{\mathfrak{S}}$ as a set of all points

$$(g, A, \omega, \eta)\in G\times\mathfrak{h}\times\mathfrak{g}\times\mathfrak{h} = T^*(G\times\mathfrak{h})$$

satisfying:

1)$g(V_k(\omega))\subset V_k(\omega)$, that is $\mathrm{Ad}_g\omega = \omega$;

2) $\det g|_{V_k(\omega)} = e^{-i<e_k,A>}$;

3) $\eta = \|\omega\|$. The notation $V_k(\omega)$ is defined in the beginning of Sec. 10, see (97).

Finally, let us denote for $A\in\mathfrak{h}$, $I_A : G\to G\times\mathfrak{h}$ the embedding $I_A(g) = (g, A)$.

We now formulate

Theorem 6.1. *There exists an object $\mathfrak{S}\in D(G\times\mathfrak{h})$ such that*

1) $SS(\mathfrak{S})\subset\Omega_{\mathfrak{S}}$;

2) $I_0^{-1}\mathfrak{S} = \mathbb{K}_{e_G}$.

6.1. **Proof of Theorem 6.1.**

6.1.1. Let $U_1, U_2\subset\mathfrak{h}$ be open convex sets. Let $a : \mathfrak{h}\times\mathfrak{h}\to\mathfrak{h}$ be addition. The map a induces a map $U_1\times U_2\to U_1+U_2$ which is well known to be a trivial smooth fibration whose fiber and base are diffeomorphic to $\mathfrak{h}$.

Let $F_k\in D(G\times U_k)$, $k = 1, 2$. Let $M : G\times U_1\times G\times U_2\to G\times U_1\times U_2$ be the map induced by the product on G. Set $F_1 *_G F_2 := RM_!(F_1\boxtimes F_2)$.

Let $a : G\times U_1\times U_2\to G\times(U_1+U_2)$ be induced by the addition on $\mathfrak{h}$.

Lemma 6.2. *Suppose that $SS(F_k)\subset\Omega_{\mathfrak{S}}\cap T^*(G\times U_k)$. Then 1) The natural map*

$$a^{-1}Ra_*(F_1 *_G F_2)\to F_1 *_G F_2$$

is an isomorphism;

2) $SS(Ra_*(F_1 *_G F_2))\subset\Omega_{\mathfrak{S}}\cap T^*(G\times(U_1+U_2))$.

Proof. Let us first estimate the microsupport of $F_1 *_G F_2 = RM_!(F_1 \boxtimes F_2)$. Since the map M is proper, we know that a point

$$\zeta := (g, A_1, A_2, \omega, \eta_1, \eta_2) \in G \times U_1 \times U_2 \times \mathfrak{g}^* \times \mathfrak{h}^* \times \mathfrak{h}^* = T^*(G \times U_1 \times U_2)$$

belongs to $\mathrm{SS}RM_!(F_1 \boxtimes F_2)$ only if there exist $g_1, g_2 \in G$ such that $M(g_1, A_1, g_2, A_2) = (g, A_1, A_2)$ (i.e. $g = g_1 g_2$) and

$$M^*\zeta|_{(g_1,A_1,g_2,A_2)} \in \mathrm{SS}(F_1 \boxtimes F_2).$$

We have

$$M^*\zeta|_{(g_1,A_1,g_2,A_2)} = (g_1, A_1, \omega, \eta_1, g_2, A_2, \mathrm{Ad}^*_{g_1}\omega, \eta_2).$$

We then have $(g_1, A_1, \omega, \eta_1), (g_2, A_2, \mathrm{Ad}^*_{g_1}\omega, \eta_2) \in \Omega_{\mathfrak{S}}$. Therefore, $\mathrm{Ad}^*_{g_1}\omega = \omega$, and we have

$$(g_k, A_k, \omega, \eta_k) \in \Omega_{\mathfrak{S}}.$$

This implies $\eta_1 = \eta_2 = \|\omega\|$. This means that any 1-form in $\mathrm{SS}(RM_!(F_1 \boxtimes F_2))$ vanishes on fibers of a. This proves part 1).

Let us now estimate $\mathrm{SS}Ra_*(F_1 *_G F_2)$. We know that $\zeta \in \mathrm{SS}Ra_*(F_1 *_G F_2)$, where $\zeta \in T^*_{(g,A)}(G \times (U_1 + U_2))$, iff for every point $(g, A_1, A_2) \in G \times U_1 \times U_2$ such that $A_1 + A_2 = A$, we have

$$a^*\zeta|_{(g,A_1,A_2)} \in \mathrm{SS}(a^{-1}Ra_*(F_1 *_G F_2)).$$

Let $\zeta = (g, A, \omega, \eta)$, then $a^*\zeta|_{(g,A_1,A_2)} = (g, A_1, A_2, \omega, \eta, \eta)$. Using the isomorphism $a^{-1}Ra_*(F_1 *_G F_2) \to F_1 *_G F_2$. and the above estimate for $\mathrm{SS}(F_1 *_G F_2)$, we get: there exist $g_1, g_2 \in G$ such that $g = g_1 g_2$ and

$$(g_k, A_k, \omega, \eta) \in \Omega_{\mathfrak{S}}.$$

It remains to show that $(g_1 g_2, A_1 + A_2, \omega, \eta) \in \Omega_{\mathfrak{S}}$. Indeed, we have $\eta = \|\omega\|$. Next, $Ad^*_{g_k}\omega = \omega$, therefore, $\mathrm{Ad}^*_{g_1 g_2}\omega = \omega$.

Finally,

$$\det g_1 g_2|_{V_k(\omega)} = \det g_1|_{V_k(\omega)} \det g_2|_{V_k(\omega)}$$

$$= e^{-i<A_1, e_{d_k(\omega)}>} e^{-i<A_2, e_{d_k(\omega)}>}$$

$$e^{-i<A_1+A_2, e_{d_k(\omega)}>}.$$

□

6.1.2. Let $b \in C^\circ_+$; $b \leq e_1/100$. Let $V_b^- := \{A \in C^\circ_- | -A << b\}$, where C°_+ is the interior of the positive Weyl chamber and $C^\circ_- = -C^\circ_+$, see Sec. 10. let $W_b^- \subset G \times V_b^-$;

$$W_b^- := \{(e^X, A); A \in V_b^-; \|X\| << -A\}.$$

Set $F^- \in D(G \times V_b^-)$;

$$F^- := \mathbb{K}_{W_b^-}[\dim G].$$

6.1.3. We will identify $TG = G \times \mathfrak{g}$; $T^*G = G \times \mathfrak{g}^* = G \times \mathfrak{g}$ via identifying $\mathfrak{g}$ with the space of all right invariant vector fields on G and $\mathfrak{g}^* = \mathfrak{g}$ with the space of all right invariant 1-forms on G. Analogously, we will identify $T(G\times\mathfrak{h}) = G\times\mathfrak{h}\times\mathfrak{g}\times\mathfrak{h}$ and $T^*(G\times\mathfrak{h}) = G\times\mathfrak{h}\times\mathfrak{g}^*\times\mathfrak{h}^* == G\times\mathfrak{h}\times\mathfrak{g}\times\mathfrak{h}$.

Lemma 6.3. *The microsupport of F^- is contained in the set of all points $(e^X, A, \omega, \eta) \in G\times V_b^{-1}\times \mathfrak{g}^* \times \mathfrak{h}^*$, where*
1) $\|X\| \leq -A$;
2) $[X, \omega] = 0$;
3) $TrX|_{V_k(\omega)} = -i < A, e_{d_k} >$;
4) $\eta = \|\omega\|$

Proof. Let $U \subset \mathfrak{g} \times V_b^-$; $U = \{(X, A) | \|X\| << -A\}$. Let

$$\exp : \mathfrak{g} \times V_b^{-1} \to G \times V_b^{-1}$$

be the exponential map. We see that exp maps U diffeomorphically onto W_b^-, hence we have an induced diffeomorhphism $\exp : T^*U \to T^*W_b^-$. It also follows that $F^- = \exp_* \mathbb{K}_U[\dim G]$ and that $\mathrm{SS}(F^-) = \exp(\mathrm{SS}\mathbb{K}_U)$.

Let us estimate $\mathrm{SS}(\mathbb{K}_U)$. $U \subset \mathfrak{g} \times V_b^-$ is an open convex subset. It follows that a point $(X, A, \omega, \eta) \in \mathfrak{g} \times V_b^- \times \mathfrak{g}^* \times \mathfrak{h}^*$ is in the microsupport of $\mathbb{K}_U$ iff 1) $\|X\| \leq -A$;

2) for all $(X', A') \in U$, $< X', \omega > + < A', \eta > < < X, \omega > + < A, \eta >$.

Fix A', then $X' \in \mathfrak{g}$ is an arbitrary element such that $\|X'\| << -A$. Lemma 10.1 implies that

$$\sup < X', \omega > = < -A'; \|\omega\| >$$

Thus, Condition 2) is equivalent to

$$< -A', \|\omega\| > + < A', \eta > \leq < X, \omega > + < A, \eta > \tag{52}$$

for all $A' \in V_b^-$. Plug $A' = A$. We will get

$$< -A, \|\omega\| > \leq < X, \omega > .$$

On the other hand $< X, \omega > \leq < \|X\|, \|\omega\| > \leq < -A, \|\omega\| >$. This implies that

$$< -A, \|\omega\| > = < X, \omega > . \tag{53}$$

According to Lemma 10.1, for all k,

$$\mathrm{Tr} X|_{V_k(\omega)} = -i < A, e_{d_k(\omega)} > .$$

Let us plug (53) into (52). We will get

$$< -A', \|\omega\| > + < A', \eta > \leq < -A, |\omega| > + < A, \eta >$$

for all $A' \in V_b^-$. As $A \in V_b^-$ and V_b^- is open, this is only possible if $\eta = \|\omega\|$. $\square$

Corollary 6.4. *We have $SS(F^-) \subset \Omega_{\mathfrak{G}} \cap T^*(G \times V_b^-)$.*

6.1.4. Let $U \subset G \times V_b^- \times V_b^-$ be given by

$$U := \{(e^X, A_1, A_2) | \|X\| << -A_1 - A_2\}$$

Lemma 6.5. *We have an isomorphism*

$$F^- *_G F^- \cong \mathbb{K}_U[\dim G].$$

Proof. Let $j_{U_1} : U_1 \hookrightarrow G \times V_b^- \times V_b^-$ be an open set defined by $U_1 = M(W_b^- \times W_b^-)$. It follows that we have an isomorphism

$$j_{U_1!}((F^- *_G F^-)|_{U_1}) \to F^- *_G F^-.$$

We have

$$U_1 = \{(e^{X_1} e^{X_2}, A_1, A_2) | A_k \in V_b^-; \|X_k\| << -A_k\}.$$

According to Lemma 10.4, we have $e^{X_1} e^{X_2} = e^Y$, where $\|Y\| \leq \|X_1\| + \|X_2\| << -A_1 - A_2$. Thus $U_1 \subset U$.

Let $j_U : U \to G \times V_b^- \times V_b^-$ be the open embedding. We then have an isomorphism

$$j_{U!}((F^- *_G F^-)|_U) \to F^- *_G F^- \tag{54}$$

Let us now study $F^- *_G F^-|_U$. Let us estimate the microsupport of this object. Similar to proof of Lemma 6.2, we see that a point

$$(g, A_1, A_2, \omega, \eta_1, \eta_2) \in G \times V_b^- \times V_b^- \times \mathfrak{g}^* \times \mathfrak{h}^* \times \mathfrak{h}^* = T^*(G \times V_b^- \times V_b^-) \tag{55}$$

is in $\mathrm{SS}(F^- *_G F^-|_U)$ iff

1) $(g, A_1, A_2) \in U$;

2) there exist $X_1, X_2 \in \mathfrak{g}$ such that $g = e^{X_1} e^{X_2}$ and $(e^{X_k}, A_k, \omega, \eta_k) \in \mathrm{SS}(F^-)$ for $k = 1, 2$.

According to Lemma 6.3, we have

$$\|X_k\| \leq -A_k;$$

$$\mathrm{Tr} X_k|_{V_l(\omega)} = -i < A_l, e_{d_l} > .$$

Hence, $e^Y = e^{X_1} e^{X_2}$ preserves the spaces $V_l(\omega)$. As $\|Y\| << \|X_1\| + \|X_2\| \leq e_1/(50N)$, it follows that all eigenvalues of $-iY$ have absolute value of less than $1/(50N)$. It then follows that Y does preserve the spaces $V_l(\omega)$ as well, and $\mathrm{Tr} Y|_{V_l(\omega)}$ has absolute value of at most $1/50$.

We also have

$$\det e^Y|_{V_l(\omega)} = \det e^{X_1}|_{V_l(\omega)} e^{X_2}|_{V_l(\omega)} = e^{-i<A_1+A_2, e_{d_l(\omega)}>}.$$

We have $| < A_1 + A_2, e_{d_l(\omega)} > | \leq 1/50$, therefore,

$$\mathrm{Tr} Y|_{V_l(\omega)} = -i < A_1 + A_2, e_{d_l(\omega)} > . \tag{56}$$

Assume $\omega \neq 0$. Then there exists a subspace $V_l(\omega)$ which is proper, i.e. $0 < d_l(\omega) < N$. On the other hand, we have $(e^Y, A_1, A_2) \in U$, meaning that $e^Y = e^{Y'}$, where $\|Y'\| << -A_1 - A_2$. We then have $\|Y\|, \|Y'\| < e_1/(50N)$ which implies $Y = Y'$ and $\|Y\| << A_1 + A_2$. This clearly contradicts to (56). Therefore, it is impossible that $\omega \neq 0$, hence $\omega = 0$. It then follows that in (55), $\eta_1 = \eta_2 = \|\omega\| = 0$. Thus, we have proven that $F^- *_G F^-|_U$ is microsupported on the zero-section, hence is locally constant. However, under the exponential map, U is a diffeomorphic image of an open convex set $\{(X, A_1, A_2) | A_k \in V_b^-; \|X\| << -A_1 + A_2\} \subset \mathfrak{g} \times V_b^- \times V_b^-$. Therefore, U is diffeomorphic to $\mathbb{R}^{\dim U}$ and $F^- *_G F^-$ is constant on U.

Let $Z := R\Gamma_c(U; F^- *_G F^-)$. We then have a natural isomorphism $F^- *_G F^-|_U \cong Z_U[\dim U]$. Because of an isomorphism (54), we have an induced isomorphism

$$\begin{aligned} R\Gamma_c(U; F^- *_G F^-) &\to R\Gamma_c(G \times V_b^- \times V_b^-; F^- *_G F^-) \\ &\cong R\Gamma_c(G \times V_b; F^-) \otimes R\Gamma_c(G \times V_b; F^-) \\ \cong \mathbb{K}[-\dim G \times V_b^-] \otimes \mathbb{K}[-\dim G \times V_b^-][2\dim G] &= \mathbb{K}[-\dim U + \dim G]. \end{aligned}$$

This implies the statement. □

Let $a : G \times V_b^- \times V_b^- \to G \times 2V_b^-$ be the addition map. The just proven Lemma as well as Lemma 6.2 imply that the natural map $a^{-1}Ra_*(F^- *_G F^-) \to F^- *_G F^-$ is an isomorphism and that

$$Ra_*(F^- *_G F^-) \cong \mathbb{K}_{\{(e^X, A) | A \in 2V_b^-; \|X\| << -A\}}[\dim G].$$

We then have an induced isomophism

$$\iota : Ra_* F^- *_G F^-|_{G \times V_b^-} \cong F^-. \tag{57}$$

6.1.5. Let $M > 0$ and let $F_M^- \in D(G \times (V_b^-)^M)$;

$$F_M^- := F^- *_G F^- *_G \cdots *_G F^-,$$

where F^- occurs M times.

Let $a_M : G \times (V_b^-)^M) \to G \times MV_b^-$ be the addition map. Lemma 6.2 implies that the natural map

$$a_M^{-1} Ra_{M*} F_M^- \to F_M^-$$

is an isomorphism.

Let $\Phi_M^- := Ra_{M*} F_M^-$.

Let us construct a map

$$I_M : \Phi_M^-|_{G \times (M-1)V_b^-} \to \Phi_{M-1}^-,$$

where $M \geq 2$, as follows.

Let $W \subset (V_b^-)^2$ be an open convex subset consisting of all points of the form (v_1, v_2), where $v_1 + v_2 \in V_b^-$. Let $W_M := (V_b^-)^{M-2} \times W \subset (V_b^-)^M$.

Let us decompose

$$\begin{aligned} \alpha_M := a_M|_{G \times W_M} : G \times W_M = G \times (V_b^-)^{M-2} \times W &\xrightarrow{a_2} G \times (V_b^-)^{M-2} \times V_b^- \\ &\xrightarrow{a_{M-1}} (M-1)V_b^-. \end{aligned}$$

It follows that $\alpha_M(W_M) = G \times (M-1)V_b^-$. We have a natural isomorphism

$$Ra_{M*} F_M^-|_{G \times (V_b^-)^{M-1}} = R\alpha_{M*} F_M^-|_{G \times W_M} \cong Ra_{M-1*} Ra_{2*} F_M^-|_{G \times W_M}.$$

We have

$$Ra_{2*} F_M^-|_{G \times W_M} \cong F_{M-2}^- *_G (R\alpha_* F^- *_G F^-|_W),$$

where $\alpha : G \times W \to G \times V_b^-$ is the addition map. We have an isomorphism (see (57))

$$R\alpha_*(F^- *_G F^-|_W) \cong (Ra_*(F^- *_G F^- *))|_{V_b^-} \overset{\iota}{\cong} F^-.$$

Hence, we have isomoprhisms

$$Ra_{2*}(F_M^-|_{G \times W_M}) \cong F_{M-1}^-$$

$$I_M : Ra_{M*} F_M^-|_{G \times (V_b^-)^{M-1}} \cong Ra_{M-1*} F_{M-1}^-.$$

Thus, we have objects $\Phi^-_M \in D(G \times MV_b^-)$ and isomorhhisms

$$I_M : \Phi^-_M|_{(M-1)V_b^-} \to \Phi^-_{M-1}.$$

It then follows that there exists an object $\Phi^- \in D(G \times C^\circ_-)$ (note that $C^\circ_- = \bigcup_M MV_b^-$) along with isomorphisms

$$J_M : \Phi^-|_{MV_b^-} \to \Phi^-_M$$

which are compatible with I_M in the obvious way.

Let $\Psi^- \in D(G \times C^\circ_-)$ be another object endowed with isomorphisms $J'_M : \Psi^-|_{MV_b^-} \to \Phi^-_M$ so that J'_M are compatible with I_M. Then there exists a (non-canonical) isomorphism $\Phi^- \to \Psi^-$ which is compatible with the isomorphisms J_M, J'_M.

Lemma 6.2 implies that $\mathrm{SS}(\Phi^-_M) \subset \Omega_{\mathfrak{G}} \cap T^*(G \times MV_b^-)$. Therefore,

$$\mathrm{SS}(\Phi^-) \subset \Omega_{\mathfrak{G}} \cap T^*(G \times C^\circ_-).$$

6.1.6. Lemma 6.2 implies that we have an isomorphism

$$A^{-1}RA_*(\Phi^- *_G \Phi^-) \to \Phi^- *_G \Phi^-$$

where $A : G \times C^\circ_- \times C^\circ_- \to G \times C^\circ_-$ is the addition.

Let us restrict this isomorphism to $G \times MV_b^- \times V_b^-$. We will then get an isomoprhism

$$A^{-1}(RA_*\Phi^- *_G \Phi^-|_{(M+1)V_b^-}) \to \Phi^-_M *_G F^- = A^{-1}\Phi^-_{M+1}.$$

Thus, we have an isomorphism

$$J'_{M+1} : RA_*\Phi^- *_G \Phi^-|_{(M+1)V_b^-} \cong \Phi^-_{M+1}$$

One can easily check that these isomoprhisms are compatible with I_M hence, there exists an isomorphism

$$J : RA_*(\Phi^- *_G \Phi^-) \cong \Phi^-$$

which is compatible with isomorphisms J_M, J'_M. Therefore, we have an isomorphism

$$I : \Phi^- *_G \Phi^- \cong A^{-1}\Phi^-. \tag{58}$$

6.1.7. Let $X^\pm \in D(G)$, $X^+ := \mathbb{K}_{\{e^{-X} | \|X\| \leq b/2\}}$; $X^- := \mathbb{K}_{\{e^X | \|X\| << b/2\}}[\dim G]$. We have an isomorphism $X^- \cong \Phi^-|_{G \times (-b/2)}$.

Lemma 6.6. *We have an isomorphism $X^- *_G X^+ \cong \mathbb{K}_e$.*

Proof. Let us first compute the microsupport of $X^- *_G X^+$. We will prove the following: *$SS(X^- *_G X^+)$ is contained in the set of all points of the form $(e^Y, \omega) \in G \times \mathfrak{g}^*$, where $Y \in \mathfrak{g}$; $\|Y\| \leq (e_1 + e_{N-1})/200$, $[Y, \omega] = 0$ and $< Y, \omega >= 0$.*

Let us first estimate $\mathrm{SS}(X^-)$. Let $\exp : \mathfrak{g} \times G$ be the exponential map. We then see that $X^- = \exp_* \mathbb{K}_U[\dim G]$, where $U \subset \mathfrak{g}$ is an open convex subset $U = \{X | \|X\| << b/2\}$.

We know that $\mathrm{SS}(\mathbb{K}_U)$ consists of all points of the form $(X, \omega) \subset \mathfrak{g} \times \mathfrak{g}^*$, where $\|X\| \leq b/2$ and

$$< X', \omega > < < X, \omega >$$

for all $X' << b/2$. Lemma 10.1 implies that this is equivalent to $< b/2, \|\omega\| > \leq < X, \omega >$ (because

$$\sup_{X' << b/2} < X', \omega >=< b/2, \|\omega\| >);$$

on the other hand

$$< X,\omega >\leq < \|b/2\|, \|\omega\| >$$

by the same Lemma 10.1. Therefore $< X,\omega >=< \|b/2\|, \|\omega\| >$. As $\|X\| \leq b/2$ this implies $[X,\omega]=0$;

$$\mathrm{Tr} X|_{V_k(\omega)} = i < b, e_{d_k(\omega)} > /2. \tag{59}$$

As $[X,\omega]=0$, we see that $\mathrm{SS}(X^-)$ consists of all points (e^X,ω), where $\|X\|\leq b/2$ and $[X,\omega]=0$ and we have (59).

Analogously, $X^+ = \exp_* \mathbb{K}_K$, where $K \subset \mathfrak{g}$ is a convex compact $K=\{X|\|-X\|\leq b/2\}$. Therefore, $\mathrm{SS}(\mathbb{K}_K)$ consists of all points (X_1,ω_1), where $\|-X_1\|\leq b/2$ and $< X',\omega_1 >\geq < X_1,\omega_1 >$ for all $X'\in K$. I.e. $< -X',\omega_1 >\leq < -X_1,\omega_1 >$. In the same way as above, we conclude that this is equivalent to $< -X_1,\omega_1 >=< b/2, \|\omega_1\| >$ which in turn is equivalent to $\|-X_1\|\leq b/2$; $[X_1,\omega_1]=0$;

$$\mathrm{Tr}(-X_1)|_{V_k(\omega_1)} = i < b/2, e_{d_k(\omega_1)} > . \tag{60}$$

Thus, $\mathrm{SS}(X^+)$ consists of all points of the form (e^{X_1},ω_1), where $[X_1,\omega_1]=0$; $\|-X_1\|\leq b/2$ and (60) is the case. Observe that we have $\|-X_1\|\leq e_1/200$ which means $\|X_1\|\leq e_{N-1}/200$.

We know that the microsupport of $X^- *_G X^+ = Rm_!(X^-\boxtimes X^+)$ is contained in the set of all points of the form (g_1g_2,ω) where $g_1,g_2\in G$; $(g_1,\omega)\in \mathrm{SS}(X^-)$; $(g_2,\mathrm{Ad}^*_{g_1}\omega)\in \mathrm{SS}(X^+)$. This means that $\mathrm{SS}(X^- *_G X^+)$ consists of all points of the form

$$(e^Xe^{X_1},\omega),$$

where $(e^X,\omega)\in \mathrm{SS}(X^-)$ and $(e^{X_1},\omega)\in\mathrm{SS}(X^+)$ (because $[X,\omega]=[X_1,\omega]=0$). According to Lemma 10.4, $e^Xe^{X_1}=e^Y$, where $\|Y\|\leq \|X\|+\|X_1\|\leq (e_1+e_{N-1})/200$. It follows that $e^YV_k(\omega)\subset V_k(\omega)$ and

$$\det e^Y|_{V_k(\omega)} = e^{i< b/2-b/2,\|\omega\|>}=1,$$

see (59), (60). As $\|Y\|\leq b$, this implies $\mathrm{Tr}Y|_{V_k(\omega)}=0$. This in turn implies that $< Y,\omega >= 0$, which we wanted.

Let $c:=(e_1+e_{N-1})/200$. Let $W:=\{X\in g; \|X\| << 2c\}$. The exponential map gives rise to an open empedding $\exp: W\to G$. The object $X^- *_G X^+$ is supported within $\exp(W)$. Consider $E\in D(W)$; $E:=\exp^{-1}(X^- *_G)X^+)$. It suffices to show that $E\cong \mathbb{K}_0$.

We see that E is microsupported within the set (X,ω), where $\|X\|\leq c$, $[\omega,X]=0$, $(\omega,X)=0$. Let D be the dilation vector field on $\mathfrak{g}$. That is D is a section of $T\mathfrak{g}=\mathfrak{g}\times\mathfrak{g}$; $D:\mathfrak{g}\to\mathfrak{g}\times\mathfrak{g}$; $D(X)=(X,X)$. It then follows that every point $(x,\omega)\in\mathrm{SS}(E)$ satisfies $i_{D_x}(\omega)=0$. Let $X\in\mathfrak{g}$; $X\neq 0$. Let $R_X:=(\mathbb{R}_{>0}.X)\cap W$ be an open segment. It then follows that $E|_{R_X}$ is a constant sheaf. However, R_X does necessarily contain points $Y\in R_X$ such that $\|Y\|>>c$, meaning that $E|_Y=0$, and $E|_{R_X}=0$. Hence E is supported at 0 and it suffices to show that $E|_0\cong\mathbb{K}$.

We have

$$E|_0=(X^- *_G X^+)|_e = R\Gamma_c(G;\mathbb{K}_{\{e^X|\|X\|<<b/2\}}\otimes \mathbb{K}_{\{e^X|\|X\|\leq b/2\}})[\dim G]$$

$$= R\Gamma_c(G;\mathbb{K}_{\{e^X|\|X\|<<b/2\}})[\dim G]=\mathbb{K},$$

because the open set $\{e^X|\|X\|<< b/2\}$ is diffeomorphic to an open ball. □

6.1.8. Let $T : C_-^\circ \to C_-^\circ$ be the shift by $-b/2$. $T(l) = l - b/2$.

Let us restrict the isomorphism (58) onto $G \times V_b^- \times (-b/2)$. We will get an isomorphism

$$\Phi^- *_G X^- \cong A^{-1}\Phi^-|_{G\times V_b^- \times(-b/2)}$$
$$= T^{-1}\Phi^-.$$

Taking the convolution with X^+ and using the previous Lemma, we will get an isomorphism

(61)
$$\Phi^- \cong (T^{-1}\Phi) *_G X^+.$$

Let $\mathbf{T}_M : (C_-^\circ + Mb/2) \to C_-^\circ$ be the shift by $-Mb/2$. Set

$$\Psi_M := \mathbf{T}_M^{-1}\Phi *_G (X^+)^{*_G^M},$$
$$\Psi_M \in D(G \times (C_-^\circ + Mb/2)).$$

We have an isomorphism

$$i_M : \Psi_M|_{C_-^\circ+(M-1)b/2} \cong \mathbf{T}_{M-1}^{-1}[(T^{-1}\Phi^- *_G X^+) *_G (X^+)^{*_G^{M-1}}]$$
$$\cong \mathbf{T}_{M-1}^{-1}((\Phi^- *_G (X^+)^{*_G^{M-1}}) = \Psi_{M-1}$$

where on the last step we have used the isomorphism (61). Similar to above, there exists an object $\mathfrak{S} \in D(G \times \mathfrak{h})$ and isomorphisms $\mathfrak{S}|_{C_-+Mb/2} \to \Psi_M$ which are compatible with isomorphisms i_M. Lemma 6.2 readily implies that $\mathrm{SS}(\mathfrak{S}) \subset \Omega_{\mathfrak{S}}$. Let us compute $\mathfrak{S}|_{G\times 0}$. We have $0 \in C_-^\circ - b/2$. Therefore, we have an isomorphism

$$\mathfrak{S}|_{G\times 0} \cong \Psi_1^-|_{G\times 0} \cong \Phi^-|_{G\times -b/2} *_G X^+ \cong X^- *_G X^+ \cong \mathbb{K}_e.$$

This proves that the object $\mathfrak{S}$ satisfies all the conditions of Theorem 6.1.

6.1.9. *Uniqueness.*

Theorem 6.7. *Let $\mathfrak{S}_1, \mathfrak{S}_2$ satisfy the conditions of Theorem 6.1. Then $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are canonically isomorphic.*

Proof. According to Lemma 6.2 (take $U_1 = U_2 = \mathfrak{h}$), we have an isomorphism

(62)
$$a^{-1}Ra_*(\mathfrak{S}_1 *_G \mathfrak{S}_2) \to \mathfrak{S}_1 *_G \mathfrak{S}_2.$$

Let $I_1, I_2 : \mathfrak{h} \to \mathfrak{h} \times \mathfrak{h}$ be as follows: $I_1(A) = (A, 0)$; $I_2(A) = (0, A)$. aApplying functors I_1^{-1}, I_2^{-1} to (62) and taking into account the isomorphisms $I_0^{-1}\mathfrak{S}_i \cong \mathbb{K}_{e_G}$, we will get the following isomorphisms

(63)
$$Ra_*(\mathfrak{S}_1 *_G \mathfrak{S}_2) \to \mathfrak{S}_2; \quad Ra_*(\mathfrak{S}_1 *_G \mathfrak{S}_2) \to \mathfrak{S}_1,$$

whence an isomorphism $\mathfrak{S}_1 \to \mathfrak{S}_2$. □

From now on we will denote by $\mathfrak{S}$ any object satisfying Theorem 6.1 (they are all canonically isomorphic).

Equations (62), (63) imply that we have an isomorphism

(64)
$$\mathfrak{S} *_G \mathfrak{S} \to a^{-1}\mathfrak{S}.$$

6.1.10. One can prove even more general result. Let $\Upsilon \subset T^*(G\times\mathfrak{h})$ consist of all points $(G, A, \omega, \|\omega\|) \in G\times\mathfrak{h}\times\mathfrak{g}^{\times}\mathfrak{h} = T^*(G\times\mathfrak{h})$. Of course, $\Omega_{\mathfrak{S}} \subset \Upsilon$. Let $C_\Upsilon \subset D(G\times\mathfrak{h})$ be the full subcategory consisting of all objects F microsupported on Υ. Let $i_0 : G \to G\times\mathfrak{h}$ be the embedding $i_0(g) = (g, 0)$. We have a functor $i_0^{-1} : C_\Upsilon \to D(G)$. We also have a functor $\Sigma : D(G) \to C_\Upsilon$; $\Sigma(F) = F *_G \mathfrak{S}$ (it is easy to show that $\mathrm{SS}(F *_G \mathfrak{S}) \subset \Upsilon$).

Theorem 6.8. *The functors i_0^{-1} anb Σ are mutually quasi-inverse equivalences.*

Proof. Let $F \in C_\Upsilon$ and consider $F *_G \mathfrak{S} \in D(G\times\mathfrak{h}\times\mathfrak{h})$. As above, let $a : G\times\mathfrak{h}\times\mathfrak{h} \to G\times\mathfrak{h}$ be the addition. Similar to above, one can show that the natural map

$$a^{-1}Ra_*(F *_G \mathfrak{S}) \to F *_G \mathfrak{S} \tag{65}$$

is an isomorphism. Let $i_1, i_2 : G\times\mathfrak{h} \to G\times\mathfrak{h}\times\mathfrak{h}$ be given by $i_1(g, A) = (g, A, 0)$; $i_2(g, A) = (g, 0, A)$. In the same spirit as above, we can apply i_1^{-1}, i_2^{-1} to (65). We will get functorial isomorphisms

$$Ra_*(F *_G \mathfrak{S}) \to F = \mathrm{Id}(F), \quad Ra_*(F *_G \mathfrak{S}) \to i_0^{-1}F *_G \mathfrak{S} = \Sigma i_0^{-1}F,$$

whence an isomorhism of functors $\mathrm{Id}_{D(G)} \cong \Sigma i_0^{-1}$.

Let us consider the composition in the opposite order:

$$i_0^{-1}\Sigma(F) = F *_G (\mathfrak{S}|_{G\times 0}) = F *_G \mathbb{K}_{e_G} = F.$$

This way we get an isomorphism $\mathrm{Id}_{D(G\times\mathfrak{h})} \cong i_0^{-1}\Sigma$. □

6.1.11. *Lemma.* These Lemma will be used in the sequel. Let $A \in \mathfrak{h}$. Let $I_A : G \to G\times\mathfrak{h}$ be given by $I_A(g) = (g, A)$. Let $S_A := I_A^{-1}\mathfrak{S}$. Let $T_A : \mathfrak{h} \to \mathfrak{h}$; $T_A(A_1) = A + A_1$ be the shift by A.

Lemma 6.9. *We have an isomorphism $T_A^{-1}\mathfrak{S} \cong S_A *_G \mathfrak{S}$.*

Proof. Apply the functor I_A^{-1} to (62). □

7. Study of $\mathcal{S}|_{\mathbf{Z}\times C_-^\circ}$

We denote by $j_{C_-^\circ} : C_-^\circ \to \mathfrak{h}$ the open embedding. We will denote by the same symbol the induced embeddings $\mathbf{Z}\times C_-^\circ \to \mathbf{Z}\times\mathfrak{h}$; $G\times C_-^\circ \to G\times\mathfrak{h}$.

We start with studying the object $j_{C_-^\circ}^{-1}\mathfrak{S}$.

7.1. Microsupport of $j_{C_-^\circ}^{-1}\mathfrak{S}$.

Lemma 7.1. *The object $j_{C_-^\circ}^{-1}\mathfrak{S}$ is microsupported within the set of points of the form $(g, A, \omega, \eta) \in G\times C_-^\circ\times\mathfrak{g}^*\times\mathfrak{h}^*$ such that there exists an $X \in \mathfrak{g}$ satisfying:*

1) $g = e^X$;
2) $\|X\| \le -A$;
3)$[X, \omega] = 0$;
4) $TrX|_{V_k(\omega)} = -i < A, e_{d_k} >$;
5)$\eta = \|\omega\|$.

Proof. As was shown in the proof of Theorem 6.1, we have $C_-^\circ = \bigcup_M MV_b$ and

$$\mathfrak{S}|_{G\times MV_b} \cong \Phi_M^-.$$

The object Φ_M^- is defined by

$$a_M^{-1}\Phi_M^- \cong F^- *_G F^- *_G \cdots *_G F^-$$

(total M copies of F^- and we use the same notation as in Sec. 6.1.)

The object $F^- *_G F^- *_G \cdots *_G F^-$ (M times) is the same as

$$Rm_!(F^-)^{\boxtimes M},$$

where $m : (G \times V_b^-)^M \to G \times (V_b^-)^M$ is induced by the product on G. The map m is proper, so we can estimate the microsupport of $Rm_!(F^-)^{\boxtimes M}$ in the standard way. Using Lemma 6.3, we conclude that $Rm_!(F^-)^{\boxtimes M}$ is microsupported within the set of points of the form

$$(e^{X_1}e^{X_2}\cdots e^{X_M}; A_1, A_2, \ldots, A_M; \omega, \eta_1, \eta_2, \ldots, \eta_M),$$

where $(X_k, A_k, \omega, \eta_k) \in \mathrm{SS}(F^-)$ (we use the the equality $[X_k, \omega] = 0$).

By Lemma 6.3, $\eta_1 = \eta_2 = \cdots = \eta_M = \|\omega\|$. This implies that the object Φ^-_M is microsupported within the set of all points of the form

$$(e^{X_1}e^{X_2}\cdots e^{X_M}; A_1 + A_2 + \cdots + A_M; \omega, \|\omega\|),$$

where $(e^{X_k}; A_k; \omega; \|\omega\|) \in \mathrm{SS}(F^-)$ for all k.

Lemma 6.3 says that $[X_k, \omega] = 0$. By Lemma 10.5, there exists $X \in \mathfrak{g}$ such that

$$e^X = e^{X_1}e^{X_2}\cdots e^{X_M}; \quad [X, \omega] = 0;$$

$$\mathrm{Tr} X|_{V_k(\omega)} = \sum_k \mathrm{Tr} X_k|_{V_k(\omega)};$$

$$\|X\| \leq \|X_1\| + \|X_2\| + \cdots + \|X_M\|$$

According to Lemma 6.3,

$$\sum_k \mathrm{Tr} X_k|_{V_k(\omega)} = -i < \sum_k A_k; e_{d_k} >;$$

$$\sum_k \|X_k\| \leq -\sum_k A_k.$$

This implies the statement. $\square$

Using this Lemma we can easily estimate the microsupport of $j^{-1}_{C^\circ_-}\mathcal{S}$.

Proposition 7.2. *The object $j^{-1}_{C^\circ_-}\mathcal{S}$ is microsupported within the set of all points of the form $(z; A; \eta) \subset \mathbf{Z} \times C^\circ_- \times \mathfrak{h}^*$, where there exists $B \in C_-$ such that*

1) $e^{-B} = z$;

2) $B \geq A$ (i.e. $\forall k :< B - A, e_k > \geq 0$);

3) $\eta \in C_+$ (i.e. $\forall k :< \eta, f_k > \geq 0$); if $< \eta, f_k > > 0$, then $< B - A, e_k >= 0$.

Proof. Let $I : Z \times C^\circ_- \hookrightarrow G \times C^\circ_-$ be the embedding. We have $j^{-1}_{C^\circ_-}\mathcal{S} = I^{-1}j^{-1}_{C^\circ_-}\mathfrak{S}[-\dim G]$. The just proven Lemma implies that $j^{-1}_{C^\circ_-}\mathfrak{S}$ is non-singular with respect to the embedding I (i.e. given a point $\zeta \in \mathrm{SS}(j^{-1}_{C^\circ_-}\mathfrak{S})$ where $\zeta \in T^*_x(G \times C^\circ_-)$, $x \in \mathbf{Z} \times C^\circ_-$, and $I^*\zeta = 0$, one then has $\zeta = 0$).

Therefore, the microsupport $I^{-1}j^{-1}_{C^\circ_-}\mathfrak{S}[-\dim G]$ consists of all points of the form $I^*\zeta$, where $\zeta \in \mathrm{SS}(j^{-1}_{C^\circ_-}\mathfrak{S})$, $\zeta \in T^*_x(G \times C^\circ_-)$, $x \in \mathbf{Z} \times C^\circ_-$.

Thus the microsupport $I^{-1}j^{-1}_{C^\circ_-}\mathfrak{S}$ is contained in the set of all points of the form

$$(e^X, A, \eta) \in \mathbf{Z} \times C^\circ_- \times \mathfrak{h}^*,$$

where there exists $\omega \in \mathfrak{g}^*$ such that $(e^X, A, \omega, \eta) \in \mathrm{SS} j_{C_-^\circ}^{-1} \mathfrak{S}$. According to the previous Lemma, this implies that $\|X\| \leq -A$; $\|\omega\| = \eta$ (so $\eta \in C_+$).

This means that $\eta = i(\lambda^1(\omega), \lambda^2(\omega), \ldots, \lambda^N(\omega))$, where $\lambda^1(\omega) \geq \lambda^2(\omega) \geq \cdots \geq \lambda^N(\omega)$ is the spectrum of $-i\omega$ (with multiplicities).

It is clear that the flag $V_\bullet(\omega)$ contains a k-dimensional subspace iff $\lambda^k(\omega) > \lambda^{k+1}(\omega)$ which is the same as $< \eta, f_k > > 0$.

Denote this k-dimensional subspace by V^k. We then know that $XV^k \subset V^k$ and $\mathrm{Tr} X|_{V^k} = -i < A, e_k >$. On the other hand we know that

$$\mathrm{Tr} - iX|_{V^k} \leq < \|X\|, e_k >,$$

for any $X \in \mathfrak{g}$. Hence,

$$< -A, e_k > \leq < \|X\|, e_k > .$$

As $\|X\| \leq -A$, this means that $< -A, e_k > = < \|X\|, e_k >$.

Let us now set $B := -\|X\|$. We see that thus defined B satisfies all the conditions. □

7.1.1. Let us reformulate the just proven Proposition.

Let $\Lambda \subset C_-$ be a discrete subset.

Let $X(\Lambda) \subset C_-^\circ \times C_+ \subset T^* C_-^\circ$ consist of all points (A, η) such that there exists a $B \in \Lambda$ satisfying:

1) $B \geq A$;

2) If $< \eta, f_k > > 0$, then $< B - A, e_k > = 0$.

For $z \in \mathbf{Z}$ let $S_z \in D(C_-^\circ)$ be the restriction

$$S_z := j_{C_-^\circ}^{-1}(\mathcal{S}|_{z \times C_-^\circ}) = \mathfrak{S}|_{z \times C_-^\circ}.$$

Let $\mathbb{L}_z^- := \{B \in C_- | e^{-B} = z\}$. $\mathbb{L}_z^-$ is an intersection of a discrete lattice in $\mathfrak{h}$ with C_-, hence is itself discrete.

Proposition 7.2 can be now reformulated as:

Proposition 7.3. *We have $SS(\mathcal{S}_z) \subset X(\mathbb{L}_z^-)$*

7.2. **Sheaves with microsupport of the form $X(\Lambda)$.** Fix a discrete subset $\Lambda \subset C_-$. One can number elements of Λ in such a way that $\Lambda = \{m_1, m_2, \ldots, m_n, \ldots\}$ and m_n is a maximum of $\Lambda \backslash \{m_1, m_2, \ldots, m_{n-1}\}$ with respect to the partial order on C_-.

For $x \in C_-$ we set $U_x^- \subset C_-^\circ$ to consist of all $y \in C_-^\circ$ such that $y << x$.

Proposition 7.4. *Let $F \in D(C_-^\circ)$ be microsupported on the set $X(\Lambda)$. Then there exists an inductive system of objects in $D(C_-^\circ)$:*

$$F = F_0 \to F_1 \to F_2 \to \cdots F_n \to \cdots ,$$

such that 1) $L\varinjlim_n F_n = 0$;

2) We have isomorphisms

$$M_n \otimes_{\mathbb{K}} \mathbb{K}_{U_{m_n}^-} \to Cone(F_{n-1} \to F_n),$$

for certain graded vector spaces M_n.

7.2.1. *Lemma.*

Lemma 7.5. *Let $U \subset V \subset \mathbb{R}^n$ be open convex sets. Let $\gamma \subset \mathbb{R}^n$be an open proper cone. Let $\gamma^\circ \subset \mathbb{R}^n$ be the dual closed cone; $\gamma^\circ = \{v| < \gamma, v >\geq 0\}$. Suppose that $V \subset U - \gamma$. Let $F \in D(V)$ be such that $SS(F) \subset V \times \gamma^\circ$. Then the restriction map $R\Gamma(V, F) \to R\Gamma(U, F)$ is an isomorphism*

Proof. Let $X \subset U \times V$ to consists of all pairs $(u, v) \in U \times V$ such that $v - u \in -\gamma$.

Let $\phi : X \times (0, 1) \to V$; $F(u, v) = (1 - t)u + tv$. We see that ϕ is a smooth fibration with contractible fiber of dimension $n + 1$. Therefore, the object $\phi^{-1}F$ is microsupported on the set of those 1-forms which are ϕ-pullbacks of 1-forms in the microsupport of F. Let $E := \mathbb{R}^n$. Identify $T^*V = V \times E^*$;

$$T^*(X \times (0, 1)) = X \times (0, 1) \times E^* \times E^* \times \mathbb{R}.$$

We then have $\mathrm{SS}(F) \subset V \times \gamma^\circ$;

$$\mathrm{SS}(\phi^{-1}F) \subset \{(u, v, t, (1 - t)\eta; t\eta; < v - u, \eta >)\},$$

where $\eta \in \gamma^\circ$.

Here we have used the formula

$$< \eta, d((1 - t)u + tv) = (1 - t) < \eta, du > +t < \eta, dv > + < \eta, (v - u) > dt.$$

As $v - u \in -\gamma, \eta \in \gamma^\circ$, we see that

$$\mathrm{SS}(\phi^{-1}F) \subset \{(u, v, t, \eta_1, \eta_2, k)|k \leq 0\}.$$

Let $S \subset X \times (0, 1)$ be any open subset such that for any $(u, v) \in X$, the set of all $t \in (0, 1)$ such that $(u, v, t) \in S$ is of the form $(0, T(u, v))$ for some $T(u, v) > 0$. It then follows that the restriction map

$$R\Gamma(X, \phi^{-1}F) \to R\Gamma(S; \phi^{-1}F)$$

is an isomorphism.

Let now $S := \phi^{-1}U$. It is easy to see that all the conditions are satisfied. It also follows that the projection $\phi_U : S \to U$ induced by ϕ is a smooth fibration with contractible fiber.

We have a commutative diagram

(66)
$$\begin{tikzcd} R\Gamma(V, F) \arrow[r] \arrow[d] & R\Gamma(U, F) \arrow[d] \\ R\Gamma(X \times (0, 1); \phi^{-1}F) \arrow[r] & R\Gamma(S; \phi^{-1}F) \end{tikzcd}$$

Coming from the Cartesian square

$$\begin{tikzcd} S \arrow[r] \arrow[d] & X \times (0, 1) \arrow[d] \\ U \arrow[r] & V \end{tikzcd} .$$

As the fibrations $S \to U$ and $X \times (0, 1) \to V$ have contractible fibers, the vertical arrows in (66) are isomorphisms. So is the low horizontal arrow. Hence the upper vertical arrow is also an isomporphism. $\square$

7.2.2.

Lemma 7.6. *We have*

$$R\,\mathrm{hom}(\mathbb{K}_{U_x^-};\mathbb{K}_{U_y^-})\cong\mathbb{K}$$

if $x\leq y$. *Othewise* $R\,\mathrm{hom}(\mathbb{K}_{U_x^-};\mathbb{K}_{U_y^-})=0$.

Proof. If $x\leq y$, we have an isomorphism

$$R\,\mathrm{hom}(\mathbb{K}_{U_x^-};\mathbb{K}_{U_y^-})=R\,\mathrm{hom}(\mathbb{K}_{U_x^-};\mathbb{K}_{U_x^-})=\mathbb{K}$$

because U_x^- is a convex hence contractible set.

If it is not true that $x\leq y$, then x does not belong to the closure of U_y^- and there exists a convex neighborhood W of x in $\mathfrak{h}$ such that W still does not intersect the closure of U_y^-. Let $V:=U_x^-\cap W$. According to the previous Lemma, we have an isomorphism

$$R\,\mathrm{hom}(\mathbb{K}_{U_x^-};\mathbb{K}_{U_y^-})\to R\,\mathrm{hom}(\mathbb{K}_V;\mathbb{K}_{U_y^-})=0.$$

Indeed, $\mathbb{K}_{U_y^-}$ is microsupported within the set $C_-^\circ\times C_+$. The dual cone to C_+ is $\gamma:=\{x|x\geq 0\}$ and $U_x^-=V-\gamma$. □

7.2.3. *Lemma.* Let E_1,E be real finitely-dimensional vector spaces and let $U\subset E_1\times E$ be an open convex set. Let $\gamma\subset E^*$ be a closed proper cone such that γ is the closure of its interior $\mathrm{Int}\gamma$. Let $\delta\subset E$ be the dual closed cone. Let $x,y\in E$, $y-x\in\mathrm{Int}\delta$. Let $V\subset E_1$ be an open subset such that $V\times((x+\delta)\cap(y-\mathrm{Int}\delta))\subset U$. Let $H:=V\times((x+\delta)\cap(y-\mathrm{Int}\delta))$.

Let us identify $T^*U=U\times E_1^*\times E^*$. Let $F\in D(U)$ be such that $\mathrm{SS}(F)\subset U\times E_1^*\times\gamma$.

Lemma 7.7. *We have* $Rhom(\mathbb{K}_H;F)=0$.

Proof. Choose vectors $e\in\mathrm{Int}\gamma$ and $f\in\mathrm{Int}\delta$. We have $<e,f>>0$. Let $E':=\mathrm{Ker}e$. We have $E=\mathbb{R}.f\oplus E'$; $E^*=\mathbb{R}.e\oplus(E')^*$. Let $\varepsilon>0$. Let $T_\varepsilon:E\to E$ be given by $T_\varepsilon|_{E'}=\mathrm{Id}$; $T_\varepsilon(f)=\varepsilon f$. Let $\delta_\varepsilon:=T_\varepsilon\delta$.

There exists a sequence of points $y_n\in E_2$, $\varepsilon_n\in(0,1)$ such that

$$(x+\delta)\cap(y_n-\delta_{\varepsilon_n})\subset(x+\delta)\cap(y_m-\mathrm{Int}\delta_{\varepsilon_m})$$

for all $n<m$ and

$$\bigcup_n(x+\delta)\cap(y_n-\mathrm{Int}\delta_{\varepsilon_n})=(x+\delta)\cap(y-\mathrm{Int}\delta).$$

We then have

$$\mathbb{K}_{(x+\delta)\cap(y-\mathrm{Int}\delta)}=\varinjlim_n\mathbb{K}_{(x+\delta)\cap(y_n-\mathrm{Int}\delta_{\varepsilon_n})}.$$

Therefore, it suffices to show that

$$R\,\mathrm{hom}(\mathbb{K}_{V\times((x+\delta)\cap(y_n-\mathrm{Int}\delta_{\varepsilon_n}))};F)=0.$$

More precisely, given $z\in E$, $\varepsilon\in(0,1)$, and any open $W_1\subset V$ such that the closure of W_1 is contained in V and $(x+\delta)\cap(z-\delta_\varepsilon)\subset(x+\delta)\cap(y-\mathrm{Int}\delta)$, we will show

$$R\,\mathrm{hom}(\mathbb{K}_{W_1\times(x+\delta\cap z-\mathrm{Int}\delta_\varepsilon)};F)=0$$

It follows that there exists an open convex $W_2\subset E$ such that $W_1\times W_2\subset U$ and

$$(x+\delta)\cap(z-\delta_\varepsilon)\subset W_2.$$

Indeed, let $\overline{V}$ be the closure of V. Then $\overline{V}\times((x+\delta)\cap(z-\delta_\varepsilon))\subset U$. As both sets in this product are compact and U is open, there exists a neighborhood W_2 of $(x+\delta)\cap(z-\delta_\varepsilon)$ such that $\overline{V}\times W_2\subset U$.

There exists $z' \in (z+\mathrm{Int}\delta_\varepsilon)\cap W_2$ such that $(x+\delta)\cap(z'-\delta_\varepsilon)\subset W_2$ Let $Z:=(z'-\mathrm{Int}\delta_\varepsilon)\cap W_2$ so that $z\in Z$ and for any $u\in Z$, $(x+\delta)\cap(u-\mathrm{Int}\delta_\varepsilon)\subset W_2$ (because $(u-\mathrm{Int}\delta_\varepsilon)\subset(z'-\mathrm{Int}\delta_\varepsilon)$).

Let $G\subset W_2\times Z$ be the following locally closed subset:

$$G=\{(w,u)|w\in x+\delta\cap u-\mathrm{Int}\delta_\varepsilon\}.$$

Let $p:W_1\times W_2\times Z\to W_1\times W_2$ and $q:W_1\times W_2\times Z\to W_1\times Z$.

Let $\Phi:=F|_{W_1\times W_2}$.

We will show $Rq_*\underline{\mathrm{Hom}}(\mathbb{K}_{W_1\times G};p^!\Phi)=0$. by computing microsupports.

Let us first study $\mathrm{SS}(\mathbb{K}_G)$, where $\mathbb{K}_G\in D(W_2\times Z)$.

We have

$$\mathbb{K}_G=\mathbb{K}_{G_1}\otimes\mathbb{K}_{G_2},$$

where $G_1,G_2\subset W_2\times Z$, $G_1=(x+\delta\cap W_2)\times Z$; $G_2=\{(w,u)|w-u\in-\mathrm{Int}\delta_\varepsilon\}$.

We have $\mathrm{SS}(\mathbb{K}_{G_1})$ is contained within the set of all points $(w,u,\eta,0)\in W_2\times Z\times E_2^*\times E_2^*$, where $\eta\in\gamma$.

Similarly, $\mathrm{SS}(\mathbb{K}_{G_2})$ is contained within the set of all points $(w,u,\zeta,-\zeta)$, where $\zeta\in\gamma_{1/\varepsilon}$ ($\gamma_{1/\varepsilon}:=T_{1/ve}\gamma$ is the dual cone to δ_ε).

Therefore, $\mathbb{K}_G$ is microsupported within the set of all points of the form

$$(w,u,\eta+\zeta,-\zeta),$$

where w,u,η,ζ are as above.

Hence $\mathrm{SS}(\mathbb{K}_{W_1\times G})$ is contained within the set of all points of the form

$$(w_1,w_2,u,0,\eta+\zeta,-\zeta)\in W_1\times W_2\times Z\times E_1^*\times E_2^*\times E_2^*.$$

The object $p^!\Phi$ is microsupported within the set of all points of the form

$$(w_1,w_2,u,\alpha,\kappa,0)\in W_1\times W_2\times Z\times E_1^*\times E_2^*\times E_2^*,$$

where $\alpha\in E_1^*$ is arbitrary and $\kappa\notin\mathrm{Int}\gamma$.

It follows that $\underline{\mathrm{Hom}}(\mathbb{K}_{W_1\times G};p^!\Phi)$ is microsuported within the set of all points of the form

$$(w_1,w_2,u,\alpha,\kappa-\eta-\zeta;\zeta),$$

where η,ζ,κ are same as before.

The map q is proper on the support of $\underline{\mathrm{Hom}}(\mathbb{K}_{W_1\times G};p^!\Phi)$, because the latter is contained within the set

$$W_1\times((x+\delta)\cap(z'-\delta_\varepsilon))\times Z,$$

and $(x+\delta)\cap(z'-\delta_\varepsilon)\subset W_2$ is compact. Therefore, $Rq_*\underline{\mathrm{Hom}}(\mathbb{K}_{W_1\times G};p^!\Phi)$ is contained within the set of all points of the form

$$(w,u,\alpha,\zeta)\in W_1\times Z\times E_1^*\times E_2^*,$$

where α is arbitrary, $\zeta\in\gamma_{1/\varepsilon}$, and there exist κ,η as above, such that $\kappa-\eta-\zeta=0$. The latter is only possible if $\zeta=0$ (otherwise $\zeta+\eta\in\mathrm{Int}\gamma$ because

$$\gamma_{1/\varepsilon}\subset\{0\}\cup\mathrm{Int}\gamma.$$

Thus, $Rq_*\underline{\mathrm{Hom}}(\mathbb{K}_{W_1\times G};p^!\Phi)$ is microsupported within the set of all points of the form $(w,u,\alpha,0)$, i.e. is locally constant along Z. There exists a convex open subset $U_0\subset Z$, $U_0\subset x-\delta$. It follows that $G\cap(W_2\times U_0)=\emptyset$. Therefore,

$$Rq_*\underline{\mathrm{Hom}}(\mathbb{K}_{W_1\times G};p^!\Phi)|_{W_1\times U_0}=0.$$

This implies that

$$Rq_*\underline{\mathrm{Hom}}(\mathbb{K}_{W_1\times G}; p^!\Phi) = 0,$$

because Z is convex, and our object is locally constant along Z.

Therefore,

$$\begin{aligned}
0 &= R\hom(\mathbb{K}_{W_1\times z}; Rq_*\underline{\mathrm{Hom}}(\mathbb{K}_{W_1\times G}; p^!\Phi)\\
&= R\hom(\mathbb{K}_{W_1\times G}\otimes \mathbb{K}_{W_1\times W_2\times z}; p^!\Phi)\\
&= R\hom(\mathbb{K}_{W_1\times(x+\delta\cap z-\mathrm{Int}\delta_\varepsilon)\times z}; p^!\Phi)\\
&= R\hom(\mathbb{K}_{W_1\times(x+\delta\cap z-\mathrm{Int}\delta_\varepsilon)}; \Phi),
\end{aligned}$$

as was required. □

7.2.4. *Lemma.*

Lemma 7.8. *Let $x, y\in C_-$, $y>x$. Let $I_x := \{k| < x, f_k >< 0\}$. There exists $k\in I_x$ such that $< y-x, e_k >> 0$.*

Proof. Assume the contrary, i.e. $< y-x, e_k >= 0$ for all $k\in I_x$. Let $z = y-x$ and let $z_k =< z, e_k >$ so that $z_k = 0$ for all $k\in I_x$. If $l\notin I_x$, then $z_l\geq 0$ and $< z, f_l >=< y, f_l >\leq 0$. On the other hand, $< z, f_l >= 2z_l - z_{l-1} - z_{l+1}$ (we set $z_0 = z_N = 0$). For $l\notin I_x$ let $[a, b]\subset [1, N-1]$ be the largest interval containing l and not intersecting with I_x. We then have $z_{A-1} = z_{B+1} = 0$;

$$0\geq -z_A\geq z_A - z_{A+1}\geq z_{A+1} - z_{A+2}\geq\cdots\geq z_B\geq 0$$

(because for any $l\notin I_x$, $2z_l - z_{l-1} - z_{l+1}\leq 0$). This implies that $z_A = z_{A+1} = \cdots = z_B = 0$. Hence, $z_l = 0$ for all l, $z = 0$, and $y = x$, which contradicts to $y > x$. □

7.2.5. *Lemma.*

Lemma 7.9. *Let $F\in D(C_-^\circ)$ be such that $SS(F)\subset X(\Lambda)$ and assume that for all $l\in\Lambda$, $R\Gamma(U_l^-; F) = 0$. Then $F = 0$.*

Proof. Consider open subsets of C_-° of the form $U\cap U_x^-$ where U is open and convex and $x\in C_-$. These sets form a base of topology of C_-°. Thus, it suffices to show $R\Gamma(U\cap U_x^-; F) = 0$ for all such U, U_x^-. By Lemma 7.5, we have an isomorphism

$$R\Gamma(U_x^-; F)\to R\Gamma(U\cap U_x^-; F).$$

Thus, it suffices to show that $R\Gamma(U_x^-; F) = 0$ for all x.

Given $x\in C_-$, let $\Lambda_x := \{l\in\Lambda | l\geq x\}$. Let $N_x = |\Lambda_x|$. Let us prove the statement by induction with respect to N_x.

If $N_x = 0$, then there are no points in $X(\Lambda)$ which project to x. Hence $x\notin \mathrm{Supp}F$. Therefore, there exists a convex neighborhood of U of x such that $F|_U = 0$. Therefore, we have an isomorphism

$$R\Gamma(U_x^-; F)\xrightarrow{\sim} R\Gamma(U\cap U_x^-; F) = 0.$$

Suppose now that $R\Gamma(U_x^-; F)$ for all x with $N_x < n$. Prove that the same is true for all x with $N_x\leq n$. Let $S\subset C_-$ be the set of all points y such that $\Lambda_y = \Lambda_x$. Let $t_k := \sup_{y\in S} < y, e_k >$. As $S\in C_-$, $t_k\geq 0$. Let

$$x' := \sum_{k=1}^{N-1} t_k f_k.$$

Let us show $x' \in C_-$. This is equivalent to $< x'; f_l > \leq 0$ for all l. We have $< x', f_l >= 2 < x', e_l >$ $- < x', e_{l-1} > - < x'; e_{l+1} >$. We have

$$2 < x', e_l >= \sup_{y \in S} 2 < y, e_l > \leq \sup_{y \in S} < y, e_{l-1} > + < y, e_{l+1} >$$
$$\leq < x', e_{l-1} > + < x', e_{l+1} > .$$

Thus, $x' \in C_-$. It then easily follows that $x' \in S$.

It is clear that $x' \geq x$. Let us show that the restriction map $R\Gamma(U^-_{x'}; F) \to R\Gamma(U_x; F)$ is an isomorphism. If $x' = x$, there is nothing to prove, so assume $x' > x$. Let $I := \{tx + (1-t)x' | 0 \leq t < 1\}$. Let $K := \{k | < x' - x, e_k >> 0\}$. Ler U' be a convex neighborhood of 0 in $\mathfrak{h}$. Let $U := C^\circ_- \cap U'$. We then see that

1) $I + U \subset C^\circ_-$ is convex and open;

2) For U' small enough the following is true. Given any $y \in I + U$, we have $\Lambda_y = \Lambda_x$; for any $l \in \Lambda_y$ and for any $k \in K$, $< l - y, e_k >> 0$.

The restriction maps

$$R\Gamma(U^-_{x'}; F) \to R\Gamma(I + U; F);$$
$$R\Gamma(U^-_x; F) \to R\Gamma(x + U; F)$$

are isomorphisms by Lemma 7.5. Hence it suffices to show that the restriction map

$$R\Gamma(I + U; F) \to R\Gamma(x + U; F) \tag{67}$$

is an isomorphism.

It follows from the definition of $X(\Lambda)$ that $F|_{I+U}$ is microsupported within the set of all points $(y, \eta) \in (I + U) \times \mathfrak{h}^*$ such that $< \eta, f_k >= 0$ for all $k \in K$. Hence, $< \eta, x' - x >= 0$. This implies that that (67) is an isomorphism.

We can now assume $x = x'$. By the construction of $x = x'$, given any point $y \in C_-$, $y > x$, the set Λ_y is a proper subset of Λ_x. If $x \in \Lambda$ there is nothing to prove. Assume $x \notin \Lambda$. Let $I_x := \{k| < x, f_k >< 0\}$. By Lemma 7.8 for any $l \in \Lambda_x$ there exists $k \in I_x$ such that $< l - x, e_k >> 0$. It follows that there exists a neighborhood U' of x in C_- such that for all $y \in U'$, $\Lambda_y \subset \Lambda_x$ and for all $l \in \Lambda_y$ there exists $k \in I_x$ such that $< l - y, e_k >> 0$. Let $U = U' \cap C^\circ_-$. It follows that $F|_U$ is microsupported within the set of all points of the form

$$(u, \eta) \in U \times \mathfrak{h}^*,$$

where $< \eta, f_k >= 0$ for some $k \in I_x$.

Let $\mathcal{V} \subset \mathfrak{h}$ be the $\mathbb{R}$-span of all f_k, $k \in K$.

It follows that there exists $\varepsilon > 0$ such that $x + \sum_{k \in I_x} t_k f_k \in U'$ if for all $k \in I_x$, $t_k \in [0, \varepsilon]$. Indeed, let $U' = W \cap C_-$, where W is a neighborhood of x in $\mathfrak{h}$. It is clear that for ε small enough, $x + \sum_{k \in I_x} t_k f_k \in W$. As $< x, f_k >< 0$ for all $k \in I_x$, for all ε small enough and for all $k' \in I_x$ we have: $< x + \sum_{k \in I_x} t_k f_k, f_{k'} >< 0$. If $\lambda \notin I_x$, then $< x + \sum_{k \in I_x} t_k f_k, f_\lambda >= \sum_{k \in I_x} t_k < f_k, f_\lambda > \leq 0$, because $< f_k, f_\lambda > \leq 0$ for all $k \neq \lambda$. Thus,

$$x + \sum_{k \in I_x} t_k f_k \in C_-.$$

Fix $\varepsilon > 0$ as above. There also exists $\varepsilon_1 > 0$ such that

$$x + \sum_{k \in I_x} t_k f_k + \sum_{\lambda=1}^{N-1} a_\lambda e_\lambda \in C^\circ_-$$

as long as $t_k \in [0,\varepsilon]$ and $0 > a_\lambda > -\varepsilon_1$.

Let $U_{\varepsilon_1} := \{x + \sum a_\lambda e_\lambda | 0 > a_\lambda > -\varepsilon_1\} \subset \mathfrak{h}$;

$$M_\varepsilon := \{\sum_{k \in I_x} t_k f_k | 0 < t_k < \varepsilon\} \subset \mathcal{V}.$$

Let $A : \mathfrak{h} \times \mathcal{V} \to \mathfrak{h}$ be the addition map. There exists an open convex neighborhood $\mathcal{U} \in \mathfrak{h} \times \mathcal{V}$ of $U_{\varepsilon_1} \times M_\varepsilon$ such that $A(\mathcal{U}) \subset U$. Let $\alpha : \mathcal{U} \to A(\mathcal{U}) \subset U$ be the map induced by A. As $\mathcal{U}$ is convex, $\alpha : \mathcal{U} \to A(\mathcal{U})$ is a smooth fibration. Let $\Phi := \alpha^!(F|_{A(\mathcal{U})})$. It follows that $\mathrm{SS}(\Phi)$ consists of pull-backs of 1-forms from $\mathrm{SS}(F)$. Thus, $\mathrm{SS}(\Phi)$ is contained in the set of all points of the form

$$(A, u, \eta, \kappa) \in \mathfrak{h} \times \mathcal{V} \times \mathfrak{h}^* \times \mathcal{V}^*,$$

where $(A, u) \in \mathcal{U}$ and there exists $k \in I_x$ such that $< \kappa, f_k >= 0$. By Lemma 7.7, we have

$$R\hom(\mathbb{K}_{U_{\varepsilon_1} \times G}; \Phi) = 0,$$

where $G = \{\sum_{k \in K} t_k f_k | 0 \le t_k < \varepsilon\}$.

For $L \subset I_x$, let $G_L := \{\sum_{l \in L} t_l f_l | 0 < t_l < \varepsilon\}$. Set $G_\emptyset := \{0\}$. We have a natural map

$$\mathbb{K}_{U_{\varepsilon_1} \times G} \to \mathbb{K}_{U_{\varepsilon_1} \times G_\emptyset}.$$

The cone of this map is obtained from sheaves $\mathbb{K}_{U_{\varepsilon_1} \times G_L}$, $L \neq \emptyset$, by means of succesive extensions.

We also have

$$R\hom(\mathbb{K}_{U_{\varepsilon_1} \times G_L}; \Phi) = R\Gamma(A(U_{\varepsilon_1} \times G_L); \Phi).$$

We have

$$A(U_{\varepsilon_1} \times G_L) \subset U^-_{x + \sum_{l \in L} \varepsilon f_l}.$$

By Lemma 7.5 the restriction map

$$R\Gamma(U^-_{x + \sum_{l \in L} \varepsilon f_l}; F) \to R\Gamma(A(U_{\varepsilon_1}, G_L); F)$$

is an isomorphism. As $x + \sum_{l \in L} \varepsilon f_l > x$ for $L \neq \emptyset$, we have

$$R\Gamma(U^-_{x + \sum_{l \in L} \varepsilon f_l}; F) = 0$$

and

$$R\hom(\mathbb{K}_{U_{\varepsilon_1} \times G_L}; \Phi) = 0$$

for all $L \neq \emptyset$. Therefore

$$R\hom(\mathrm{Cone}(\mathbb{K}_{U_{\varepsilon_1} \times G} \to \mathbb{K}_{U_{\varepsilon_1} \times G_\emptyset}); \Phi) = 0.$$

Therefore

$$0 = R\hom(\mathbb{K}_{U_{\varepsilon_1} \times G_\emptyset}; \Phi) = R\hom(U^-_x; F)$$

$\square$

7.2.6. *Proof of Proposition 7.4.* Let us construct objects $F_n \in D(C_-^\circ)$ by induction. Set $F_0 = F$. Set $M_n := R\Gamma(U_{m_n}^-; F_{n-1})$ and

$$F_n := \mathrm{Cone}(\alpha_n : M_n \otimes \mathbb{K}_{U_{m_n}^-} \to F_{n-1}),$$

where α_n is the natural map.

We have structure maps $i_n : F_{n-1} \to F_n$ so that the sheaves F_n form an inductive system. This system stabilizes on any compact $K \subset C_-^\circ$ because for n large enough, $K \cap U_{m_n}^- = \emptyset$.

Let $G := L\varinjlim_n F_n$. It follows that $\mathrm{SS}(G) \subset X(\Lambda)$ (because $\mathrm{SS}(F_n) \subset X(\Lambda)$).

Let U_n be a neighborhood of m_n in C_-° such that the closure of U_n in C_-° is compact. We have

$$R\Gamma(U_{m_n}^-; G) \cong R\Gamma(U_{m_n}^- \cap U_n; G)$$

$$\cong R\Gamma(U_{m_n}^- \cap U_n; F_N) \cong R\Gamma(U_{m_n}^-; F_N)$$

for N large enough.

Let $S^i := R\Gamma(U_{m_n}^-; F_i)$ As follows from Lemma 7.6, $S^i = S^{i+1}$ for all $i \geq n$; also, by construction, $S^n = 0$. Thus $S^N = 0$ for $N \geq n$. Therefore,

$$R\Gamma(U_{m_n}^-; G) = 0$$

for all n and $G = 0$ by Lemma 7.9.

Next, $\mathrm{Cone}(F_n \to F_{n-1})$ is isomorphic to $M_n \otimes \mathbb{K}_{U_{m_n}^-}$. This proves the proposition.

7.3. **Invariant definition of the spaces M_n.** The goal of this section is to define spaces M_n from Proposition 7.4 in a more invariant way.

7.3.1. *Lemma.* As in the previous Lemma, let $x \in C_-$ and let $I_x := \{k| < x, f_k >< 0\}$. As was shown in the previous Lemma, there exists $\varepsilon > 0$ such that $x + \sum_{k \in I_x} t_k f_k \in C_-$ as long as all $t_k \in [0, \varepsilon]$. Fix such a $\varepsilon > 0$.

Set

$$V := V(x, \varepsilon) := \{y \in C_-^\circ | \forall k \in I_x :< y - x, e_k > \in [0, \varepsilon); \forall l \notin I_x :< y - x, e_l >< 0.\}$$

Lemma 7.10. *1) We have*

$$R\hom(\mathbb{K}_V; \mathbb{K}_{U_x^-}) \cong \mathbb{K}[-|I_x|].$$

2) Let $y \in C_-$. Suppose there exists $k \in \{1, 2, \ldots, N-1\}$ such that either $k \in I_x$ and $< y - x, e_k > \notin [0, \varepsilon]$ or $k \notin I_x$ and $< y, e_k > << x, e_k >$. Then

$$R\hom(\mathbb{K}_V; \mathbb{K}_{U_y^-}) = 0.$$

Proof. For $L \subset I_x$ set

$$f_L := \varepsilon \sum_{l \in L} f_l.$$

For every $k \in I_x$ we have a natural map

$$\mathbb{K}_{U^-_{x + f_{I_x - \{k\}}}} \to \mathbb{K}_{U^-_{x + f_{I_x}}}.$$

Let C_k be the corresponding 2-term complex, we put $\mathbb{K}_{U^-_{x+f_{I_x}}}$ into degree 0.

Consider the complex

$$D := \bigotimes_{k \in I_x} C_k$$

We have

$$D^{-i} = \bigoplus_L \mathbb{K}_{U^-_{x+f_L}},$$

where the sum is taken over all $|I_x| - i$-element subsets L of I_x.

In particular $D^0 = \mathbb{K}_{U^-_{x+f_{I_x}}}$. As $V \subset U^-_{x+f_{I_x}}$ is a closed subset, we have a natural map

$$\mathbb{K}_{U^-_{x+f_{I_x}}} \to \mathbb{K}_V.$$

This map defines a map of complexes $D \to \mathbb{K}_V$ which is a quasi-isomrorphism. Therefore, we have an isomorphism

$$R\hom(\mathbb{K}_V; \mathbb{K}_{U^-_y}) \to R\hom(D; \mathbb{K}_{U^-_y}).$$

Let $y = x$, then, according to Lemma 7.6, $R\hom(\mathbb{K}_{U^-_{x+f_L}}; \mathbb{K}_{U^-_x}) = 0$ for all $L \neq \emptyset$. For $L = \emptyset$, we have

$$R\hom(\mathbb{K}_{U^-_x}; \mathbb{K}_{U^-_x}) = \mathbb{K}.$$

Therefore, we have an isomorphism

$$R\hom(D, \mathbb{K}_{U^-_x}) \cong \mathbb{K}[-|I_x|].$$

Let now $y \in C_-$ and $k \in I_x$ be such that $< y - x, e_k > \notin [0, \varepsilon]$.

Let $D_k := \bigotimes_{l \neq k} C_l$ so that we have $D = D_k \otimes C_k$. I.e

$$D \cong \mathrm{Cone}(D_k \otimes \mathbb{K}_{U^-_{x+f_{I_x - \{k\}}}} \to D_k \otimes \mathbb{K}_{U^-_{x+f_{I_x}}}), \tag{68}$$

where the map is induced by the natural map

$$\mathbb{K}_{U^-_{x+f_{I_x - \{k\}}}} \to \mathbb{K}_{U^-_{x+f_{I_x}}}.$$

We have

$$D_k^{-i} \otimes \mathbb{K}_{U^-_{x+f_{I_x - \{k\}}}} = \bigoplus_L \mathbb{K}_{U^-_{x+f_L}},$$

where the sum is taken over all $|I_x| - i - 1$-element subsets $L \subset I_x - \{k\}$.

Analogously,

$$D_k^{-i} \otimes \mathbb{K}_{U^-_{x+f_{I_x}}} = \bigoplus_L \mathbb{K}_{U^-_{x+f_L}}$$

where the sum is taken over all $|I_x| - i$-element subsets $L \subset I_x$ such that $k \in L$.

In view of these identifications, the $-i$-th degree component of the map in (68) is induced by the natural maps

$$\mathbb{K}_{U_{x+f_L}} \to \mathbb{K}_{U_{x+f_{L \cup \{k\}}}}.$$

If $< y - x, e_k > \notin [0, \varepsilon]$, then these maps induce isomorphism

$$R\hom(\mathbb{K}_{U_{x+f_{L \cup \{k\}}}}; \mathbb{K}_{U^-_y}) \to R\hom(\mathbb{K}_{U_{x+f_L}}; \mathbb{K}_{U^-_y})$$

.

Hence, the map in (68) induces an isomorphism

$$R\hom(D_k \otimes \mathbb{K}_{U^-_{x+f_{I_x}}}; \mathbb{K}_{U^-_y}) \to R\hom(D_k \otimes \mathbb{K}_{U^-_{x+f_{I_x - \{k\}}}}; \mathbb{K}_{U^-_y})$$

Therefore,

$$R\hom(D, \mathbb{K}_{U^-_y}) = 0,$$

as was stated.

If there exists $k \notin I_x$ such that $< y, e_k > << x, e_k >$, then it follows that $R\hom(\mathbb{K}_{U^-_{x+f_L}}; \mathbb{K}_y) = 0$ for all L (because it is not true that $x + f_L \leq y$). $\square$

7.3.2.

Lemma 7.11. *Let $l \in \Lambda$. There exists $\varepsilon > 0$ such that for any $l' \in \Lambda$, $l' \neq l$:*
— either there exists $k \in I_l$ such that $< l' - l, e_k > \notin [0, \varepsilon]$
— or there exists $k \notin I_l$ such that $< l', e_k ><< l, e_k >$.

Proof. If there exists $k \in \{1, 2, \dots, N-1\}$ such that $< l' - l, e_k >< 0$, then one of the conditions is satisfied. If such a k does not exist, then $l' \geq l$. There are only finitely many $l' \in \Lambda$ with this property. Hence, the statement follows from Lemma 7.8. $\square$

7.3.3. Let m_n be a numbering of Λ as in Proposition 7.4. Let ε be as in the proof of the previous Lemma.

Lemma 7.12. *Let $\varepsilon' \in (0, \varepsilon)$. We have*

$$M_n \cong R\hom(\mathbb{K}_{V(m_n, \varepsilon')}; F)[|I_{m_n}|]$$

Proof. Follows from Proposition 7.4 and two previous Lemmas. $\square$

7.4. **The sheaf $\mathcal{S}_z$.** Proposition 7.4 and Lemma 7.12 applies to $j^{-1}_{C^\circ_-}\mathcal{S}_z$ with $\Lambda = \mathbb{L}^-_z$. We would like to rewrite the expression from Lemma 7.12 in a more convenient way.

Let $x \in \mathfrak{h}$ and $I \subset \{1, 2, \dots, N-1\}$. let $W(I, x) \subset \mathfrak{h}$ be given by

$$W(I, x, \varepsilon) = \{y : \forall k \in I :< y - x, e_k > \in [0, \varepsilon); \forall k \notin I :< y - x, e_k >< 0\}.$$

For $x \in C_-$ and ε as in Sec. 7.3.1, we have

$$V := V(x, \varepsilon) = W(I_x, x, \varepsilon) \cap C^\circ_-,$$

Set $W := W(I_x, x, \varepsilon)$. Set $I := I_x$.

For any $F \in D(\mathfrak{h})$ we have an induced map of sheaves

$$R\hom_{\mathfrak{h}}(\mathbb{K}_W; F) \to R\hom_{\mathfrak{h}}(\mathbb{K}_V; F) = R\hom_{C^\circ_-}(\mathbb{K}_V; j^{-1}_{C^\circ_-}F). \tag{69}$$

Lemma 7.13. *Suppose that $SS(F) \subset \mathfrak{h} \times C_+$. Then the map (69) is an isomorphism*

Proof. For $z \in \mathfrak{h}$ set $U_z = \{y \in \mathfrak{h} | y << z\}$. Lemma 7.5 implies that for any $z \in C_-$, the restriction map

$$R\hom(\mathbb{K}_{U_z}; F) \to R\hom(\mathbb{K}_{U^-_z}; F)$$

is an isomorphism.

For $k \in I$ consider the following 2-term complex C'_k

$$\mathbb{K}_{U_{x+f_{I-\{k\}}}} \to \mathbb{K}_{U_{x+f_I}},$$

where we use the notation from proof of Lemma 7.10. Let

$$D' := \bigotimes_{k \in I} C'_k. \tag{70}$$

Similar to D, we have a quasi-isomorphism

$$D' \to \mathbb{K}_W.$$

We also have

$$(D')^{-i} = \bigoplus_L \mathbb{K}_{U_{x+f_L}},$$

where the sum is taken over all $|I| - i$-element subsets of I. We have natural maps $C_k \to C'_k$ which induce maps $D \to D'$. The latter map is induced by maps

$$\mathbb{K}_{U^-_{x+f_L}} \to \mathbb{K}_{U_{x+f_L}}$$

According to Lemma 7.5, the induced map

$$R\hom(\mathbb{K}_{U_{x+f_L}}; F) \to R\hom(\mathbb{K}_{U^-_{x+f_L}}; F)$$

is an isomorphism for all F such that $\mathrm{SS}(F) \subset \mathfrak{h} \times C_+$. This implies the statement. □

7.4.1.

Lemma 7.14. *Let $F \in D(\mathfrak{h})$ be constant along fibers of projection $\mathfrak{h} \to \mathfrak{h}/\mathbb{R}.f_k$ for some k. Then for all $I \subset \{1, 2, \ldots, N-1\}$ such that $k \in I$ and for all $\varepsilon > 0$, we have*

$$R\hom(\mathbb{K}_{W(I,x,\varepsilon)}; F) = 0$$

Proof. Follows easily from the quasi-isomorphism $D' \to \mathbb{K}_{W(I,x,\varepsilon)}$. □

7.5. **Periodicity.** Let us get back to the object $j^{-1}_{C^\circ_-}\mathcal{S}_z$. In this case $\Lambda = \mathbb{L}^-_z$. There exists $\varepsilon > 0$ such that the condition of Lemma 7.11 is satisfied for all $l \in \mathbb{L}^-_z$. Fix such a ε throughout. Proposition 7.4 applies to $F = \mathcal{S}_z$. by Lemmas 7.12 and 69 we have an isomorphism

$$M_n = R\hom(\mathbb{K}_{V(m_n,\varepsilon)}; \mathcal{S}_z)[-|I_{m_n}|] = R\hom(\mathbb{K}_{W(I_{m_n},m_n,\varepsilon)}; \mathcal{S}_z)[-|I_{m_n}|].$$

For $z \in \mathfrak{h}$ and $I \subset \{1, 2, \ldots, N-1\}$ and $F \in D(\mathfrak{h})$

$$\Delta_{I;z}(F) := R\hom(\mathbb{K}_{W(I,z,\varepsilon)}; F)[|I|]$$

Our goal is to prove the following theorem

Theorem 7.15. *For any $m \in \mathfrak{h}$, any $I \subset \{1, 2, \ldots, N-1\}$ and any $k \in I$ there exists a quasi-isomorphism*

$$\Delta_{I;m}\mathcal{S}_z \to \Delta_{I;m-2\pi e_k}\mathcal{S}_{ze^{-2\pi e_k}}[-D_k]$$

where $D_k = 2k(N-k)$.

The rest of the current subsection will be devoted to proving this Theorem.

In the next two subsections we will prove the main auxiliary result towards the proof.

7.5.1. *Sheaves $\mathfrak{S}|_{G\times -2\pi e_k}$.* Recall that $\mathfrak{S} \in D(G \times \mathfrak{h})$. Let $\Sigma_k := \mathfrak{S}|_{G\times -2\pi e_k}$, so that $\Sigma_k \in D(G)$.

Lemma 7.16. *We have an isomorphism $\Sigma_k = \mathbb{K}_{W_k}$, where $W_k \subset G$ is an open subset consisting of all points of the form*

$$W_k = \{e^{-Y} | \|Y\| < 2\pi e_k\}.$$

Proof. As follows from the proof of Theorem 6.1 Σ_k can be constructed as follows. Let us decompose $-2\pi e_k = A_1 + A_2 + \cdots A_M$, where $A_i \in V_b^-$. For $A \in C_-^\circ$ set $U(A) \subset G$; $U(A) := \{e^X | X \in \mathfrak{g}; \|X\| << -A\}$. One then has

$$\Sigma_k \cong \mathbb{K}_{U_{A_1}} *_G \mathbb{K}_{U_{A_2}} *_G \cdots *_G \mathbb{K}_{U_{A_M}}[M \dim \mathfrak{g}]$$

Let $g \in G$. It follows that $\Sigma_k|_g \neq 0$ only if there exist $X_k \in \mathfrak{g}$; $\|X_k\| << -A_k$ such that $g = e^{X_1} e^{X_2} \cdots e^{X_M}$. According to Lemma 10.4, this implies that $g = e^Y$, where $\|Y\| << -(A_1 + \cdots + A_M) = 2\pi e_k$. Thus, fibers of Σ_k at any point outside of W_k are zeros.

Let $H := \Sigma_k|_{W_k}$. It then suffices to prove that $H \cong \mathbb{K}[\dim \mathfrak{g}]$.

Let us find $\mathrm{SS}(H)$. Observe that the exponential map identifies W_k with $\{X \in \mathfrak{g} | \|X\| << 2\pi e_k\}$. Lemma 7.1 implies that $(g, \omega) \in \mathrm{SS}(H)$ only if there exists $X \in \mathfrak{g}$ such that $g = e^X$; $\|X\| \leq 2\pi e_k$, $[X, \omega] = 0$; $< \|X\| - 2\pi e_k, e_{d_r(\omega)} >= 0$ for all r. As $g \in U_{-2\pi e_k}$ and $\|X\| \leq 2\pi e_k$ we must have $\|X\| << 2\pi e_k$, so that $< \|X\| - 2\pi e_k, e_l >< 0$ for all l. This means that $\omega = 0$.

Thus, H *is a constant sheaf.*

Let us now find $H|_e$. We have $H|_e = \mathcal{S}_e|_{-2\pi e_k}$.

However, as follows from Proposition 7.2, $\mathcal{S}_e$ is constant in the domain consisting of all $A \in C_-$ such that there is no $l \in \mathbb{L}_0^-$, $l \neq 0$, $A \geq l$. Both $-2\pi e_k$ and $-e_1/100$ lie in this domain. Thus we have an isomorphism

$$\mathcal{S}_e|_{-2\pi e_k} = \mathcal{S}_e|_{-e_1/100} = \mathbb{K}[\dim \mathfrak{g}].$$

This finishes the proof. ☐

Let us compute $H_k := R\hom(\mathbb{K}_{e^{-2\pi e_k}}; \Sigma_k)$.

Let us choose a small neighborhood U of $e^{-2\pi e_k}$ in G so that $U = \{e^{-X} e^{-2\pi e_k} | \|X\| << b\}$. Let us describe the set $U_k := U \cap W_k$. Let $g \in U \cap W_k$. As $g \in W_k$, we have $g = e^{-Y}$ where $\|Y\| << 2\pi e_k$ which simply means that $\lambda_1(Y) < 2\pi(N-k)/N$; $\lambda_N(Y) > -2\pi k/N$, where

$$\lambda_1(Y) \geq \lambda_2(Y) \geq \cdots \geq \lambda_N(Y)$$

is the spectrum of a Hermitian matrix Y/i.

As $g \in U$, there must exist X, $\|X\| << b$ such that $e^{-Y} = e^{-X} e^{-2\pi e_k}$, or

$$e^Y = e^{2\pi e_k} e^X.$$

Observe that $e^{2\pi e_k} = e^{-2\pi k/N}\mathrm{Id}$. Therefore, one can number the spectrum of X/i in such a way that $\lambda^j(X) - 2\pi k/N - \lambda_j(Y) \in 2\pi\mathbb{Z}$, $j = 1 \ldots N$. In other words, there exist integers m_i such that $\lambda_j(Y) = -2\pi k/N + \lambda^j(X) + 2\pi m_i$, where m_i are integers.

As $-2\pi k/N < \lambda_j(Y) < 2\pi(N - k/N)$ and $\lambda^j(X)$ are small we see that $m_j = 0$ or $m_j = 1$. Since $\mathrm{Tr}(Y) = \mathrm{Tr}(X) = 0$, $\sum m_j = k$. Since $\lambda_1(Y) \geq \lambda_2(Y) \geq \cdots$, we conclude that $m_1 = \cdots = m_k = 1$; $m_{k+1} = m_{k+2} = \cdots m_N = 0$. We then see that

$$0 > \lambda^1(X) \geq \lambda^2(X) \geq \cdots \geq \lambda^k(X);$$

$$\lambda^{k+1}(X) \geq \cdots \geq \lambda^N(X) > 0.$$

In other words, the set W_k consists of all elements of the form $e^{-2\pi e_k} e^{-X}$ where$\|X\| << b$ and X/i has k negative eigenvalues and $N-k$ positive eigenvalues (and no 0 eigenvalues). Let $H_k \subset \mathfrak{g}$ be an open subset consisting of all matrices A such that A/i has k negaitive and $N-k$ positive eigenvalues. It now follows that

$$R\hom_G(\mathbb{K}_{e^{-2\pi e_k}}; \Sigma_k) \cong R\hom_{\mathfrak{g}}(\mathbb{K}_0; \mathbb{K}_{H_k}[\dim \mathfrak{g}]).$$

Let $M^\circ \subset M \subset E \subset G(k,N) \times \mathfrak{g}$ be defined as follows:

$$E = \{(V,X)|XV \subset V\};$$
$$M = \{(V,X)|XV \subset V; X/i|_V \geq 0; X/i|_{V^\perp} \leq 0\};$$
$$M^\circ = \{(V,X)|XV \subset V; X/i|_V > 0; X/i|_{V^\perp} < 0\}.$$

It follows that $M \subset E \subset G(k,N) \times \mathfrak{g}$ are closed embeddings and that $M^\circ \subset M$ is an open embedding. The projection $\pi : E \to \mathfrak{g}$ is proper. The natural projection $p_E : E \to G(k,N)$ is a complex unitary bundle; $E = S \otimes \overline{S} \oplus S^\perp \otimes \overline{S^\perp}$, where S is the k-dimensional tautological bundle over $G(k,N)$.

Let $j : M^\circ \to E$ be the open inclusion. Then $k_{H_k} = R\pi_! j_! \mathbb{K}_{M^\circ} = R\pi_* j_! \mathbb{K}_{M^\circ}$. Therefore,

$$R\hom_{\mathfrak{g}}(\mathbb{K}_0; \mathbb{K}_{H_k}[\dim \mathfrak{g}]) = R\hom_{\mathfrak{g}}(\mathbb{K}_0; R\pi_* j_! \mathbb{K}_{M^\circ}[\dim \mathfrak{g}])$$
$$= R\hom_M(\pi^{-1}\mathbb{K}_0; j_! \mathbb{K}_{M^\circ}[\dim \mathfrak{g}])$$

Let $i : G(k,N) \to E$; $i(V) = (V,0)$ be the zero section. We then have

$$R\hom_{\mathfrak{g}}(\mathbb{K}_0; \mathbb{K}_{H_k}[\dim \mathfrak{g}]) = R\hom_M(i_* \mathbb{K}_{G(k,N)}; j_! \mathbb{K}_{M^\circ}[\dim \mathfrak{g}]).$$

It is easy to see that the natural map

$$R\hom_M(i_* \mathbb{K}_{G(k,N)}; j_! \mathbb{K}_{M^\circ}[\dim \mathfrak{g}]) \to R\hom_M(i_* \mathbb{K}_{G(k,N)}; \mathbb{K}_E[\dim \mathfrak{g}]) = R\Gamma(G(k,N); i^! \mathbb{K}_E)[\dim \mathfrak{g}]$$

is a quasi-isomorphism. We have a natural isomorphism $i^! \mathbb{K}_E \cong \mathrm{or}_E[-\dim_{\mathbb{R}} E]$ where or_E is the sheaf of orientations on E which is canonically trivial on every complex bundle. Thus $i^! k_E[\dim \mathfrak{g}] = \mathbb{K}_{G(k,N)}[-\dim E + \dim \mathfrak{g}] = \mathbb{K}_{G(k,N)}[\dim G(k,N)] \cong D_{G(k,N)}$, where $D_{G(k,N)}$ is the dualizing sheaf on $G(k,N)$. Finally we have $R\Gamma(G(k,N); D) \cong H_*(G(k,N); k)$. Thus we have established

Proposition 7.17. *There is a natural isomorphism*

$$R^{-\bullet}\hom(\mathbb{K}_{e^{-2\pi e_k}}; \Sigma_k) \cong H_\bullet G(k,N).$$

7.5.2. Let $D_k := \dim_{\mathbb{R}} G(k,N) = 2k(N-k)$. Let $\beta \in H_{D_k}(G(k,N))$ be the fundamental class.

According to the previous Proposition, the element β defines a map $B_k : \mathbb{K}_{e^{-2\pi e_k}} \to \Sigma_k[-D_k]$ in $D(G)$. Let $C_k := \mathrm{Cone} B_k$.

Proposition 7.18. *The singular support of the sheaf C_k is confined within the set*

$$\{(g,\omega)| < |\omega|, f_k >= 0\}$$

Proof. First, consider the case $g \neq e^{-2\pi e_k}$.

Then $(g,\omega) \in \mathrm{SS}(C_k)$ iff $(g,\omega) \in \mathrm{SS}(\Sigma_k)$. The sheaf Σ_k is microsupported within the set

$$(e^X, \omega),$$

where $\|-X\| \leq 2\pi e_k$ and if $<\omega, f_j> \neq 0$, then $<\|-X\|, e_j> = 2\pi <e_k, e_j>$ for all j.

Therefore, it suffices to show that $<\|-X\|/2\pi, e_k> < <e_k, e_k>$. Assume the contrary and let $\eta := \|-X\|/2\pi$. Let $\eta_l := <\eta, e_l>$; $\varepsilon_l = <e_k, e_l>$. Set $\eta_0 = \eta_N = \varepsilon_0 = \varepsilon_N = 0$. We have $0 \leq <\eta, f_l> = 2\eta_l - \eta_{l-1} - \eta_{l+1}$. Therefore, $\eta_l - \eta_{l-1} \geq \eta_{l+1} - \eta_l$. These convexity inequalities imply

$$\eta_l \geq l/k\eta_k$$

for all $l \leq k$;

$$\eta_l \geq (N-l)/(N-k)\eta_k,$$

for all $l \geq k$.

If $\eta_k = \varepsilon_k$, these inequalities mean that $\eta_l \geq \varepsilon_l$ for all l. However, we know that $\eta \leq \varepsilon$. Hence, $\eta = \varepsilon$ and $\|-X\| = 2\pi e_k$, hence $e^X = e^{-2\pi e_k}$ which is a contradiction.

Thus, $< \|-X\|, e_k > < < 2\pi e_k, e_k >$, therefore, $< \eta, f_k >= 0$.

Let us now consider the case $g = e^{-2\pi e_k}$. It suffices to consider the restriction $\Sigma_k|_{U\cap W_k}$ as in the previous theorem. Let $V := \{X \in \mathfrak{g} | |X| << b\}$ We then have an identification $I : V \to U$; $X \mapsto e^{-X} e^{-2\pi e_k}$. we know that $I^{-1}\Sigma_k \cong R\pi_* \mathbb{K}_{M^\circ}[\dim \mathfrak{g}]|_{V_k}$ an the map $\mathbb{K}_{e^{-2\pi e_k}} \to \Sigma_k[d_k]$ is induced by a certain map $\mathbb{K}_0 \to R\pi_*\mathbb{K}_{M^\circ}[\dim \mathfrak{g}]$. Namely, this map comes from the identification

$$\begin{aligned}\hom(\mathbb{K}_0; R\pi_*\mathbb{K}_{M^\circ}[\dim \mathfrak{g}]) \to \hom(\mathbb{K}_{G(k,N)}; \mathbb{K}_{M^\circ}[\dim \mathfrak{g}]) \to \hom(\mathbb{K}_{G(k,N)}; \mathbb{K}_E[\dim \mathfrak{g}])\\ = H_*(G(k,N)).\end{aligned}$$

Note that the sheaves $\mathbb{K}_0$ and $R\pi_*\mathbb{K}_{M^\circ}$ are dilation invariant, so we may study their Fourier-Sato transforms. Let us find $(R\pi_*\mathbb{K}_{M^\circ})^\vee$. Let $E^* \cong E$ be the dual bundle over $G(k,N)$; let $M^* \subset E^*$ be the closed cone dual to the open convex cone $M^\circ \subset E$. Upon the identification $E^* = E$ by means of the scalar product, we identify M^* with the set of all pairs $(X,V) \in \mathfrak{g} \times G(k,N)$ such that $XV = V$ and the smallest eigenvalue of $X/i|_V$ is greater or equal to the largest eigenvalues of $X/i|_{V^\perp}$.

Let $P : \mathfrak{g} \times G(k,N) \to E^*$ be the map dual to $\pi : E \to \mathfrak{g}$. Let $p_\mathfrak{g} : \mathfrak{g} \times G(k,N) \to \mathfrak{g}$ be the projection. We then have

$$R\pi_*\mathbb{K}_{M^\circ}^\vee = p_{\mathfrak{g}!}P^{-1}\mathbb{K}_{M^*}[-\dim_\mathbb{R} E/G(k,N)]$$

We then see that

$$P^{-1}\mathbb{K}_{M^*} = \mathbb{K}_Z,$$

where $Z \subset \mathfrak{g} \times G(k,N)$;

$$Z = \{(X,V) | XV \subset V; \lambda_{\min} X|_V \geq \lambda_{\max} X|_{V^\perp}\}.$$

Thus, $R\pi_*\mathbb{K}_{M^\circ}[\dim \mathfrak{g}]^\vee = Rp_{\mathfrak{g}!}\mathbb{K}_Z[\dim \mathfrak{g} - \dim E/G(k,N)] = Rp_{\mathfrak{g}!}\mathbb{K}_Z[\dim G(k,N)]$. Next, $\mathbb{K}_0^\vee = \mathbb{K}_\mathfrak{g}$. The map B_k induces a map of Fourier-Sato transforms:

$$B^\vee : \mathbb{K}_\mathfrak{g} \to Rp_{\mathfrak{g}!}\mathbb{K}_Z$$

Let us specify this map. By the conjugacy (since $p_\mathfrak{g}$ is proper), one can instead specify a map

$$B^\vee_{\text{conj}} : p_\mathfrak{g}^{-1}\mathbb{K}_\mathfrak{g} = \mathbb{K}_{\mathfrak{g}\times G(k,N)} \to \mathbb{K}_Z.$$

One can show that this map is simply the natural map induced by the closed embedding $Z \subset \mathfrak{g} \times G(k,N)$.

Let us now consider an open set $U \subset \mathfrak{g}$ consisting of all $X \in \mathfrak{g}$ such that $\lambda_k(X) > \lambda_{k+1}(X)$. We then see that the projection $Z \times_\mathfrak{g} U \to U$ is a homeomorphism. Therefore, $\text{Cone}B^\vee|_U = 0$ that is $\text{Cone}B^\vee = (\text{Cone}B)^\vee$ is supported on the complement of U which is precisely the set of all $X \in \mathfrak{g}$ such that $< \|X\|, f_k >= 0$. This proves the statement. □

7.5.3. Let $l \in \mathfrak{h}$. Let $T_l : G \times \mathfrak{h} \to G \times \mathfrak{h}$ be the shift in l: $T_l(g,X) = (g, X+l)$. We know that $T_l^{-1}\mathfrak{S} = \mathfrak{S}|_{G\times l} *_G \mathfrak{S}$ (Lemma 6.9). Therefore, the maps B_k induce maps

$$B_k' : \mathbb{K}_{e^{-2\pi e_k}} *_G \mathfrak{S} \to \mathfrak{S}|_{G\times e^{-2\pi e_k}} *_G \mathfrak{S}[-D_k] = T^{-1}_{-2\pi e_k}\mathfrak{S}[-D_k], \tag{71}$$

where $D_k = \dim G(k,N)$. The previous Proposition implies that

Corollary 7.19. *Cone$B_{k'}$ is locally constant on the fibers of the projection $G \times \mathfrak{h} \to G \times \mathfrak{h}/f_k$.*

Proof. We have
$$\mathrm{Cone}B_{k'} \cong C_{k'} *_G \mathfrak{S}.$$
Using the previous Proposition as well as Theorem 6.1 one can easily show that 1-forms from $\mathrm{SS}(C_{k'} *_G \mathfrak{S})$ do vanish on the fibers of the projection $G \times \mathfrak{h} \to G \times \mathfrak{h}/f_k$. $\square$

Let $z \in \mathbf{Z}$ and restrict (71) onto $ze^{-2\pi e_k} \in G$. We will get a map
$$B_k^g : \mathcal{S}_z \to T^{-1}_{-2\pi e_k}\mathcal{S}_{ze^{-2\pi e_k}}[-D_k].$$
It follows that $\mathrm{Cone}B_k^g$ is also constant along the fibers of the projection $\mathfrak{h} \to \mathfrak{h}/f_k$.

7.5.4. The map B_k^g induces a map
$$\Delta_{I,m}(\mathcal{S}_z) \to \Delta_{I,m}T^{-1}_{-2\pi e_k}\mathcal{S}_{ze^{-2\pi e_k}}[-D_k].$$
for all I and m. This is the same as a map
$$\Delta_{I,m}\mathcal{S}_z \to \Delta_{I,m-2\pi e_k}\mathcal{S}_{ze^{-2\pi e_k}}[-D_k]. \tag{72}$$

Proposition 7.20. *If $k \in I$, the above map is a quasi-isomorphism.*

Proof. Follows from Lemma 7.14. $\square$

Theorem 7.15 now follows directly from the previous Proposition.

7.5.5. *Corollary from Theorem 7.15.* Let $u \in \mathfrak{h}$, $u = 2\pi \sum x_i e_i$, set $D(u) := -\sum x_k D_k$.
We then see:

Corollary 7.21. *Let $z \in \mathbf{Z}$, $m \in \mathbb{L}_z \cap C_-$. Then there exists an isomorphism*
$$\Delta_{m,I_m}\mathcal{S}_z \cong \Delta_{0,I_m}\mathcal{S}_e[D(m)]. \tag{73}$$

Proof. Follows directly from Theorem 7.15. $\square$

7.6. **Computing $\Delta_{0,I}\mathcal{S}_e$.** Let $I := \{j_1 < j_2 < \cdots < j_r\}$. Let $\mathcal{FL}(I)$ be the partial flag manifold with dimensions of the subspaces being $j_1, j_2, \ldots, j_r$. We will show

Proposition 7.22.
$$\Delta_{0,I}\mathcal{S}_e \cong H^\bullet(\mathcal{FL}(I)).$$

Proof. Let b be as in Sec. 6. Let $Z \in C_-^\circ$; $-Z << b$. One can choose ε so small that $Z + \sum_k a_k f_k \in C_-^\circ$ if $0 \leq a_k \leq \varepsilon$.

For $A \in \mathfrak{h}$, set $S_A := \mathfrak{S}|_{G\times A}$. We have
$$\Delta_{0,I}\mathcal{S}_e = R\hom_{\mathfrak{h}}(\mathbb{K}_{W(I,0,\varepsilon)}; \mathcal{S}_e)[|I|]$$

For $\delta > 0$, let
$$W(I,0,\varepsilon,\delta) \subset \mathfrak{h}$$
be the set of all points A such that for all $k \in I$, $< A, e_k > \in [0,\varepsilon)$; for all $k \notin I$, $-\delta << A, e_k >< 0$.

We have a natural map $\mathbb{K}_{W(I,0,\varepsilon,\delta)} \to \mathbb{K}_{W(I,0,\varepsilon)}$. Using the complex D' from 70 one can easily prove that for any object in $D(\mathfrak{h})$ whose microsupport is contained within $\mathfrak{h} \times C_+$, in particular, for $\mathcal{S}_e$, the natural map
$$R\hom_{\mathfrak{h}}(\mathbb{K}_{W(I,0,\varepsilon)}; \mathcal{S}_e) \to R\hom_{\mathfrak{h}}(\mathbb{K}_{W(I,0,\varepsilon,\delta)}; \mathcal{S}_e)$$
is an isomorphism.

One can choose ε, δ so small that $Z + W(I,0,\varepsilon,\delta) \subset V_b \cap C_-^\circ$. Set $W := W(I,0,\varepsilon,\delta)$.

By definition, we have:

$$R\hom(\mathbb{K}_W;\mathcal{S}_e)$$
$$= R\hom_{G\times\mathfrak{h}}(\mathbb{K}_e\boxtimes\mathbb{K}_W;\mathfrak{S}).$$

We have a endofunctors on $D(G\times\mathfrak{h})$: $E_\pm : F\mapsto S_{\pm Z}*_G F$. The composition

$$E_+E_-(F)=S_Z *_G S_{-Z}*_G F=S_{Z-Z}*_G F=S_0 *_G F=F$$

is isomorphic to the identity (we have use an isomorophism $S_{Z_1}*_G S_{Z_2}=S_{Z_1+Z_2}$ which follows directly from (64).) Thus, $E_+E_-\cong \mathrm{Id}$; likewise $E_-E_+\cong\mathrm{Id}$, so $E_\pm$ are quasi-inverse autoequivalences of $D(G\times\mathfrak{h})$. Hence, we have

$$R\hom_{G\times\mathfrak{h}}(\mathbb{K}_e\boxtimes\mathbb{K}_W;\mathfrak{S})=R\hom_{G\times\mathfrak{h}}(S_Z\boxtimes\mathbb{K}_W;T_Z^{-1}\mathfrak{S})$$
$$=R\hom_{G\times\mathfrak{h}}(S_Z\boxtimes\mathbb{K}_{Z+W};\mathfrak{S}),$$

where the last equality follows from Lemma 6.9. As $Z+W\subset C_-^\circ\cap V_b$, we have:

$$R\hom(\mathbb{K}_W;\mathcal{S}_e)=R\hom_{G\times(C_-^\circ\cap V_b)}(S_Z\boxtimes\mathbb{K}_{Z+W};\mathfrak{S}|_{G\times(C_-^\circ\cap V_b)})$$

Let $V:=C_-^\circ\cap V_b$. As follows from the proof of Theorem 6.1, we have

$$\mathfrak{S}|_{G\times V}=\mathbb{K}_{\{(e^X,v)|\|X\|<<-v\}}[\dim\mathfrak{g}],$$

Analogously, $S_Z:=\{e^X|\|X\|<<-Z\}[\dim\mathfrak{g}]$.

Let $V_Z\subset\mathfrak{g}$, $V_Z:=\{X|\|X\|<<-Z\}$. Let $\Omega\subset\mathfrak{g}\times\mathfrak{h}$;

$$\Omega:=\{(X,A)|\|X\|<<-A\}.$$

We then have

$$\Delta_{I,0}\mathcal{S}_e=R\hom_{\mathfrak{g}\times\mathfrak{h}}(\mathbb{K}_{V_Z}\boxtimes\mathbb{K}_{Z+W};\mathbb{K}_\Omega)[\|I\|]$$

Let $\mathcal{O}$ be the closure of Ω in $\mathfrak{g}\times\mathfrak{h}$. As Ω is an open proper cone, we have

$$\mathbb{K}_\Omega=R\underline{\mathrm{Hom}}(\mathbb{K}_\mathcal{O};\mathbb{K}_{\mathfrak{g}\times\mathfrak{h}}).$$

Therefore,

$$\Delta_{I,0}\mathcal{S}_e=R\hom_{\mathfrak{g}\times\mathfrak{h}}((\mathbb{K}_{V_Z}\boxtimes\mathbb{K}_{Z+W})\otimes\mathbb{K}_\mathcal{O};\mathbb{K}_{\mathfrak{g}\times\mathfrak{h}})[|I|]$$

Let $A:=(V_Z\times(Z+W))\cap\mathcal{O}$ so that

$$(\mathbb{K}_{V_Z}\boxtimes\mathbb{K}_{Z+W})\otimes\mathbb{K}_\mathcal{O}=\mathbb{K}_A.$$

Let $p:\mathfrak{g}\times\mathfrak{h}\to\mathfrak{g}$ be the projection. We have $\mathbb{K}_{\mathfrak{g}\times\mathfrak{h}}=p^!\mathbb{K}_\mathfrak{g}[-\dim\mathfrak{h}]$. Hence, by the conjugacy,

$$\Delta_{I,0}\mathcal{S}_e=R\hom_\mathfrak{g}(Rp_!\mathbb{K}_A;\mathbb{K}_\mathfrak{g})[-\dim\mathfrak{h}+|I|]. \tag{74}$$

Let $X\in\mathfrak{g}$ and consider

$$(Rp_!\mathbb{K}_A)|_X=R\Gamma_c(\mathfrak{h};\mathbb{K}_{A\cap X\times\mathfrak{h}}).$$

Let $X_k=<\|X\|,e_k>$; $Z_k=<Z,e_k>$. We see that $A\cap X\times\mathfrak{h}$ is non-empty only if $X\in V_Z$, i.e. $X_k+Z_k<0$ for all k. In this case we see that $A\cap X\times\mathfrak{h}$ consists of all points of the form $(X,Z+\sum_{k=1}^N t_kf_k)$, where $0\le t_k<\varepsilon$ for all $k\in I$; $-\delta<t_k<0$ for all $k\notin I$; $Z_k+t_k+X_k\le 0$ for all k. Let L be the set of all $k\in I$ such that $Z_k+X_k>-\varepsilon$. One then sees that these conditions are equivalent to

$$0\le t_k\le -X_k-Z_k$$

for all $k\in L$;

$$0 \le t_k < \varepsilon$$

for all $k \in I \backslash L$;

$$-\delta < t_k < 0.$$

for all $k \notin I$.

It follows that $R\Gamma_c(\mathfrak{h}; \mathbb{K}_{A \cap X \times \mathfrak{h}}) = 0$ if $L \neq I$. Thus, the object $Rp_! \mathbb{K}_A$ is supported on an open subset $E_\varepsilon \subset \mathfrak{g}$ consisting of all points X such that $X_k + Z_k < 0$ for all $k \notin I$ and $-\varepsilon < X_k + Z_k < 0$ for all $k \in I$.

Let $F_\varepsilon := E_\varepsilon \times \mathfrak{h} \cap A$. It follows that the natural map $Rp_! \mathbb{K}_{F_\varepsilon} \to Rp_! \mathbb{K}_A$ is an isomorphism.

One also has a natural isomorphism

$$Rp_! \mathbb{K}_{F_\varepsilon} = \mathbb{K}_{E_\varepsilon}[|I| - N + 1] = \mathbb{K}_{E_\varepsilon}[|I| - \dim \mathfrak{h}].$$

We can substitute this into (74):

$$\Delta_{I,0} \mathcal{S}_e = R \hom_{\mathfrak{g}}(\mathbb{K}_{E_\varepsilon}; \mathbb{K}_{\mathfrak{g}})$$

which can be rewritten as

$$\Delta_{I,0} \mathcal{S}_e[\|I\|] = H^\bullet(E_\varepsilon),$$

because $E_\varepsilon \subset \mathfrak{g}$ is an open subset.

Lemma 7.23. *For $\varepsilon > 0$ small enough, we get:*

$$\forall Y \in E_\varepsilon; \forall i \notin I :< Y, f_i >< 0.$$

Proof. We have $< Y, f_i >=< X + \sum t_j f_j, f_i ><< X, f_i > + t_i < f_i, f_i >$, because $< f_i, f_j >\le 0$ for all $i \neq j$. Next,

$$< X, f_i > + t_i < f_i, f_i >\le< X, f_i > + 2\varepsilon < 0$$

for ε small enough. □

This implies that for any $X \in E_\varepsilon$ and for every $k \in I$, we have a well-defined k-dimensional eigenspace space $V^k(X)$ spanned by the eigenvectors of X/i with top k eigenvalues. The spaces $V^\bullet(X)$ form a flag from $\mathcal{FL}(I)$. Thus we have a map $P : E_\varepsilon \to \mathcal{FL}(I)$; $P(X) := V^\bullet(X)$.

Let $\mathcal{E} \to \mathcal{FL}(I)$ be the vector bundle whose fiber at $V^\bullet \in \mathcal{FL}(I)$ consists of all unitary matrices preserving $V^\bullet$. One can easily check that $E_\varepsilon \subset \mathcal{E}$ is an open convex subset. Therefore, P induces an isomorphism $H^\bullet(E_\varepsilon) = H^\bullet(\mathcal{FL}(I))$ so that

$$\Delta_{I,0} \mathcal{S}_e[|I|] \cong H^\bullet(\mathcal{FL}(I)).$$

□

7.6.1. *The sheaf $j_{C_-^\circ}^{-1} \mathcal{S}$, up to an isomorphism.* Let us combine Proposition 7.4, Corollary 7.21, and Proposition 7.22. We will then get the following statement:

Proposition 7.24. *Let $g \in \mathbf{Z}$. There exists an inductive system of sheaves on C_-:*

$$j_{C_-^\circ}^{-1} \mathcal{S}_g = F^0 \to F^1 \to \cdots \to F^n \to \cdots$$

such that

$$L \varinjlim_n F_n = 0;$$

$$Cone(F_{n-1} \to F_n) \cong \mathbb{K}_{U^-{}_{m_n}} \otimes H^*(\mathcal{FL}(I_{m_n}))[D(m_n)],$$

where the sequence $m_1, m_2, \ldots, m_n, \ldots$ consists of all elements of $\mathbb{L}_g \cap C_-$, each term occuring once.

It turns out that this Proposition allows us to recover the isomorpism type of $j_{C_-^\circ}^{-1}\mathcal{S}_g$.

Let $A_n := \mathbb{K}_{U^-{}_{m_n}} \otimes H^*(\mathcal{FL}(I^c_{m_n}))[D(m_n)]$.

Lemma 7.25. *There exist maps*

$$i_n : A_n \to F_0$$

such that for every n the triangle

$$\bigoplus_{n' \le n} A_{n'} \to F_0 \to F_n \tag{75}$$

is exact.

Proof. Let us prove the statement by induction in n.For $n = 1$ we have a natural map $i_1 : A_1 \to j_{C_-^\circ}^{-1}\mathcal{S}_g = F_0$ whose cone is F_1; this proves the base.

Let us now proceed to the incuction step.

Suppose we have aready constructed an exact triangle as in (75) for some n. Let us apply to this triangle the functor $R\,\hom(A_{n+1}, \cdot)$.

We will then get an exact sequence

$$R^0\hom(A_{n+1}; j_{C_-^\circ}^{-1}\mathcal{S}_g) \to R^0\hom(A_{n+1}; F_n) \to \bigoplus_{n' \le n} R^1\hom(A_{n+1}; A_{n'}). \tag{76}$$

Observe that the last arrow in this sequence is 0: because of Lemma 7.6 and because all the spaces M_i are concentrated in the even degrees, therefore, $R^{\mathrm{odd}}\hom(A_i, A_j) = 0$ for all i, j.

Therefore, the left arrow in (76) is surjective. Next, we have a map $E_{n+1} : A_{n+1} = \mathrm{Cone}(F_n \to F_{n+1}) \to F_n$. Let $i_{n+1} : A_{n+1} \to \mathcal{S}_g$ be the lifting of E_{n+1} (which exists precisely because of surjectivity of the left arrow in (76). It is straigtforward to see that so chosen i_{n+1} satisfies the conditions □

Theorem 7.26. *There exists an isomorphism*

$$\bigoplus_{l \in \mathbb{L}_g \cap C_-} A_l \to j_{C_-^\circ}^{-1}\mathcal{S}_g,$$

where $A_l := \mathbb{K}_{U^-{}_l} \otimes H^*(\mathcal{FL}(I_l))[D(l)]$

Proof. Indeed, the previous Lemma implies that the map $\bigoplus_n i_n : \bigoplus_n A_n \to j_{C_-}^{-1}\mathcal{S}_g$ is an isomorphism, whence the statement. □

8. B-sheaves

For a manifold X let $\mathbf{Complexes}_X$ be the dg-category of complexes of sheaves on X.

Suppose X is equipped with an action of the monoid $\mathbb{L}_-$. Let $T_l : X \to X$ be the translation by $l \in \mathbb{L}_-$. In all our examples all T_l will be open embeddings.

Let $F \in \mathbf{Complexes}_X$ and $l \in \mathbb{L}_-$. Set $A(l) := A_F(l) := \hom(F, T_l^{-1}F)$. These complexes obviously form a $\mathbb{L}_-$-graded dg-algebra to be denoted by $A = A_F$.

Let B be another $\mathbb{L}_-$-graded dg-algebra. We define *a B-sheaf structure on F* as a $\mathbb{L}_-$-graded dg-algebra homomorphism $B \to A_F$. B-sheaves form a triangulated dg-category in the obvious way.

We will only use algebras B of a special type. Namely, We will assume that:

—$B(l)$ is concentrated in degrees $\leq -D(l)$;

—the cohomology $H^\bullet(B(l))$ is concentrated in degree $-D(l)$ and is one dimensional;

—one can choose generators $b_l \in H^{-D(l)}(B(l))$ which are stable under the product induced by the product on B.

Call such a B *homotopically standard*.

Let $\mathbf{b}$ be a $\mathbb{L}_-$-graded dg-algebra defined by setting $\mathbf{b}(l) = k[D(l)]$. Let $1_l := 1 \in k[D(l)]^{-D(l)}$ be generators.

We then define the product on $\mathbf{b}$ by setting $1_l 1_m = 1_{l+m}$. It follows that we have a unique $\mathbb{L}_-$-graded dg-algebra homomorphism $B \to \mathbf{b}$ such that the induced map $H^\bullet(B) \to H^\bullet(\mathbf{b}) = \mathbf{b}$ sends b_l to 1_l.

We call a B-sheaf F *acyclic* if it is acyclic as a complex of sheaves on X (i.e. for each $x \in X$ the complex of fibers F_x is acyclic).

Following [2] we can produce the derived dg-category by taking the quotient with respect to the full subcategory of acyclic objects.

However, in our situation one can prove

Proposition 8.1. *The category of B-sheaves has enough injective objects.*

Remark By an injective object we mean any B-sheaf X such that for any acyclic B-sheaf Z, the complex $\hom(Z, X)$ is acyclic.

Proof. Let A be a B-sheaf. Let β_A be another B-sheaf such that $\beta_A := \prod_{l \in \mathbb{L}_-} \hom(B(l); T_l^{-1}A)$ We then get a B-structure on β_A and a natural map of B-sheaves $A \to \beta_A$. Let now $A \to A'$ be a termwise injective map in the category of complexes of sheaves on X (we forget the B-structure) such that A' is injective. We then have a termwise injective map of B-sheaves

$$A \hookrightarrow \beta(A) \hookrightarrow \beta_{A'}$$

One sees that $\beta_{A'}$ is injective: given any B- sheaf T on X we have

$$\hom(T, \beta_{A'}) = \hom(T, A'),$$

where hom on the RHS is in the category of complexes of sheaves on X. As A' is injective, we see that $\hom(T, A') \sim 0$ as long as T is acyclic.

□

As we know, in this case, the derived category is equivalent to the full subcategory of injective objects.

We will only need the homotopy category of the derived cateogory of B-sheaves. Denote this category by $DB\mathrm{Sh}_X$.

Let $f : X_1 \to X_2$ be a $\mathbb{L}_-$-equivariant map We then have a right derived functor of f_*: $Rf_* : DB\mathrm{Sh}_{X_1} \to DB\mathrm{Sh}_{X_2}$: if we choose the category of injective B-sheaves on X_1 as a model for $DB\mathrm{Sh}_{X_2}$ then Rf_* is given by the termwise application of the functor f_*. Similarly, one defines functors $Rf_!, f^{-1}$. One can also define a functor $f^!$ as a right afjoint to $Rf_!$, but we won't need this functor.

Recall that we have a natural map $p : B \to \mathbf{b}$. This map induces an obvious functor p^{-1} from the category of $\mathbf{b}$-sheaves to the category of B-sheaves on X and one sees that this map has a right adjoint p_*. This functor admits a right derived $\pi := Rp_* : DB\mathrm{Sh}_X \to D\mathbf{b}\mathrm{Sh}_X$. This functor is an equivalence.

8.0.2. *A B-sheaf structure on the sheaves $\mathfrak{S}$ and $\mathcal{S}$.* Let $\mathfrak{S} \in D(G \times \mathfrak{h})$ be as in Theorem 6.1. Choose an injective representative for $\mathfrak{S}$, to be denoted by the same symbol $\mathfrak{S}$. Define a diagonal $\mathbb{L}_-$-action on $G \times \mathfrak{h}$ by setting $l.(g, A) := (e^l g, A + l)$. For $l \in \mathbb{L}_-$ consider the complex $B'(l) := \hom_{G\times\mathfrak{h}}(\mathfrak{S}; T_l^{-1}\mathfrak{S})$ and compute its cohomology:

$$H^\bullet(B'(l)) = R^\bullet \hom(\mathfrak{S}; T_l^{-1}\mathfrak{S}).$$

Let $i_0 : G \to G \times \mathfrak{h}$, $i_0(g) = (g, 0)$. By Theorem (6.8) we have

$$R^\bullet \hom(\mathfrak{S}; T_l^{-1}\mathfrak{S}) = R^\bullet \hom_G(i_0^{-1}\mathfrak{S}; i_0^{-1}T_l^{-1}\mathfrak{S}).$$

We know that $i_0^{-1}\mathfrak{S} = \mathbb{K}_{e_G}$. Thus,

$$R^\bullet \hom_G(\mathfrak{S}; T_l^{-1}\mathfrak{S}) = R^\bullet \hom_G(\mathbb{K}_e; i_0^{-1}T_l^{-1}\mathfrak{S}).$$

As $T_l^{-1}\mathfrak{S}$ is non-singular along $i_0(G) \subset G\times\mathfrak{h}$, we have an isomoprphism $i_0^{-1}T_l^{-1}\mathfrak{S} \cong i_0^! T_l^{-1}\mathfrak{S}[\dim\mathfrak{h}] = i_0^! T_l^! \mathfrak{S}[\dim\mathfrak{h}]$. Thus,

$$R^\bullet \hom_G(\mathbb{K}_e; i_0^{-1}T_l^{-1}\mathfrak{S}[\dim\mathfrak{h}]) = R^\bullet \hom_G(\mathbb{K}_e; i_0^! T_l^! \mathfrak{S}[\dim\mathfrak{h}])$$

$$= i^!_{(e,0)} T_l^! \mathfrak{S}[\dim\mathfrak{h}] = i^!_{(e^l,l)}\mathfrak{S}[\dim\mathfrak{h}]$$

$$= i_l^! \mathcal{S}_l[\dim\mathfrak{h}].$$

Here $i_{(e,0)}; i_{(e^l,l)}$ denote embeddings of the points specified into $G \times \mathfrak{h}$, and, likewise, i_l is the embedding of the point l into $\mathfrak{h}$.

Theorem 7.26 implies that $H^{<D(l)} i_l^! \mathcal{S}_{e^l} = 0$ and $H^{D(l)} i_l^! \mathcal{S}_{e^l}$ is one dimensional. Indeed, one sees that given $l' \in C_-$, we have: $i_l^! \mathbb{K}_{U^-{}_{l'}} \cong \mathbb{K}[-\dim\mathfrak{h}]$ for all $l' \geq l$; otherwise $i_l^! \mathbb{K}_{U^-{}_{l'}} = 0$. Therefore,

$$i_l^! \mathcal{S}_{e^l}[\dim\mathfrak{h}] \cong \bigoplus_{l' \in \mathbb{L}_{e^l}, l' \geq l} H^\bullet(\mathcal{FL}(I_{l'}))[D(l')],$$

and the lowest degree contribution comes from $H^0(\mathcal{FL}(I_l))[D(l)] = \mathbb{K}[D(l)]$.

Set $B(l) := \tau_{\leq -D(l)} B'(l)$. It then follows that B is a homotopically standard $\mathbb{L}_-$-graded algebra. We thus automatically get a B-sheaf structure on $\mathfrak{S}$. Let $I_Z : \mathbf{Z} \times \mathfrak{h} \to G \times \mathfrak{h}$ be the embedding. This embedding is $\mathbb{L}_-$-equivariant, where $\mathbb{L}_-$-action on $\mathbf{Z} \times \mathfrak{h}$ is defined by

$$T_l(c, A) = (e^l c; A + c).$$

Hence we get a B-sheaf structure on $\mathcal{S} := I_Z^! \mathfrak{S}$ (as $\mathfrak{S}$ is injective and I_Z is a closed embedding one can compute $I_Z^!$ by taking sections supported on $\mathbf{Z} \times \mathfrak{h} \subset G \times \mathfrak{h}$.

8.1. **A B-sheaf $j_{C_-^\circ}^{-1}\mathcal{S}$ on $\mathbf{Z}\times C_-^\circ$.** Let $j_{C_-^\circ}:\mathbf{Z}\times C_-^\circ\to\mathbf{Z}\times\mathfrak{h}$ be the open embedding. The $\mathbb{L}_-$-action on $\mathbf{Z}\times\mathfrak{h}$ preserves $\mathbf{Z}\times C_-^\circ$, thus making the embedding $j_{C_-^\circ}$ to be $\mathbb{L}_-$-equivariant.

We then have a B-sheaf $j_{C_-^\circ}^{-1}\mathcal{S}$. Let $p:B\to\mathbf{b}$ be the canonical map. Let us choose an injective model for $Rp_*\mathcal{S}$, to be still denoted by $\mathcal{S}$.

Let us study the $\mathbf{b}$-structure on $j_{C_-^\circ}^{-1}\mathcal{S}$. Let $I\subset\{1,2,\ldots,N-1\}$. Let $e_I:=\sum_{i\in I}e_i$.

According to Theorem 7.26 we have a map

$$i_I:H^*(\mathcal{FL}(I))[D(-2\pi e_I)]\otimes\mathbb{K}_{e^{-2\pi e_I}\times U^-_{-2\pi e_I}}\to j_{C_-^\circ}^{-1}\mathcal{S}$$

Set $H(I):=H^\bullet(\mathcal{FL}(I))$. For $I\subset J$ we have a tautological projection

$$\mathcal{FL}(J)\to\mathcal{FL}(I)$$

hence an induced map $H(I)\to H(J)$. Hence H is a functor from the poset of subsets of $\{1,2,\ldots,N-1\}$ to the category of graded vector spaces.

One can show that this functor is actually free, i.e:

Lemma 8.2. *There exist graded vector spaces $G(I)$, where $I\subset\{1,2,\ldots,N-1\}$such that we have decompositions*

$$H(I)=\bigoplus_{J\subset I}G(J),\tag{77}$$

which are compatible with the structure maps $H(I_1)\to H(I_2)$, $I_1\subset I_2$ in the obvious way.

Proof. Let us use Schubert cellular decomposition of partial flag varieties $\mathcal{FL}(I)$. Let $\mathbf{f}\subset\mathcal{FL}(I)$ be the flag such that $\mathbf{f}^r\subset\mathbb{C}^N$ consists of all points $(v_1,v_2,\ldots,v_N)\in\mathbb{C}^N$ such that $v_k=0$ for all $k>i_r$.

Let $H:=\mathrm{GL}_N(\mathbb{C})$. Let $P(I)\subset H$ be the standard parabolic subgroup, namely the stabilizer of $\mathbf{f}$. We have $\mathcal{FL}(I)=H/P(I)$. Let $W\subset G$ be the standard Weyl group. For any $w\in W/W\cap P(I)$ let $[w]\in H/P(I)$ be the image of $[w]$ and let $C_{I,w}:=C_w:=B.[w]$ where $B\subset H$ is the standard Borel subgroup of upper-triangular matrices. It is well known that the cells C_w, $w\in W/W\cap P(I)$ form a cellular decomposition of $\mathcal{FL}(I)$. We have $\dim_{\mathbb{R}}C_w=2D_I(w)$, where $D_I(w)$ is defined as follows. Let $\pi_I:\{1,2,\ldots,N\}\to\{1,2,\ldots,|I|\}$ be defined by letting $\pi_I(k)$ be the minimal number r such that $i_r\geq k$.

In particular, for any $M\in P(I)$, we have $M_{ij}=0$ as long as $\pi_I(i)>\pi_I(j)$. Let $w'\in W$ be any representative of $w\in W/W\cap P(I)$.

One then has that $D_I(w)$ is equal to the number of all pairs (i,j) such that $i,j\in\{1,2,\ldots,N\}$, $i<j$ and $\pi_I(w^{-1}(i))>\pi_I(w^{-1}j)$.

Thus we have a basis of $H_*(\mathcal{FL}(I))$ labelled by the cells C_w. Let $c_w\in H_{2D_I(w)}\mathcal{FL}(I)$ be the class corresponding to C_w.

We see that the map $p_{IJ}:\mathcal{FL}(I)\to\mathcal{FL}(J)$ is cellular. We have $p_{IJ}C_w\subset C_{w'}$ where w' is the image of $w\in W/W\cap P(I)$ in $W/W\cap P(J)$. One sees that $\dim C_{w'}\leq\dim C_w$. It then follows that $p_{IJ*}c_w=c_{w'}$ is $D_I(w)=D_J(w')$. Otherwise $p_{IJ*}(c_w)=0$.

Let us describe the dual map p^*_{IJ}. Let $c^w\in H^\bullet(\mathcal{FL}(I)$ be the element dual to c_w. Let us identify $W/W\cap P(I)$ with the set $V(I)$ of partitions $\{1,2,\ldots,N\}=A_1\sqcup A_2\sqcup A_{|I|}$ where $|A_r|=i_r-i_{r-1}$ and we assume $i_0=0$, $i_{|I|}=N$. We have a map $Q^{JI}:V(J)\to V(I)$ defined as follows. Pick $t\leq N$. Let $i_m=j_{t-1}$; $i_M=j_t$. Order A_t and subdivide it into several subsets, such that the first subset consits of the first $i_{m+1}-i_m$ elements of A_t; the second subset consists of the next$i_{m+2}-i_{m+1}$

elements of A_t, etc. This way we get a partition $Q^{JI}A$. One sees that $Q^{JI}=p^*_{IJ}$. For $A\in V(I)$ let $\sim_A$ be an equivalence relation on I given by $i_1\sim_A i_2$ if for all $j_1<j_2$, $j_1,j_2\in[i_1,i_2]$, $A_{j_1}<A_{j_2}$. Call $A\in V(I)$ elementary if $\sim_A$ is trivial. One then can set $G(I)$ to be the span of all elementary $A\in V(I)$. $\square$

Let us now consider through maps

$$j_I: G(I)\otimes \mathbb{K}_{e^{-2\pi e_I}\times U^-_{-2\pi e_I}}[D(-2\pi e_I)]\to H(I)\otimes \mathbb{K}_{e^{-2\pi e_I}\times U^-_{-2\pi e_I}}[D(-2\pi e_I)]\to j^{-1}_{C^\circ_-}\mathcal{S}.$$

Introduce a notation: for $l\in\mathbb{L}_-$, set $\mathcal{U}_l:=e^l\times U^-{}_l\subset \mathbf{Z}\times C^\circ_-$. Denote $\mathcal{G}_I:=G(I)\otimes \mathbb{K}_{e^{-2\pi e_I}\times \mathcal{U}_{-2\pi e_I}}[D(-2\pi e_I)]$. The $\mathbf{b}$-structure on $j^{-1}_{C^\circ_-}\mathcal{S}$ gives rise to maps

$$T_{l*}\mathcal{G}_I\otimes \mathbf{b}(l)\to T_{l*}j^{-1}_{C^\circ_-}\mathcal{S}\otimes\mathbf{b}(l)\to T_{l*}T_l^{-1}j^{-1}_{C^\circ_-}\mathcal{S}=j^{-1}_{C^\circ_-}\mathcal{S}$$

for all $l\in\mathbb{L}_-$. Take the direct sum:

$$\iota:\bigoplus_{I\subset\{1,2,\ldots,N-1\};l\in\mathbb{L}_-}T_{l*}\mathcal{G}_I[D(l)]\to j^{-1}_{C^\circ_-}\mathcal{S} \tag{78}$$

(we have replaced $\mathbf{b}(l)=k[D(l)]$). The sheaf on the LHS has an obvious structure of a $\mathbf{b}$-sheaf and the map ι is a map of $\mathbf{b}$-sheaves.

Furthermore the $\mathbf{b}$-sheaf on the LHS splits into a direct sum of $\mathbf{b}$-sheaves

$$\mathbb{S}^I:=\bigoplus_{l\in\mathbb{L}_-}T_{l*}\mathcal{G}_I[D(l)] \tag{79}$$

thus we have a map of $\mathbf{b}$-sheaves

$$\iota:\bigoplus_{I\subset\{1,2,\ldots,N-1\}}\mathbb{S}^I\to\mathcal{S} \tag{80}$$

For future purposes, let us rewrite the definition of $\mathbb{S}^I$. We have

$$\mathbb{S}^I:=\bigoplus_{l\in\mathbb{L}_-}T_{l*}\mathcal{G}_I[D(l)]$$

$$=G_I[D(-2\pi e_I)]\otimes[\bigoplus_{l\in\mathbb{L}_-}\mathbb{K}_{\mathcal{U}_{-2\pi e_I+l}}[D(l)]]$$

$$=G_I[D(-2\pi e_I)]\otimes T_{-2\pi e_I*}[\bigoplus_{l\in\mathbb{L}_-}\mathbb{K}_{\mathcal{U}_l}[D(l)]]$$

Let

$$\mathcal{X}:=\bigoplus_{l\in\mathbb{L}_-}\mathbb{K}_{\mathcal{U}_l}[D(l)] \tag{81}$$

with the obvious $\mathbf{b}$-structure. We then have an isomorphism of $\mathbf{b}$-sheaves:

$$\mathbb{S}^I\cong G_I[D(-2\pi e_I)]\otimes T_{-2\pi e_I*}\mathcal{X}. \tag{82}$$

Proposition 8.3. *The map (80) is a quasi-isomorphism.*

Proof. For any $z \in \mathbf{Z}$ and any $F \in D(\mathbf{Z} \times C_-^\circ)$ we set $F_z \in D(C_-^\circ)$; $F_z := F|_{z \times C_-^\circ}$. We have induced maps

$$\iota_z : \bigoplus_I \mathbb{S}_z^I \to j_{C_-^\circ}^{-1} \mathcal{S}_z,$$

and it suffices to show that these maps are isomorphisms for all $z \in \mathbf{Z}$. We know (Proposition 7.3) that $\mathrm{SS}(j_{C_-^\circ}^{-1} \mathcal{S}_z) \subset X(\mathbb{L}_-^z)$. One can easily check that $\mathbb{S}_z^I \in X(\mathbb{L}_-^z)$ for all I. As follows from Proposition 7.4 and Lemma 7.12, it suffices to show that the induced maps

$$R\hom_{C_-^\circ}(\mathbb{K}_{V_{x,\varepsilon}}; \bigoplus_I \mathbb{S}_z^I) \to R\hom_{C_-^\circ}(\mathbb{K}_{V_{x,\varepsilon}}; j_{C_-^\circ}^{-1} \mathcal{S}_z) \tag{83}$$

are isomorphisms for $\varepsilon > 0$ small enough and for all $x \in \mathbb{L}_-^z$. Let $F \in D(\mathbf{Z} \times C_-^\circ)$ and $x \in \mathbb{L}_-$. Set

$$\Delta_x(F) := R\hom(\mathbb{K}_{V_{x,\varepsilon}}; F|_{e^x}).$$

Let now F be a $\mathbf{b}$-sheaf on $\mathbf{Z} \times \mathfrak{h}$. The $\mathbf{b}$-structure gives rise to maps

$$\Delta_x(F) \to \Delta_{x+l}(F)[-D(l)],$$

for all $l \in \mathbb{L}_-$. Set $\delta_F(x) := \Delta_x(F)[-D(x)]$. Introduce a partial order $\preceq$ on $\mathbb{L}_-$ by setting $l_1 \preceq l_2$ if $l_2 - l_1 \in \mathbb{L}_-$. We see that δ_F is a functor from this poset, viewed as a category, to the category of graded $\mathbb{K}$-vector spaces. As follows from Corollary 7.21 and Proposition 7.22, we have $\delta_{\mathcal{S}}(l) = H^\bullet(\mathcal{FL}(I_l))$. Let $l_1 \preceq l_2$. As follows from the proof of Proposition 7.22, the induced map $\delta_{\mathcal{S}}(l_1) \to \delta_{\mathcal{S}}(l_2)$ is induced by the projection $\mathcal{FL}(I_{l_2}) \to \mathcal{FL}(I_{l_1})$ coming from the embedding $I_{l_1} \subset I_{l_2}$. It then follows from Lemma 8.2 that $\delta_{\mathcal{S}}$, as a functor, is freely generated by subspaces

$$G(I) \subset H^\bullet(\mathcal{FL}(I)) = \delta_{\mathcal{S}}(-2\pi e_I) \tag{84}$$

for all $I \subset \{1, 2, \ldots, N-1\}$.

One can easily check that $\delta_{\oplus_I \mathbb{S}^I}$ is freely generated by the subspaces

$$G(I) = \delta_{\mathbb{S}(I)}(-2\pi e_I) \subset \delta_{\oplus_I \mathbb{S}^I}(-2\pi e_I). \tag{85}$$

The map ι preserves the generating supspaces (84), (85). Hence, the maps (83) are isomorphisms, which proves the Proposition. □

8.2. **Strict B-sheaves.** Let F be a B-sheaf on $\mathfrak{h}$. Let $v_k \in B(-e_k)$ be a representative of $u_k \in H^{D(-e_k)}B(-e_k)$. We then have induced maps

$$a_k : F \to T_{-e_k}^{-1} F[D(-e_k)]$$

induced by v_k.

Let

$$\mathbf{Con}_k := \mathrm{Cone}\, a_k; \tag{86}$$

let $p_k : \mathfrak{h} \to \mathfrak{h}/\mathbb{R}f_k$. We call F *strict* if

1) for all k, the natural map $p_k^{-1} Rp_{k*}\mathbf{Con}_k \to \mathbf{Con}_k$ is an isomorphism in $D(\mathfrak{h})$ (that is, $\mathbf{Con}_k$ is constant along fibers of p_k);

2) F is microsupported on $\mathfrak{h} \times C_+ \subset \mathfrak{h} \times \mathfrak{h}^*$.

Denote the full subcategory of $DB\mathrm{Sh}_{\mathfrak{h}}$ consisting of all strict B-sheaves on $\mathfrak{h}$ by $DB\mathrm{Sh}_{\mathfrak{h}}^{\mathrm{strict}}$.

Analogously, let F be a sheaf on C_-°. Let us define a_k and $\mathbf{Con}_k$ in the same way as above.

Let $C_-^\circ/\mathbb{R}f_k$ be the image of C_-° under the map $C_-^\circ \to \mathfrak{h} \to \mathfrak{h}/\mathbb{R}f_k$. Let $p_k : C_-^\circ \to C_-^\circ/\mathbb{R}f_k$ be the projection.

As above, let us call F *strict* if

1) the natural map $p_k^{-1}Rp_{k*}\mathbf{Con}_k \to \mathbf{Con}_k$ is an isomorphism in $D(C_-^\circ)$ for all k;

2) F is microsupported on $C_-^\circ \times C_+ \subset C_-^\circ \times \mathfrak{h}^*$.

Denote the full subcategory of $DB\mathrm{Sh}_{C_-^\circ}$ consisting of all strict B-sheaves on C_-° by $DB\mathrm{Sh}_{C_-^\circ}^{\mathrm{strict}}$.

Let $\lambda \in \mathfrak{h}$ and condider a shifted open set $C_-^\circ + \lambda \subset \mathfrak{h}$. We then have a notion of a B-sheaf and of a strict B-sheaf on $C_-^\circ + \lambda$ via an identification $C_-^\circ + \lambda \cong C_-^\circ$ via the shift T_λ. Hence we have categories $DB\mathrm{Sh}_{C_-^\circ+\lambda}$; $DB\mathrm{Sh}_{C_-^\circ+\lambda}^{\mathrm{strict}}$.

8.2.1. Let $\lambda \in \mathfrak{h}$ and let $j_\lambda : C_-^\circ + \lambda \to \mathfrak{h}$ be an open embedding. We then see that the functor j_λ^{-1} transforms strict sheaves on $\mathfrak{h}$ into strict sheaves on $C_-^\circ + \lambda$

Theorem 8.4. *The functor*

$$j_\lambda^{-1} : DBSh_{\mathfrak{h}}^{strict} \to DBSh_{C_-^\circ+\lambda}^{strict}$$

is an equivalence.

8.3. **Proof of the theorem.**

8.3.1. *First reductions.* Without loss of generality one can put $\lambda = 0$. We also set $j := j_0$.

Let $\pi : B \to \mathbf{b}$ be the projection. As the functor $R\pi_*$ is an equivalence, without loss of generality, one can assume $B = \mathbf{b}$.

Let $I \subset \{1, 2, \ldots, N-1\}$. Let $\mathcal{C}(I, \mathfrak{h}) \subset D\mathbf{b}\mathrm{Sh}_{\mathfrak{h}}$ be the full subcategory consisting of all sheaves F satisfying:

1) for all $i \in I$, we have: $\mathbf{Con}_i(F) = 0$;

2) for all $i \notin I$ the natural map $p_i^{-1}Rp_{i*}F \to F$ is an isomorphism.

It is clear that every object of $\mathcal{C}(I, \mathfrak{h})$ is strict.

Let us define the category $\mathcal{C}(I, C_-^\circ)$ in a similar way.

Lemma 8.5. *Every strict* $\mathbf{b}$*-sheaf on* C_- *(resp.* $\mathfrak{h}$*) is quasi-isomorphic to a complex of objects from* $\bigsqcup_I \mathcal{C}(I, \mathfrak{h})$ *(resp.* $\bigsqcup_I \mathcal{C}(I, C_-^\circ)$*).*

Proof. We will prove Lemma for strict sheaves on C_-°. The proof for $\mathfrak{h}$ is similar.

Let us first consider a through map

$$\pi_I : C_-^\circ \to \mathfrak{h} \to \mathfrak{h}/(\mathbb{R} < f_j >_{j \notin I})$$

let C_I be the image of π_I.

We also have a through map

$$\sigma_I : \mathbb{R}_{<0} < e_i >_{i \in I} \hookrightarrow \mathfrak{h} \to \mathfrak{h}/(\mathbb{R} < f_j >_{j \notin I})$$

Sublemma 8.6. *The map* σ_I *is an open embedding whose image is the same as the image of* π_I

Proof. (of sublemma) It is easy to see that the vectors $f_j, j \notin I$; $e_i; i \in I$ form a basis of $\mathfrak{h}$. Therefore, the vectors $e_i; i \in I$ (more precisely, their images) form a basis of $\mathfrak{h}/(\mathbb{R} < f_j >_{j \notin I})$. Let $x \in C_-^\circ$. Let us expand

$$x = \sum_{i \in I} a_i e_i + \sum_{j \notin I} b_j f_j$$

Then $p_I(x) = \sum_{i \in I} a_i e_i$.

We have: for all $j \notin I$:

$$< x, f_j >= \sum_{k \notin I} b_k < f_j, f_k > \leq 0.$$

Let $J := \{1, 2, \ldots, N-1\} \backslash I$ and let us decompose J into intervals as follows: $J = J_1 \sqcup J_2 \sqcup \cdots \sqcup J_s$ where each $J_t = [k_t; l_t]$, $k_t \leq l_t < k_{t+1} - 1$. Set $b_k^t = b_k$ if $k \in J_t$; otherwise set $b_k^t = 0$. We then have $2b_k^t - b_{k-1}^t - b_{k+1}^t \leq 0$ for all $k \in J_t$. Let $D_k^t := b_k^t - b_{k-1}^t$. We then know that $D_{k+1}^t \geq D_k^t$ if $k, k+1 \in J_t$. We then have $b_k = D_{k_l}^t + \cdots + D_k^t$. Assume $b_k > 0$. Then $D_k^t > 0$ (because $D_{k_l}^t \leq D_{k_l+1}^t \leq \cdots \leq D_k^t$). Hence, $0 \leq D_k^t \leq D_{k+1}^t \leq \cdots$ and $0 < b_k^t < b_{k+1}^t < \cdots < b_{l_t+1}^k = 0$. Contradiction. Thus, $b_k^t \leq 0$ for all k. Therefore, for all k, $b_k \leq 0$.

For every $i \in I$ we have

$$0 >< x, f_i >= a_i + \sum_{j \notin I} b_j < f_i, f_j > .$$

Hence,

$$< x, f_i > - \sum_{i \in I} b_i < f_i, f_j >= a_j.$$

For $i \in I$, $j \notin I$, we have $i \neq j$ and $< f_i; f_j > \leq 0$. As $b_i \leq 0$, we see that $0 >< x, f_j > \geq a_j$. Hence, Image $(\pi_k) \subset$ Image (σ_k). Let us prove the inverse inclusion. Let $g := \sum_{i \in I} a_i e_i - b \sum_{j \notin I} f_j$ We see that for $a_j > 0$ and $0 < b << 1$,we have $g \in C_-^\circ$ and $\pi_k(g) = \sum_{i \in I} a_i e_i$. □

Let $\Gamma_I := R_{<0} < e_i >_{i \in I}$.

We then have a surjection $P_I : C_-^\circ \to \Gamma_I$. It is easy to see that P_I is a trivial bundle whose fiber is homeomorphic to $\mathbb{R}^{N-1-|I|}$. Let $J \subset I$. It follows that we have projections $P_{IJ} : \Gamma_I \to \Gamma_J$ so that $P_J = P_{IJ} P_I$.

Let F be a strict sheaf on C_-°. Let $F_J := P_J^{-1} P_{J*} F$. It is easy to see that F_J is a strict sheaf on C_-°. For $J \subset I$ we have a natural map $F_J \to F_I$. Let Subsets be the poset of all subsets of $\{1, 2, \ldots, N-1\}$; view this subset as a category. We then see that $I \mapsto F_I$ is a functor from Subsets to the dg category of B-sheaves whose image lies in the full subscategory of strict B-sheaves.

For a subset $I \subset \{1, 2, \ldots, N-1\}$ consider the standard complex

$$K(I, F) = \bigoplus_{J : J \subset I} F_J \otimes \Lambda^{\mathrm{top}}(\mathbb{K}[I \backslash J])[|I \backslash J|]$$

with the standard differential. We then see that 1) $K(I, F)$ is a strict B-sheaf on C_-°;

2) $p_I^{-1} R p_{I*} K(I, F) \to K(I, F)$ is an isomorphism;

3) Let $J \subset I$ and $J \neq I$. Then $R p_{J*} K(I, F) = 0$.

2) and 3) imply that

4) for any J which intersects I, $R p_{J*} K(I, F) = 0$.

Let $k \in I$. Then we know that $p_k^{-1} R p_{k*} \mathbf{Con}_k \to \mathbf{Con}_k$ is an isomorphism. On the other hand, 4) implies that $R p_{k*} \mathbf{Con}_k(K(I, F)) = 0$ Hence,

5) $\mathbf{Con}_k K(I, F) = 0$ for all $k \notin I$.

Thus, $K(I, F) \in \mathcal{C}(I, C_-^\circ)$, which proves Lemma for C_-°. The proof for $\mathfrak{h}$ is similar. □

8.3.2. It is clear that the functor j^{-1} takes $\mathcal{C}(I, \mathfrak{h})$ to $\mathcal{C}(I, C_-^\circ)$ for all I.

We will prove:

Lemma 8.7. *Let X be a $\mathbf{b}$-sheaf on $\mathfrak{h}$ and let $Y \in \mathcal{C}(\mathfrak{h}; I)$ for some $I \subset \{1, 2, \ldots, N-1\}$. Then the natural map*

$$R\hom_{D\mathbf{b}Sh_{\mathfrak{h}}}(X, Y) \to R\hom_{D\mathbf{b}Sh_{C_-^\circ}}(j^{-1}X; j^{-1}Y)$$

is an isomorphism

Proof. We see that $j_! j^{-1} X$ is a $\mathbf{b}$-sheaf on $\mathfrak{h}$ and that

$$R\hom_{D\mathbf{b}\mathrm{Sh}_{C^{\circ}_{-}}}(j^{-1}X; j^{-1}Y) = R\hom_{D\mathbf{b}\mathrm{Sh}_{\mathfrak{h}}}(j_! j^{-1}X; Y).$$

We also have a natural map $j_! j^{-1} X \to X$ of $\mathbf{b}$-sheaves on $\mathfrak{h}$. Let Z be the cone of this map. The statement of the Lemma is equivalent to $R\hom_{D\mathbf{b}\mathrm{Sh}_{\mathfrak{h}}}(Z, Y) = 0$

For every $k \in I$ we have a structure map

$$Z \to T^{-1}_{-2\pi e_k} Z[D(-2\pi e_k)] \tag{87}$$

Sublemma 8.8. *The natural map*

$$R\hom_{D\mathbf{b}Sh_{\mathfrak{h}}}(T^{-1}_{-2\pi e_k} Z[D(-2\pi e_k)]; Y) \to R\hom_{D\mathbf{b}Sh_{\mathfrak{h}}}(Z; Y)$$

is an isomorphism.

Proof. (of Sublemma) Let W be the cone of the map (87).We are to show that $R\hom_{D\mathbf{b}\mathrm{Sh}_{\mathfrak{h}}}(W, Y) = 0$. It follows that the structure map $W \to T^{-1}_{-e_k} W[D(-e_k)]$ is homotopy equivalent to 0.

Choose an injective representative of Y and consider a $\mathbb{L}_-$- graded complex $H(l) := \hom(W; T_l^{-1} Y)$. This complex is a $\mathbb{L}_-$-graded $\mathbf{b}$-bimodule. We also have a $\mathbb{L}_-$-graded $\mathbf{b}$-bimodule structure on $\mathbf{b}$. We then have

$$R\hom_{D\mathbf{b}\mathrm{Sh}_{\mathfrak{h}}}(W, Y) = R\hom_{\mathbf{b}-\mathrm{bimod}}(\mathbf{b}; H).$$

Let $R_k := 1 \otimes 1_{-e_k} \in \mathbf{b} \otimes \mathbf{b}$; $L_k = 1_{-e_k} \otimes 1 \in \mathbf{b} \otimes \mathbf{b}$. We then see that the action of R_k on H is a quasi-isomorphism, whereas the action of L_k is homotopy equivalent to 0. Hence the action of $R_k - L_k$ on H is a quasi-isomorphism. The action of $R_k - L_k$ on $\mathbf{b}$ is zero. Hence an induced action of $R_k - L_k$ on $R\hom_{\mathbf{b}-\mathrm{bimod}}(\mathbf{b}; H)$ is simultaneously 0 and an isomorphism, meaning that $R\hom_{\mathbf{b}-\mathrm{bimod}}(\mathbf{b}; H) = 0$, whence the statement ☐

Let $g = -\sum_{i \in I} 2\pi e_i$. Consider an inductive system of $\mathbf{b}$-sheaves on $\mathfrak{h}$:

$$Z \to T_g^{-1}[D(g)]Z \to T_{2g}^{-1}[D(2g)]Z \to \cdots \to T_{ng}^{-1}[D(ng)]Z \to \cdots$$

and let $L(Z)$ be the derived direct limit of this system. We have a natural map $Z \to L(Z)$. The previous Lemma easily implies that the induced map

$$R\hom_{D\mathbf{b}\mathrm{Sh}_{\mathfrak{h}}}(L(Z); Y) \to R\hom_{D\mathbf{b}\mathrm{Sh}_{\mathfrak{h}}}(Z; Y)$$

is an isomorphism.

It also follows that the natural map

$$R\hom(L(Z); Y) \to R\hom(Rp_{I!} L(Z); Rp_{I!} Y)$$

is an isomorphism (because Y is locally constant along fibers of p_I). Thus, the statement of our Lemma reduces to showing that

$$Rp_{I!} L(Z) = 0$$

Let $x \in \mathfrak{h}_I$ and show that $Rp_{I!} L(Z)|_x = 0$. We have

$$Rp_{I!} L(Z)|_x = R\Gamma_c(p_I^{-1} x; L(Z)|_{p_I^{-1} x})$$

Let $U_x \subset \mathfrak{h}$; $U_x := p_I^{-1} x$. By definition, we have

$$R\Gamma_c(p_I^{-1} x; L(Z)|_{p_I^{-1} x}) = \varinjlim_n R\Gamma_c(U_{x+ng}; Z|_{U_{x+ng}}[D(ng)])$$

where the spaces $R\Gamma_c(U_{x+ng}; Z|_{U_{x+ng}})$ form an inductive system by means of the structure maps $Z \to T_g^{-1}Z[D(g)]$. Next, we have

$$R\Gamma_c(U_{x+ng}; Z|_{U_{x+ng}}) = \mathrm{Cone}[R\Gamma_c(U_{x+ng} \cap C_-^\circ; X|_{U_{x+ng}}) \to R\Gamma_c(U_{x+ng}; X|_{U_{x+ng}})].$$

We have maps

$$U_x \to U_{x+g} \to U_{x+2g} \to \cdots \to U_{x+ng} \to \cdots$$

induced by the shifts T_g. Let $U := \bigcup_n T_{ng}^{-1}(U_{x+ng} \cap C_-^\circ)$. We then have

$$U_{x+ng} \cap C_-^\circ \subset T_{ng}U \subset U_{x+ng}.$$

It follows that U consists of all vectors $v = \sum_{i\in I} x_i e_i + \sum_{j \notin I} y_j f_j$, where

$$x = \sum_{i \in I} x_i e_i \tag{88}$$

and for all $l \notin I$,

$$< \sum_{j \notin I} y_j f_j, f_l >< 0. \tag{89}$$

It also follows that the natural maps

$$\varinjlim_n R\Gamma_c(U_{x+ng} \cap C_-^\circ; X|_{U_{x+ng}}][D(ng)) \to \varinjlim_n R\Gamma_c(T_{ng}U; X|_{U_{x+ng}}[D(ng)]) \tag{90}$$

is an isomorphism. Indeed, set $Z_n := T_{ng}^{-1}Z|_{T_{ng}U}[D(ng)]$, $Z_n \in D(U)$. The objects Z_n form an inductive system. Set $U_n := T_{ng}^{-1}(U_{x+ng} \cap C_-^\circ) \subset U$. We see that $U_0 \subset U_1 \subset U_2 \subset \cdots$ and $\bigcup_n U_n = U$ We then see that our inductive systems and their map can be rewritten as

$$\varinjlim_n R\Gamma_c(U_n; Z_n) \to \varinjlim_n R\Gamma_c(U; Z_n)$$

Let $K_n := U \backslash U_n$. We then see that $\cap_n K_n = 0$ and that the cone of the above map is isomorphic to

$$\varinjlim_n R\Gamma_c(K_n; Z_n|_{K_n}). \tag{91}$$

We see that for each m, the natural map

$$R\Gamma_c(K_m; Z_m|_{K_m}) \to \varinjlim_n R\Gamma_c(K_n; Z_n|_{K_n}).$$

factors as

$$R\Gamma_c(K_m; Z_m|_{K_m}) = \varinjlim_{n>m} R\Gamma_c(K_m \backslash K_n; Z_m|_{K_m}) \to \varinjlim_n R\Gamma_c(K_n; Z_n|_{K_n})$$

hence it is 0, which means that the space (91) is 0 and the map (90) is an isomorphism.

Therefore, our original statement now reduces to showing that

$$\mathrm{Cone}(R\Gamma_c(T_{ng}U; X|_{U_{x+ng}}) \to R\Gamma_c(U_{x+ng} : X|_{U_{x+ng}})) = 0 \tag{92}$$

for all $n > 0$.

let $A := \mathbb{R} < f_j >_{j \neq I}$. We have an identification

$$\alpha : A \to U_{x+ng}, \quad a \mapsto \sum_{i\in I} x_i e_i + ng + a,$$

where x_i are the same as in (88). Let $B \subset A$ be an open subset specified by the condition (89). It follows that $\alpha(B) = T_{ng}U$. Let $Y \in D(A)$, $Y := \alpha^{-1}X|_{U_{x+ng}}$. We can rewrite (92) as

$$\mathrm{Cone}(R\Gamma_c(B, Y) \to R\Gamma_c(A, Y))$$

Let us estimate the microsupport of Y. We know that $\mathrm{SS}(X) \subset \mathfrak{h} \times C_+$. Using Proposition (11.8) one can show that Y is microsupported on the set $A \times \beta^*(C_+)$, where $\beta^* : \mathfrak{h}^* \to A^*$ is dual to the embedding $\beta : A \to \mathfrak{h}$; $\beta(f_j) = f_j$. let $\varepsilon^j \in A^*$ be the basis dual to f^j. One sees that

$$\beta^*(C_+) = \mathbb{R}_{\geq} 0 < \varepsilon^j >_{j \notin I} .$$

Let $\gamma \subset A$ be the dual cone to $\beta^*(C_+)$; $\gamma = \mathbb{R}_{>0} < f_j >_{j \notin I}$. One can check $B + \gamma = A$. As $\mathrm{SS}(Y) \subset A \times \beta^*(C_+)$, the Lemma follows. $\square$

It now follows that the functor $j^{-1} : D\mathbf{b}\mathrm{Sh}_{\mathfrak{h}} \to D\mathbf{b}\mathrm{Sh}_{C_-^\circ}$ is conservative (the natural map $R\hom(F, G) \to R\hom(j^{-1}F; j^{-1}G)$ is an isomorphism). We only need to check the essential surjectivity of j^{-1}. It suffices to check that for each $I \subset \{1, 2, \ldots, N-1\}$, the functor $j^{-1} : \mathcal{C}(I, \mathfrak{h}) \to \mathcal{C}(I, C_-^\circ)$ is essentially surjective. Let $F \in \mathcal{C}(I, C_-^\circ)$ and consider a $\mathbf{b}$-sheaf $G := Rp_I^! Rp_{I!} L(j_! F) := Rp_I^{-1} Rp_{I!} L(j_! F)[N - 1 - |I|]$, where L is the same as in the proof of Lemma. One easily checks that $j^{-1}G \cong F$. This completes the proof of the theorem.

8.3.3. Let us check that the $\mathbf{b}$- sheaf $\mathcal{S}$ is strict. Indeed, it follows that the structure map

$$b_{-2\pi e_k} : \mathcal{S} \to T^{-1}_{-2\pi e_k} \mathcal{S}$$

is induced by the correponding map

$$b^{\mathfrak{S}}_{-2\pi e_k} : \mathfrak{S} \to T^{-1}_{-2\pi e_k} \mathfrak{S} = \mathfrak{S}|_{G \times -2\pi e_k} *_G \mathfrak{S}$$

which is in turn induced by the map

$$\beta_{-e_k} : \mathbb{K}_e \to \mathfrak{S}|_{G \times -e_k}$$

as in Proposition 7.18. let $B_k := \mathrm{Cone} b_k$. We then get

$$\mathrm{Cone} b^{\mathfrak{S}}_{-2\pi e_k} = B_k * \mathfrak{S}.$$

According to Proposition 7.18, $\mathrm{SS}(B_k) \subset \{(g, \omega) | :< \|\omega\|, f_k >= 0\}$ Standard computation shows that the sheaf $B_k * \mathfrak{S}$ is microsupported on the set

$$\{(g, A, \omega, \eta) | (\eta, f_k) = 0\}$$

meaning that $\mathrm{Cone} b^{\mathfrak{S}}_{-2\pi e_k} = B_k * \mathfrak{S}$ is constant along the fibers of the projection $G \times \mathfrak{h} \to G \times (\mathfrak{h}/f_k)$. Hence, $\mathrm{Cone} b_{-2\pi e_k} = i^{-1} b^{\mathfrak{S}}_{-2\pi e_k}$ is constant along the fibers of the projection

$$\mathbf{Z} \times \mathfrak{h} \to \mathbf{Z} \times \mathfrak{h}/f_k$$

It then follows that the sheaf $j^{-1}_{C_-^\circ} \mathcal{S}$ is a strict $\mathbf{b}$-sheaf on C_-°. We know (see (80) that $j^{-1}_{C_-^\circ} S \cong \bigoplus_{I \subset \{1,2,\ldots,N-1\}} \mathbb{S}_I|_{C_-^\circ}$. It then easily follows that each $\mathbb{S}_I$ is a strict $\mathbf{b}$-sheaf on C_-°. Indeed, $\mathrm{Cone} b_k^{\mathcal{S}} = \bigoplus_I \mathrm{Cone} b_k^{\mathbb{S}_I}$. Let $C := \mathrm{Cone} b_k^{\mathcal{S}}$ and $C_I := \mathrm{Cone} b_k^{\mathcal{S}'_I}$. Let $p_k : \mathbf{Z} \times C_-^\circ \to \mathbf{Z} \times C_-^\circ / f_k$. One then sees that the natural map

$$p_k^{-1} R p_{k*} C \to C$$

is isomorphic to the direct sum of natural maps

$$p_k^{-1} p_{k*} C_I \to C_I$$

As the map $p_k^{-1} R p_{k*} C \to C$ is an isomorphism, so is each of its direct summands, i.e. all maps $p_k^{-1} p_{k*} C_I \to C_I$ are isomorphisms meaning that all sheaves $\mathcal{S}'_I$ are strict.

Remark One can also prove that the sheaves $\mathbb{S}_I$ are strict directly from the definition (79).

According to Theorem 8.4, there exist strict $\mathbf{b}$-sheaves on $\mathbf{Z}\times\mathfrak{h}$, to be denoted by $\mathcal{S}_I$ such that $i^!_{C^\circ_-}\mathcal{S}_I \cong \mathbb{S}_I$ and the sheaves $\mathcal{S}_I$ are unique up-to a unique isomorphism. Same theorem implies that we should have an isomorphism

$$\mathcal{S}\cong\bigoplus_I \mathcal{S}_I.$$

9. Identifying the sheaf $\mathcal{S}$

One can check that the $\mathbf{b}$- sheaf $\mathcal{X}$ on $\mathbf{Z}\times C^\circ_-$ as in (81) is strict. Indeed, this follows from the fact that the $\mathbf{b}$- sheaf $\mathcal{S}_\emptyset = G_\emptyset\otimes\mathcal{X}$ is strict, or it can be checked directly.

It then follows that there exists a strict $\mathbf{b}$-sheaf $\mathcal{Y}$ on $\mathbf{Z}\times\mathfrak{h}$ such that $j^{-1}_{C^\circ_-}\mathcal{Y}=\mathcal{X}$. As $\mathbb{S}_I\cong G_I[D(-2\pi e_I)]\otimes T_{-2\pi e_I *}\mathcal{X}$, it then follows that we have an isomorphism $\mathcal{S}_I\cong G_I[D(-2\pi e_I)]\otimes T_{-2\pi e_I *}\mathcal{Y}$ which is induced by the obvious isomorphism

$$G_I[D(-2\pi e_I)]\otimes T_{-2\pi e_I *}\mathcal{X}|_{(C^\circ_- -2\pi e_I)} = G_I[D(-2\pi e_I)]\otimes T_{-2\pi e_I *}\mathcal{Y}|_{(C^\circ_- -2\pi e_I)}.$$

Thus, we have an isomorphism

$$\mathcal{S}\cong\bigoplus_I G_I[D(-2\pi e_I)]\otimes T_{-2\pi e_I *}\mathcal{Y}. \tag{93}$$

It now remains to identify the $\mathbf{b}$-sheaf $\mathcal{Y}$.

9.1. Identifying $\mathcal{Y}$.

9.1.1. For a subset $J\subset\{1,2,\ldots,N-1\}$ and $l\in L$ let $K(J,l)\subset e^l\times\mathfrak{h}\subset\mathbf{Z}\times h$ be defined as follows:

$$K(J,l):=\{(e^l,x)\in\mathbf{Z}\times\mathfrak{h}|\forall j\in J:< x-l,e_j>\geq 0\}.$$

Let $V(J,l):=\mathbb{K}_{K(J,l)}[D(l)]$. Let $\mathbb{L}_J=\{l\in\mathbb{L}|\forall i\notin J:< l,f_j>\leq 0\}$ Let $\Psi^J:=\bigoplus\limits_{l\in\mathbb{L}_J}V(J,l)$. Let us endow Ψ^J with a $\mathbf{b}$-structure. Let $\lambda\in\mathbb{L}_-$. We have

$$T_\lambda^{-1}V(J,l)=\mathbb{K}_{T_\lambda^{-1}K(J,l)}[D(l)];$$

$$\begin{aligned}T_\lambda^{-1}K(J,l') &= \{(e^{l'},x)|\forall j\in J: e^\lambda e^{l'}=e^l;< x+\lambda-l,e_j>\geq 0\}\\ &= K(J,l-\lambda).\end{aligned}$$

Thus,

$$T_\lambda^{-1}V(J,l)=\mathbb{K}_{K(J,l-\lambda)}[D(l)]=V(J,l-\lambda)[D(\lambda)].$$

It is clear that if $l\in\mathbb{L}_J$, then $l+\lambda\in\mathbb{L}_J$. We then can define the map $b_\lambda:\Psi^J\otimes\mathbf{b}(\lambda)\to T_\lambda^{-1}\Psi^J$ as a direct sum of maps

$$V(J,l)\otimes\mathbf{b}(\lambda)=V(J,l)[D(\lambda)]=T_\lambda^{-1}V(J,l+\lambda).$$

Let us now check that Ψ^J are strict $\mathbf{b}$-sheaves.

Let $j\notin J$. Then it is clear that Ψ^J is constant along the fibers of the map $p_j:\mathbf{Z}\times\mathfrak{h}\to\mathbf{Z}\times\mathfrak{h}/f_j$. Therefore so is the cone of b_{-e_j}. Let $j\in J$. it is then easy to see that the map b_{-e_j} is an isomoprhism, whence the statement.

Let $J_1\subset J_2$. Construct a map of $\mathbf{b}$-sheaves

$$I_{J_1J_2}:\Psi^{J_1}\to\Psi^{J_2}.$$

It is defined as the direct sum of the natural maps

$$V(J_1,l)\to V(J_2,l)$$

for all $l \in \mathbb{L}_{J_1} \subset \mathbb{L}_{J_2}$. These maps come from the closed embeddings $K(J_2, l) \subset K(J_1, l)$.

Let Subsets be the poset (hence the category) of all subsets of $\{1, 2, \ldots, N-1\}$. We then see that Ψ is a functor from Subsets to the category of $\mathbf{b}$-sheaves on $\mathbf{Z} \times \mathfrak{h}$. We then construct the standard complex $\Phi^\bullet$ such that

$$\Phi^k := \bigoplus_{I, |I|=k} \Psi^I \tag{94}$$

and the differential $d_k : \Phi^k \to \Phi^{k+1}$ is given by

$$d_k = \sum (-1)^{\sigma(J_1, J_2)} I_{J_1 J_2}, \tag{95}$$

where the sum is taken over all pairs $J_1 \subset J_2$ such that $|J_1| = k$ and $|J_2| = k+1$. The set $J_2 \backslash J_1$ then consists of a single element e and $\sigma(J_1 J_2)$ is defined as the number of elements in J_2 which are less than e.

The constructed complex defines an object in $DB\mathrm{Sh}^{\mathrm{strict}}_{\mathbf{Z} \times \mathfrak{h}}$, to be denoted by Φ.

We will show $\Phi \cong \mathcal{Y}$. To this end it suffices to prove:

Lemma 9.1. *We have* $j_{C_-^\circ}^{-1} \Psi \cong \mathcal{X}$.

Proof. We have a natural map $\iota : \mathcal{X} \to j_{C_-^\circ}^{-1} \Phi^0 = j_{C_-^\circ}^{-1} \Psi^\emptyset$. Indeed,

$$\mathcal{X} = \bigoplus_{l \in \mathbb{L}_-} \mathbb{K}_{\mathcal{U}_l}[D(l)]$$

and

$$j_{C_-^\circ}^{-1} \Psi^\emptyset = \bigoplus_{l \in \mathbb{L}_-} \mathbb{K}_{e^l \times C_-^\circ}[D(l)].$$

The map ι is defined as a direct sum of the obvious maps

$$\mathbb{K}_{\mathcal{U}_l}[D(l)] \to \mathbb{K}_{e^l \times C_-^\circ}[D(l)]$$

coming from the open embeddings $\mathcal{U}_l \subset e^l \times C_-^\circ$.

It is clear that $I_{\emptyset, J} \iota = 0$ for all nonempty J. Hence the map ι defines a map $\mathcal{X} \to j_{C_-^\circ}^{-1} \Phi$. Let us show that this map is an isomorphism.

For each $l \in L$ set

$$\Phi_l^n := \bigoplus_{J | l \in \mathbb{L}_J; |J|=n} V(J, l)[D(l)].$$

It is clear that for each l, $\Phi_l^\bullet \subset \Phi$ is a subcomplex (in the category of comlexes of sheaves on $\mathbf{Z} \times \mathfrak{h}$) and

$$\Phi = \bigoplus_{l \in \mathbb{L}} \Phi_l$$

The map ι takes values in $\bigoplus_{l \in \mathbb{L}_-} j_{C_-^\circ}^{-1} \Phi_l$ and splits into a direct sum of maps $\iota_l : \mathbb{K}_{\mathcal{U}_l} \to j_{C_-^\circ}^{-1} \Phi_l$.

We thus need to show that 1) complexes $j_{C_-^\circ}^{-1} \Phi_l$ are acyclic for all $l \notin \mathbb{L}_-$;

2) the maps ι_l are quasi-isomorphisms.

Let us first study the complexes Φ_l. Let us identify $\mathfrak{h} = \mathbb{R}^{N-1}$ by means of the basis $f_1, f_2, \ldots, f_{N-1}$. Let $X_j : \mathbf{Z} \times \mathfrak{h} \to \mathbf{Z} \times \mathbb{R}$ be defined by

$$X_j(c, A) = (c, x_j(A)),$$

where $A=\sum_j x_j(A)f_j$. Let $l_i=<l,f_i>$. Let $J_l:=\{i|l_i>0\}$. It follows that $l\in\mathbb{L}_J$ iff $J\supset J_l$. We also have

$$V(J,l)=T_{l*}(\bigotimes_{j\in J}X_j^{-1}\mathbb{K}_{e\times[0,\infty)}\otimes\bigotimes_{i\notin J}X_j^{-1}\mathbb{K}_{e\times\mathbb{R}})[D(l)],$$

where $e\in\mathbf{Z}$ is the unit. Let E be the following complex of sheaves on $\mathbf{Z}\times\mathbb{R}$:

$$\mathbb{K}_{e\times\mathbb{R}}\to\mathbb{K}_{e\times[0,\infty)}.$$

We then have an isomorphism of complexes

$$\Phi_l=(T_{l*}\bigotimes_{j\in J_l}X_j^{-1}\mathbb{K}_{e\times[0,\infty)}\otimes\bigotimes_{i\notin J_l}X_i^{-1}E)[D(l)+|J_l|].$$

We have a quasi-isomorhism $\mathbb{K}_{e\times(-\infty,0)}\to E$ which induces a quasi-isomorphism

$$\Phi_l\cong(T_{l*}\bigotimes_{j\in J_l}X_j^{-1}\mathbb{K}_{e\times[0,\infty)}\otimes\bigotimes_{i\notin J_l}X_i^{-1}\mathbb{K}_{e\times(-\infty,0)})[D(l)+|J_l|]$$

$$=T_{l*}\mathbb{K}_{W_J}[D(l)+|J_l|],$$

where

$$W_J=\{(e,A)\in\mathbf{Z}\times\mathfrak{h}|j\in J\Rightarrow x_j(A)\geq 0;i\notin J\Rightarrow x_i(A)<0\}$$

Let us now prove 1) It follows that Φ_l is supported on the set $\overline{T_l(W_{J_l})}=\overline{W_{J_l}}+(e^l,l)$. It suffices to prove that $\overline{T_l(W_{J_l})}\cap\mathbf{Z}\times C_-^\circ=0$. Suppose $z'\in\overline{T_l(W_{J_l})}\cap\mathbf{Z}\times C_-^\circ$. Let $z'=(e^l,z)$, $z\in\mathfrak{h}$.

Let $z=A+l$, $(e^l,A)\in W_{J_l}$. Let $A_j=(A,f_j)$ and $l_j=(l,f_j)$. We also set $A_0=A_N=l_0=l_N=0$. Set $x_j:=x_j(A)$. We then know that $l_j>0$ for all $j\in J_l$; $l_j\leq 0$ otherwise. We also have

$$A_j=<A,f_j>=<A,2e_j-e_{j-1}-e_{j+1}>=2x_j-x_{j-1}-x_{j+1}.$$

As $A+l\in C_-^\circ$, we have $A_j+l_j<0$. Therefore, if $j\in J$, then $A_j<0$, thus $2x_j-x_{j-1}-x_{j+1}<0$. We also know that if $j\in J$, then $x_j\geq 0$.

If $j\notin J$, then we know that $x_j\leq 0$. For $j\in J$ let $j_1<j$ be the largest number such that $j_1\notin J$, if it does not exist, set $j_1=0$. Similarly, let $j_2>j$ be the smallest number such that $j_2\notin J$, if it does not exist set $j_2=N$.

We then have $x_{j_1}\leq 0$; $x_{j_2}\leq 0$; for all j such that $j_1<j<j_2$; $2x_j-x_{j-1}-x_{j+1}<0$, hence $x_j-x_{j-1}<x_{j+1}-x_j$, and $x_j\geq 0$.

Therefore, we have

$$0\leq x_{j_1+1}-x_{j_1}<x_{j_1+2}-x_{j_1+1}<\cdots<x_{j_2}-x_{j_2-1}\leq 0.$$

Observe that $j_2-j_1\geq 2$, therefore, we get $0<0$, which is a contradiction. Thus, indeed, $j_{C_-^\circ}^{-1}\Phi_l\cong 0$ for all $l\notin\mathbb{L}_-$.

2) If $l\in\mathbb{L}_-$, then $J_l=\emptyset$ and we have a quasi-isomorphism $\mathbb{K}_{(e^l,x)|x<<l}[D(l)]\to\Phi_l$. Therefore we have an induced quasi-isomorphism $\mathbb{K}_{\mathcal{U}_l}[D(l)]\to j_{C_-^\circ}^{-1}\Phi_l$. One can easily check that this map is isomorphic to ι, whence the statement. $\square$

From now on we set $\mathcal{Y}=\Phi$.

Let us summarize our results:

Theorem 9.2. *Let $\mathcal{Y}=\Phi$, where Φ is as in (94),(95). Then we have an isomoprhism (93)*

This theorem is equivalent to Theorem 5.6.

10. Appendix 1:SU(N) and its Lie algebra: notations and a couple of Lemmas

Let us introduce notation we will use when working with $G = \mathrm{SU}(N)$. Let $\mathfrak{g}$ be the Lie algebra of G; it is naturally identified with the space of all skew-hermitian traceless $N \times N$ matrices. Let $\mathfrak{h} \subset \mathfrak{g}$ be the Cartan subalgebra of $\mathfrak{g}$ consisting of all diagonal matrices from $\mathfrak{g}$. The abelian Lie algebra $\mathfrak{h}$ consists of all matrices of the form $i\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)$, where $\lambda_i \in \mathbb{R}$ and $\sum_i \lambda_i = 0$.

Let $C_+ \subset \mathfrak{h}$ be the positive Weyl chamber consisting of all matrices $i\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$. For every $X \in \mathfrak{g}$ there exists a unique element $\|X\| \in C_+$ such that X is conjugate with $\|X\|$.

We have an invariant positive definite inner product $<,>$ on $\mathfrak{g}$ such that $< X, Y >= -\mathrm{Tr}(XY)$. By means of this product we identify $\mathfrak{g} = \mathfrak{g}^*$, $\mathfrak{h} = \mathfrak{h}^*$.

We will use the basis of roots in $\mathfrak{h}^*$ which consists of vectors $f_1, f_2, \ldots, f_{N-1}$, where

$$f_k(i\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)) = \lambda_k - \lambda_{k+1}.$$

Via identification $\mathfrak{h} = \mathfrak{h}^*$, the vector $f_k \in \mathfrak{h}^*$ corresponds to a vector in $\mathfrak{h}$ denoted by the same symbol, and we have

$$f_k = i\mathrm{diag}(0, 0, \ldots, 0, 1, -1, 0, \ldots, 0)$$

where 1 is at the k-th position.

We also have the dual basis of coroots $e_1, e_2, \ldots, e_N$ determined by $< f_k, e_l >= \delta_{kl}$. One has

$$e_k = i\mathrm{diag}((N-k)/N, (N-k)/N, \ldots, (N-k)/N, -k/N, -k/N, \ldots, -k/N) \tag{96}$$

where there are total k entries equal to $(N-k)/N$. One can check that $f_k = 2e_k - e_{k-1} - e_{k+1}$ for $k = 1, 2, \ldots, N-1$ and we assume $e_0 = e_N = 0$.

One can rewrite $e_k = i\mathrm{diag}(1, 1, \ldots, 1, 0, 0, \ldots, 0) - ik/N\mathrm{diag}(1, 1, 1, \ldots, 1)$, where it is assumed that we have k entries of 1 in $\mathrm{diag}(1, 1, \ldots, 1, 0, 0, \ldots, 0)$. In particular, we have

$$< e_k, i\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) >= \lambda_1 + \lambda_2 + \cdots + \lambda_k.$$

One also sees that C_+ consists of all $X \in \mathfrak{h}$ such that $< X, f_k >\geq 0$. Therefore,

$$C_+ = \{\sum_{k=1}^{N} L_k e_k | L_k \geq 0\}.$$

We have a partial order on $\mathfrak{h}$: $X \geq Y$ means $< X - Y, e_k >\geq 0$ for all k.

We also write $X >> Y$ if $< X - Y, e_k >> 0$ for all k.

Let $\omega \in \mathfrak{g}$. The matrix $-i\omega$ is hermitian and let $\lambda_1(\omega) > \lambda_2(\omega) > \cdots > \lambda_r(\omega)$ be eigenvalues of $-i\omega$. Let $V^k(\omega)$ be the eigenspace of $-i\omega$ of eigenvalue λ_k. Let

$$V_k(\omega) = V^1(\omega) \oplus V^2(\omega) \oplus \cdots \oplus V^k(\omega).$$

We then get a partial flag

$$0 \subset V_1(\omega) \subset \cdots \subset V_r(\omega) = \mathbb{C}^N. \tag{97}$$

Let $d_k(\omega) := \dim V_k(\omega)$.

10.0.2. In the future, we will need

Lemma 10.1. *Let $X, \omega \in \mathfrak{g}$. Let $\|X\| = i diag(A_1, A_2, \ldots, A_N) \in C_+$ and let*

$$0 \subset V_1(\omega) \subset \cdots \subset V_r(\omega) = \mathbb{C}^N$$

be the flag as in (97).

Then

$$< \omega, X > \leq (\|\omega\|, \|X\|).$$

The equality takes place iff

a) $[X, \omega] = 0$ (hence $XV_k(\omega) \subset V_k(\omega)$ for all k, and

b) $TrX|_{V_k} = i(A_1 + A_2 + \cdots A_{d_k(\omega)}) = i < e_{d_k(\omega)}; \|X\| >$.

Proof. Let $\mu_k = \lambda_k(\omega) - \lambda_{k+1}(\omega)$; $k < r$. Let us also set $\mu_r = \lambda_r(\omega)$. We then have

$$\omega = i \sum_{k=1}^{r} \mu_k \mathbf{pr}_{V_k(\omega)},$$

where $\mathbf{pr}$ denotes the orthogonal projector;

$$< \omega, X > = \sum_{k=1}^{r-1} \mu_k \mathrm{Tr}(-iX\mathbf{pr}_{V_k(\omega)}).$$

We know that $\mathrm{Tr}(-iX\mathbf{pr}_{V_k(\omega)}) \leq A_1 + A_2 + \cdots + A_{d_k(\omega)}$ (this is a particular case of the general factP: given any hermitian matrix Y on $\mathbb{C}^N$ (in our case $-iX$) and a vector subspace $V \subset \mathbb{C}^N$ of dimension n (in our case $V_k(\omega)$), the maximal value of $\mathrm{Tr}(Y\mathbf{pr}_V)$ equals the sum of top n eigenvalues of Y).

Hence

$$< \omega, X > \leq \sum_{k=1}^{r-1} \mu_k (A_1 + \cdots + A_{d_k(\omega)}) = \sum_{j=1}^{r} A_j \sum_{k | j \leq d_k(\omega)} \mu_k$$

$$= \sum_{j=1}^{r} A_j \lambda_j(\omega) = < \|\omega\|, \|X\| > .$$

The equality is only possible if for all k $\mathrm{Tr}(-iX\mathbf{pr}_{V_k(\omega)}) = A_1 + \cdots A_{d_k(\omega)}$. As $A_1, \ldots, A_{d_k(\omega)}$ are top $d_k(\omega)$ eigenvalues of $-iX$, the equality occurs iff $V_k(\omega)$ is the span of eigenvectors of $-iX$ with eigenvalues $A_1, \ldots, A_{d_k(\omega)}$, which implies the statement b) of Lemma. □

10.0.3.

Lemma 10.2. *Let $X, Y \in \mathfrak{g}$. We have $\|X + Y\| \leq \|X\| + \|Y\|$.*

Proof. We need to show that for every k,

$$< \|X + Y\|, e_k > \leq < \|X\|, e_k > + < \|Y\|, e_k > .$$

For a Hermitian operator A on a finite-dimensional Hermitian vector space V we set $\mathbf{n}(A) := \max_{|v|=1} < Av, v >$, where $<,>$ is the hermitian inner product on V. We see that

$$\mathbf{n}(A + B) \leq \mathbf{n}(A) + \mathbf{n}(B) \tag{98}$$

and that $\mathbf{n}(A)$ equals the maximal eigenvalue of A.

Let ε_k be the standard representation of $\mathfrak{g}$ on $\Lambda^k\mathbb{C}^N$. Let $X \in \mathfrak{g}$ and let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$ be the spectrum of a Hermitian matrix $-iX$. This means that $\|X\| = i\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)$.

Eigenvalues of $-i\varepsilon_k(X)$ are of the form $\lambda_{i_1}+\lambda_{i_2}+\cdots+\lambda_{i_k}$ where $i_1<i_2<\ldots<i_k$. Therefore, the maximal eigenvalue of $-i\varepsilon_k(X)$ is $\lambda_1+\lambda_2+\ldots+\lambda_k$, i.e.

$$\mathbf{n}(-i\varepsilon_k(X)) = <\|X\|, e_k> .$$

As follows from (98),

$$\mathbf{n}(-i\varepsilon_k(X+Y)) \le \mathbf{n}(-i\varepsilon_k(X)) + \mathbf{n}(-i\varepsilon_k(Y)),$$

hence

$$<\|X+Y\|, e_k> \le <\|X\|, e_k> + <\|Y\|, e_k>,$$

as was required. □

10.0.4. Let $[a,b]\subset\mathbb{R}$, $a\le b$, be a segment. Let $g\in \mathrm{SU}(N)$. Write $g\sim[a,b]$ if every eigenvalue of g is of the form $e^{i\phi}$, where $\phi\in[a,b]$.

Lemma 10.3. *Let $g_k\sim[a_k,b_k]$, $k=1,2$. Then $g_1g_2\sim[a_1+a_2,b_1+b_2]$.*

Proof. If $b_1+b_2-(a_1+a_2)\ge 2\pi$, there is nothing to prove, because $x\sim[a_1+a_2,b_1+b_2]$ for any element $x\in\mathrm{SU}(N)$. Let now $b_1+b_2-(a_1+a_2)<2\pi$. Let $c_k=(a_k+b_k)/2$ and $d_k=(b_k-a_k)/2$. We have $d_1+d_2<\pi$, hence $d_k<\pi$, $k=1,2$.

Let $h_k=e^{-ic_k}g_k$. We have $h_k\sim[-d_k,d_k]$. Let $S\subset\mathbb{C}^N$ be the unit sphere. Let ρ be the standard metric on S; $\rho(v,w)=\arccos\mathrm{Re}<v,w>$; $\rho(v,w)\in[0,\pi]$. For $g\in\mathrm{SU}(N)$, set

$$\mathbf{N}(g):=\max_{v\in S}\rho(gv,v).$$

It follows $\mathbf{N}(g_1g_2)\le\mathbf{N}(g_1)+\mathbf{N}(g_2)$ for all $g_1,g_2\in\mathrm{SU}(N)$.

Let us estimate $\mathbf{N}(h_k)$. Let $e_1,e_2,\ldots,e_N$ be an eigenbasis of h_k. We have $h_k(e_s)=e^{i\alpha_{ks}}e_s$, where $\alpha_{ks}\in[-d_k,d_k]$. Let $v=\sum_s v_se_s$, $v\in S$, so that $1=\sum_s|v_s|^2$. We have

$$h_kv=\sum_s v_se^{i\alpha_{ks}}e_s;$$

$$<h_kv,v>=\sum_s|v_s|^2e^{i\alpha_{ks}};$$

$$\mathrm{Re}<h_kv,v>=\sum_s|v_s|^2\cos\alpha_{ks}.$$

As $\alpha_{ks}\in[-d_k,d_k]$ and $0\le d_k<\pi$, we have $\cos\alpha_{ks}\ge\cos d_k$. Therefore,

$$\mathrm{Re}<h_kv,v>\ge\sum_s|v_s|^2\cos d_k=\cos d_k.$$

Therefore,

$$\mathbf{N}(h_k)=\rho(h_kv,v)=\arccos\mathrm{Re}<h_kv,v>\le d_k.$$

Therefore, $\mathbf{N}(h_1h_2)\le\mathbf{N}(h_1)+\mathbf{N}(h_2)\le d_1+d_2$. It then follows that $h_1h_2\sim[-d_1-d_2;d_1+d_2]$. Indeed, assuming the contrary, we have an eigenvalue $e^{i\phi}$ of h_1h_2, where $d_1+d_2<|\phi|\le\pi$. let $h_1h_2v=e^{i\phi}v$, $|v|=1$. We then have $\rho(h_1h_2v,v)=|\phi|>d_1+d_2$, which is a contradiction.

Finally, we have $g_1g_2=e^{c_1+c_2}h_1h_2$, which implies that

$$g_1g_2\sim[c_1+c_2-d_1-d_2;c_1+c_2+d_1+d_2]=[a_1+a_2;b_1+b_2].$$

□

10.0.5. Fix $b \in C_+^\circ$; $b < e_1/(100N)$. Here and below $\circ$ means the interior.

Lemma 10.4. *Let $X, Y \in \mathfrak{g}$ and $\|X\|, \|Y\| \le b$. Then $e^X e^Y = e^Z$, where $\|Z\| \le \|X\| + \|Y\|$.*

Proof. We have

$$e_1 = ((N-1)/N, -1/N, -1/N, \dots, -1/N) = (1, 0, 0, \dots, 0) - 1/N(1, 1, \dots, 1).$$

Let $b = i\mathrm{diag}(b_1, b_2, \dots, b_N)$. Since $b \in C_+^\circ$, we have $b_1 > b_2 > \dots > b_N$. We have $< b, e_k >< < e_1/(100N), e_k >$ for all k. In particular, $b_1 =< b, e_1 > \le (N - 1/N) \cdot (1/100N) < 1/(100N)$; $b_1 + b_2 + \dots + b_{N-1} = (b, e_{N-1}) \le < e_1, e_{N-1} > /(100N) = 1/(100N^2) < 1/(100N)$. As $\sum_k b_k = 0$, we have $b_N > -1/(100N)$. Thus, $\forall k, |b_k| \le 1/(100N)$. Let $\|X\| = i\mathrm{diag}(X_1, X_2, \dots, X_N)$. As $\|X\| \le b$, $|X_k| \le 1/(100N)$.

Therefore, one has $e^X \sim [-1/(100N); 1/(100N)]$. Analogously, $e^Y \sim [-1/(100N); 1/(100N)]$. Lemma 10.3 implies that

$$e^X e^Y = [-2/(100N); 2/(100N)] = [-1/(50N); 1/(50N)].$$

Let $u_1, u_2, \dots, u_N$ be the eigenbasis for $e^X e^Y$. It then follows that $e^X e^Y = e^{i\phi_s} u_s$, where $|\phi_s| \le 1/(50N)$. We have $1 = \det(e^X e^Y) = e^{i\sum_s \phi_s}$. Therefore $\sum_s \phi_s = 2\pi n$, $n \in \mathbb{Z}$. However, $|\sum_s \phi_s| \le 1/50 < 2\pi$. Hence, $n = 0$ and $\sum_s \phi_s = 0$. Let Z be a skew-hermitian matrix defined by $Zu_s = i\phi_s u_s$. As $\sum_s \phi_s = 0$, $Z \in \mathfrak{su}(N) = \mathfrak{g}$. We also have $e^X e^Y = e^Z$. Let us prove that $\|Z\| \le \|X\| + \|Y\|$.

Let Λ_k (resp. ε_k) be the standard representation of $G = \mathrm{SU}(N)$ (resp. $\mathfrak{g} = \mathfrak{su}(N)$) on $\Lambda^k \mathbb{C}^N$. We then have

$$e^{\varepsilon_k(Z)} = e^{\varepsilon_k(X)} e^{\varepsilon_k(Y)}.$$

Let $\|Z\| = i\mathrm{diag}(Z_1, Z_2, \dots, Z_N)$. As was shown above, we have $|Z_j| \le 1/(50N)$.

We then see that the spectrum of $\varepsilon_k(Z)$ consists of all numbers of the form

$$i(Z_{j_1} + Z_{j_2} + \cdots Z_{j_k}),$$

where $j_1 < j_2 < \dots < j_k$. We have

$$|Z_{j_1} + Z_{j_2} + \cdots Z_{j_k}| \le k/(50N) \le 1/50. \tag{99}$$

Let $\|X\| = i\mathrm{diag}(X_1, X_2, \dots, X_N)$. the spectrum of $e^{\lambda_k(X)}$ consists of numbers of the form

$$e^{i(X_{j_1} + X_{j_2} + \dots + X_{j_k})},$$

where $j_1 < j_2 < \ldots < j_k$. Therefore

$$e^{\lambda_k(X)} \sim [X_{N-k+1} + X_{N-k+2} + \dots + X_N; X_1 + X_2 + \dots + X_k].$$

We have $X_{N-k+1} + X_{N-k+2} + \dots + X_N = -(X_1 + X_2 + \dots + X_{N-k}) = - < X, e_{N-k} >$. Therefore,

$$e^{\lambda_k(X)} \sim [- < \|X\|, e_{N-k} >; < \|X\|, e_k >].$$

Analogously,

$$e^{\lambda_k(Y)} \sim [- < \|Y\|, e_{N-k} >; < \|Y\|, e_k >].$$

By Lemma 10.3, we have

$$e^{\lambda_k(Z)} = e^{\lambda_k(X)} e^{\lambda_k(Y)} \sim [- < \|X\| + \|Y\|, e_{N-k} >; < \|X\| + \|Y\|, e_k >].$$

As was shown above, we have $|X_j|, |Y_j| \le 1/(100N)$ for all j. Therefore, $| < \|X\|, e_{N-k} > | \le (N-k)/(100N) < 1/100$. Analogously

$$| < \|X\|, e_k > |, | < \|Y\|, e_k > |, < \|Y\|, e_{N-k} < 1/100.$$

Therefore

$$[- < \|X\| + \|Y\|, e_{N-k}; < \|X\| + \|Y\|, e_k >] \subset [-1/50; 1/50].$$

According to (99), all eigenvalues of $\lambda_k(Z)$ are of the form it, $|t| \leq 1/50$. It now follows that all eigenvalues of $\lambda_k(Z)$ are of the form it, where

$$t \in [- < \|X\| + \|Y\|, e_{N-k} >; < \|X\| + \|Y\|, e_k >].$$

(otherwise, $e^{i\lambda_k(Z)}$ is not of the form e^{it}, where $t \in [- < \|X\| + \|Y\|, e_{N-k} >; < \|X\| + \|Y\|, e_k >]$, as follows from our estimates). In particular,

$$< \|Z\|, e_k >= Z_1 + Z_2 + \cdots + Z_k \in [- < \|X\| + \|Y\|, e_{N-k}; < \|X\| + \|Y\|, e_k >],$$

whence

$$< \|Z\|, e_k > \leq < \|X\| + \|Y\|, e_k > .$$

As k is arbitrary, it follows that $\|Z\| \leq \|X\| + \|Y\|$. □

For our future purposes we will need a stronger result.

10.0.6.

Lemma 10.5. *Let $X_1, X_2, \dots X_n \in \mathfrak{g}$; $\|X_i\| \leq b$. Let $V_1 \subset V_2 \subset \cdots V_r = \mathbb{C}^N$ be a flag which is preserved by all X_i Then there exists an $X \in \mathfrak{g}$ such that:*
1) $e^{X_1} e^{X_2} \cdots e^{X_n} = e^X$;
2)$XV_k \subset V_k$ and $TrX|_{V_k} = \sum_k TrX_k|_{V_k}$ for all k;
3) $\|X\| \leq \sum_k \|X_i\|$

Proof. 1) Fix an Ad- invariant Hilbert norm $\mathbf{N}$ on $\mathfrak{g}$ (such an $\mathbf{N}$ is unique up-to a scalar multiple). It follows that $\mathbf{N}(Z) \leq \mathbf{N}(Y_1) + \mathbf{N}(Y_2)$, the equality being possible only if Y_1 and Y_2 are proportional with non-negative coefficient(indeed: $\mathbf{N}(Z)$ is the length of the geodesic from the unit to e^Z; $\mathbf{N}(Y_1) + \mathbf{N}(Y_2)$ is the length of a broken line, the equality is possible only if this broken line is actually a geodesic).

2) Suppose $Y_1, Y_2 \in \mathfrak{g}$; $\|Y_1\|, \|Y_2\| \leq b$. According to Lemma 10.4 there exists a unique $Z := Z(Y_1, Y_2) \in \mathfrak{g}$; $\|Z\| \leq \|Y_1\| + \|Y_2\|$ such that $e^Z = e^{Y_1} e^{Y_2}$. We see that $e^Z V_k = V_k$, hence $(e^Z - \mathrm{Id})V_k \subset V_k$. We can express Z as a convergent series in powers of $e^Z - \mathrm{Id}$, therefore, $ZV_k \subset V_k$. The equality

$$\det e^Z|_{V_k} = \det e^{Y_1}|_{V_k} \det e^{Y_2}|_{V_k}$$

implies that $e^{\mathrm{Tr}Z|_{V_k}} = e^{\mathrm{Tr}(Y_1+Y_2)|_{V_k}}$. As $\|Z\| \leq 2b$, this implies that $\mathrm{Tr}Z|_{V_k} = \mathrm{Tr}(Y_1 + Y_2)|_{V_k}$.

3) Let $(Y_1, Y_2, \cdots Y_n)$ be a sequence of elements $Y_i \in \mathfrak{g}$; $|Y_i| \leq b$. Let

$$S_k(Y_1, Y_2, \dots, Y_n) := (Y_1, \dots, Y_{k-1}, Z/2, Z/2, Y_{k+2}, \dots, Y_n),$$

where $k = 1, 2, \dots, n-1$, $Z = Z(Y_k, Y_{k+1})$ is as explained above.

Let $\mathcal{X} \subset \mathfrak{g}^n$ be the set consisting of all sequences of the form

$$S_{k_1} S_{k_2} \cdots S_{k_R}(X_1, X_2, \dots, X_n)$$

for all R and all $k_1, k_2, \dots, k_R$. Let μ be the infimum of $\mathbf{N}(Y_1) + \mathbf{N}(Y_2) + \cdots \mathbf{N}(Y_n)$ where $(Y_1, Y_2, \dots, Y_n) \in \mathcal{X}$.

Let $(Y_1(k), Y_2(k), \dots, Y_n(k)) \in \mathcal{X}$, $k = 1, 2, \dots$, be a sequence such that $\mathbf{N}(Y_1(k)) + \cdots \mathbf{N}(Y_n(k)) \to \mu$ as $k \to \infty$. As $|Y_i(k)| \leq b$, one can choose a convergent subsequence, hence without loss of generality, one can assume that our sequence converges:

$$\lim_{k\to\infty} Y_i(k) = Z_i.$$

Then for all $(Y_1, Y_2, \dots, Y_n) \in X$,

$$\mathbf{N}(Y_1) + \cdots \mathbf{N}(Y_n) \geq \mathbf{N}(Z_1) + \cdots \mathbf{N}(Z_n).$$

Let us show that there exists $Z \in \mathfrak{g}$ such that each Z_i is proportional to Z with a non-negative coefficient. If not then there are $i < j$ such that

1) for all $i < k < j$, $Z_k = 0$;

2) Z_i and Z_j are not proportional to each other with a non-negative coefficient. Let $(Z'_1, \dots, Z'_n) = T_{j-1} \cdots T_{i+1} T_i (Z_1, Z_2, \dots, Z_n)$. We then have $\mathbf{N}(Z'_1) + \cdots \mathbf{N}(Z'_n) < \mathbf{N}(Z_1) + \cdots \mathbf{N}(Z_n)$. Hence there exists a k such that

$$\mathbf{N}(Y'_1) + \cdots + \mathbf{N}(Y'_n) < \mathbf{N}(Z_1) + \mathbf{N}(Z_2) + \cdots + \mathbf{N}(Z_n),$$

where

$$(Y'_1, Y'_2, \dots, Y'_n) = T_{j-1} \cdots T_i (Y_1(k), Y_2(k), \dots, Y_n(k)).$$

But $(Y'_1, Y'_2, \dots, Y'_n) \in \mathcal{X}$, so we get a contradiction.

Thus all Z_i are proportional with non-negative coefficients. Let us now set $X = Z_1 + Z_2 + \dots Z_n$. Such an X satisfies all the conditions □

11. Appendix 2: Results from [1] on functorial properties of microsupport

Despite the results to be quoted here are proved in [1] for the bounded derived category, the same arguments work for the unbounded derived category so that we will omit the proofs.

11.0.7. Let $S \subset X$ be a subset and $x \in X$. Following [1] Definition 5.3.6, one can define subsets $N(S) \subset TX$ and $N^*(S) \subset T^*X$. As explained on p 228, these subsets can be characterized as follows. Let $x \in X$. A non-zero vector $\theta \in T_xX$ belongs to $N_x(S)$ if and only if, in a local chart near x, there exists an open cone γ containing θ and a neighborhood U of x such that $U \cap ((S \cap U) + \gamma) \subset S$.

One then defines $N^*_x(S) \subset T^*_xX$ as the dual cone to $N_x(S)$. Finally one sets $N(S) = \cup_x N_x S$; $N^*(S) = \cup_x N^*_x(S)$. If $S \subset X$ is a closed submanifold, then $N^*(S) = T^*_S(X)$

Let now $x \in X$ and let U be a neighborhood of x. Suppose that $S \cap U$ is defined by an inequality $f > 0$ (or f
$geq0$), where $f : U \to \mathbb{R}$ is a smooth function and $d_xf \neq 0$. In this case $N^*_x(S) = \mathbb{R}_{\geq 0} \cdot d_xf$.

For a subset $K \subset T^*X$ we set $K^a \subset T^*X$ to consist of all vectors ω such that $-\omega \in K$.

Proposition 11.1. *([1], Proposition 5.3.8) Let X be a manifold, Ω an open subset and Z a closed subsets. Then $SS(\mathbb{K}_\Omega) = N^*(\Omega)^a$; $SS(\mathbb{K}_Z) = N^*(Z)$*

11.0.8.

Proposition 11.2. *([1],Proposition 5.4.1) Let $F \in D(X)$ and $G \in D(Y)$. Then*

$$SS(F \boxtimes G) \subset SS(F) \times SS(G).$$

(Note that since our ground ring is a field $\mathbb{K}$, the bifunctor $\boxtimes$ is exact)

11.0.9. Let $q_1 : X \times Y \to X$; $q_2 : X \times Y \to Y$ be the projections.

Proposition 11.3. *([1], Proposition 5.4.2) Let $F \in D(X)$; $G \in D(Y)$. Then:*

$$SSR\underline{Hom}(q_2^{-1}G; q_1^{-1}F) \subset SS(F) \times SS(G)^a,$$

where $SS(G)^a \subset T^*Y$ consists of all points ω such that $-\omega \in SS(G)$.

11.0.10. Let $f : Y \to X$ be a morphism of manifolds. We have natural maps

$$(f^t) : T^*X \times_X Y \to T^*Y$$

and $f_\pi : T^*x \times_X Y \to T^*X$.

Thus, $T^*X \times_X Y$ is a correspondence between T^*X and T^*Y. Using this correspondence, one can transport sets from T^*Y to T^*X and vice versa. Indeed, given a subset $A \subset T^*Y$ one has a subset $f_\pi(f^t)^{-1}A \subset T^*X$. Given a subset $B \subset T^*X$, one has a subset $(f^t)f_\pi^{-1}(B) \subset T^*Y$.

Proposition 11.4. *([1],Proposition 5.4.4) Let $f : Y \to X$ be a morphism of manifolds, $G \in D(Y)$, and assume f is proper on Supp(G). Then*

$$SS(Rf_*G) \subset f_\pi((f^t)^{-1}(SS(G))).$$

Observe that under the hypothesis of this Proposition, the natural map $Rf_!G \to Rf_*G$ is an isomorphism. Therefore, the Proposition remains true upon replacement of Rf_* with $Rf_!$.

11.0.11. Let $f : Y \to X$ be a morphism of manifolds and $A \subset T^*X$ a closed conic subset. We say that f is non-characteristic for A if $f_\pi^{-1}A \cap T_Y^*X \subset Y \times_X T_X^*X$. Here $T_Y^*X \subset T^*X \times XY$ is the kernel of (f^t) viewed as a linear map of vector bundles.

Proposition 11.5. *([1],Proposition 5.4.13) Let $F \in D(X)$ and assume $f : Y \to X$ is non-characteristic for $SS(F)$. Then*

$$\textit{(i)} \quad SS(f^{-1}F) \subset (f^t)(f_\pi^{-1}(SS(F)));$$

(ii) The natural morphism $f^{-1}F \otimes \omega_{Y/X} \to f^!F$ is an isomorphism.

11.0.12.

Proposition 11.6. *([1],Proposition 5.4.14) Let $F, G \in D(X)$.*
(i) Assume $SS(F) \cap SS(G)^a \subset T_X^*X$. Then $SS(F \otimes G) \subset SS(F) + SS(G)$;
(ii) Assume $SS(F) \cap SS(G) \subset T_X^*X$. Then $SS(R\underline{Hom}(G, F) \subset SS(F) - SS(G)$.

11.0.13. We need a notion of Witney sum of two conic closed subsets $A, B \subset T^*X$. We will reproduce a definition in terms of local coordinates from [1] Remark 6.2.8 (ii).

Let (x) be a system of local coordinates on X, (x, ξ) the asssociated coordinates on T^*X. Then $x_o, \xi_o \in A \hat{+} B$ iff there exist sequences $\{(x_n, \xi_n)\}$ in A and $\{(y_n, \eta_n)\}$ in B such that $x_n \to x_o$, $y_n \to y_o$, $\xi_n + \eta_n \to \xi_o$, and $|x_n - y_n||\xi_n| \to 0$.

Proposition 11.7. *([1],Theorem 6.3.1). Let Ω be an open subset of X and $j : \Omega \to X$ the embedding. Let $F \in D(X)$. Then $SS(Rj_*F) = SS(F)\hat{+}N^*(\Omega)$; $SS(j_!F) \subset SS(F)\hat{+}N^*(\Omega)^a$.*

11.0.14. Let $f: Y \to X$ be a morphism of manifolds and $A \subset T^*X$ be a closed conic subset. One can define a closed conic subset $f^{\#}(A) \subset T^*M$ ([1], Definition 6.2.3 (iv)).

Proposition 11.8. *([1], Corollary 6.4.4) Let $F \in D(X)$. Then $SS(f^{-1}F) \subset f^{\#}(SS(F))$.*

In a particular case when f is a closed embedding, the set $f^{\#}(A)$ admits an explicit description in local coordinates [1], Remark 6.2.8, (i). That's the only case we will need.

Let (x', x'') be a system of local coordinates on X such that $Y = \{(x', 0)\}$. Then $(x''_o; x''_o) \in f^{\#}(A)$ iff there exists a sequence of points $(x'_n, x''_n, \xi'_n, \xi''_n) \in A$ such that $x'_n \to 0$; $x''_n \to x''_o$; $\xi''_n \to \xi''_o$, and $|x'_n||\xi'_n| \to 0$.

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