

# PROOF OF SWISS CHEESE VERSION OF DELIGNE'S CONJECTURE

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*To the beautiful country of Confoederatio Helvetica.*

ABSTRACT. For an associative algebra  $A$  we consider the pair “the Hochschild cochain complex  $C^\bullet(A, A)$  and the algebra  $A$ ”. There is a natural 2-colored operad which acts on this pair. We show that this operad is quasi-isomorphic to the singular chain operad of Voronov’s Swiss Cheese operad. This statement is the Swiss Cheese version of the Deligne conjecture formulated by M. Kontsevich in [22].

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## CONTENTS

1. Introduction	2
1.1. Remarks on higher dimensional versions	3
1.2. Organization and layout of the paper	4
2. The operad $\mathcal{O}$ of natural operations on the objects $C^n(A, A), n = 0, 1, 2, \dots; A$	5
3. Two combinatorial descriptions of the operad $\mathcal{O}$	7
3.1. Finite ordinals instead of natural numbers	7
3.2. Planar trees	7
3.3. Replacing trees with sequences	10
3.4. The whole operad $\mathcal{O}$ in terms of sequences	13
3.5. Operadic structure on $\mathfrak{s}\mathcal{O}$	13
4. Review of 2-operads	15
4.1. Colored 2-operads	16
4.2. Unital colored 2-operads	17
4.3. Symmetrization	18
4.4. Batanin’s theorem	19
5. Swiss Cheese (SC) Operads	19
5.1. Symmetric Swiss Cheese type operads	20
5.2. SC 2-operads	21
5.3. Desymmetrization	23
5.4. Symmetrization	23
6. Linking the operad $\mathfrak{s}\mathcal{O}$ with 2 operads: a 2-operad $\mathfrak{seq}$	24
6.1. 2-operad $\mathfrak{seq}$ (cochain part)	24
6.2. SC version of $\mathfrak{seq}$	25
6.3. Totalization: A dg 2-operad $ \mathfrak{seq} $ and a dg operad $ \mathfrak{s}\mathcal{O} $	25
6.4. Extension to the SC-setting: a dg SC 2-operad $\mathfrak{seq}^{\text{SC}}$ and an SC operad $ \mathfrak{s}\mathcal{O} $	26
7. The SC 2-operad $\mathfrak{br}$ and the SC operad $\mathfrak{braces}$	27
7.1. An increasing filtration on the colored 2-operad $\mathfrak{seq}$	27
7.2. Extension of the filtration onto $\mathfrak{seq}^{\text{SC}}$	27

7.3. Compatibility of the filtration with the operadic structure	28
7.4. Definition of the (SC) 2-operad <b>br</b>	31
7.5. An increasing filtration on $\mathfrak{sO}$	32
7.6. Definition of the (SC) operad <b>braces</b>	34
8. Proof of Theorem 7.9	35
8.1. Proof of Proposition 8.1	37
9. Proof of Theorem 2.1	46
Appendix	49
References	51

## 1. INTRODUCTION

The interest to various versions [6], [8], [18], [20], [23], [24], [25], [27], [31], [34], [36] of the Deligne conjecture on Hochschild complex is motivated by generalizations [11], [12], [30], [33], [35] of the famous Kontsevich’s formality theorem [21]. Thus, in recent preprint [24] M. Kontsevich and Y. Soibelman proposed a proof of the chain version of Deligne’s conjecture for Hochschild complexes of an  $A_\infty$ -algebra. This is an important step in proving the formality for the homotopy calculus algebra of Hochschild (co)chains [12].

Let  $A$  be an associative algebra and  $C^\bullet(A, A)$  be the Hochschild cochain complex of  $A$ . The original version of Deligne’s conjecture says that the operad of natural operations on  $C^\bullet(A, A)$  is quasi-isomorphic to the singular chain operad of the operad  $E_2$  of little discs [10], [26]. This statement is not very precise because there are different choices of what one may call “the operad of natural operations on  $C^\bullet(A, A)$ .” One may use the so-called minimal operad of M. Kontsevich and Y. Soibelman [23] or the operad of braces [16], [19] as in [27] and [37] or the “big operad” of M. Batanin and M. Markl [5]. Due to works of various people [5], [8], [23], [27], [32], and [37] it is now known that all these operads are quasi-isomorphic to the singular chain operad of the operad  $E_2$ .

The topological operad  $E_2$  of little discs admits a natural extension to a 2-colored topological operad which is called the Swiss Cheese operad  $SC_2$ . This operad was proposed by A. Voronov in [38].

In [38] A. Voronov also described the homology operad  $H_{-\bullet}(SC_2)$ . More precisely, he showed that an algebra over the operad  $H_{-\bullet}(SC_2)$  is a pair of graded vector spaces  $(V_1, V_2)$ , where  $V_1$  is a Gerstenhaber algebra<sup>1</sup>, and  $V_2$  is an associative algebra equipped with a module structure over the commutative algebra  $V_1$

$$(1.1) \quad V_1 \otimes V_2 \rightarrow V_2,$$

satisfying the following condition

$$(1.2) \quad (u_1 \cdot v_1) \dots (u_n \cdot v_n) = (u_1 \dots u_n) \cdot (v_1 \dots v_n),$$

where  $u_i \in V_1$ ,  $v_i \in V_2$ , and for the multiplication of the corresponding elements we use either the associative algebra structure in  $V_2$  or the commutative algebra structure in  $V_1$ .

It is not hard to prove the following proposition:

**Proposition 1.1.** *If  $A$  is an associative algebra and  $HH^\bullet(A, A)$  is its Hochschild cohomology then the pair  $(HH^\bullet(A, A), A)$  forms an algebra over the operad  $H_{-\bullet}(SC_2)$ .*

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<sup>1</sup>In particular, it means that  $V_1$  is a commutative algebra.

*Proof.* Indeed the associative algebra structure on  $A$  is already given.  $HH^\bullet(A, A)$  is a Gerstenhaber algebra due to [14]. Finally, to define the module structure on  $A$  over the commutative algebra  $HH^\bullet(A, A)$  we use the fact that the zeroth Hochschild cohomology  $HH^0(A, A)$  is the center  $Z(A)$  of  $A$ . Namely, we declare

$$z \cdot a = \begin{cases} z a, & \text{if } z \in HH^0(A, A) = Z(A), \\ 0, & \text{otherwise.} \end{cases}$$

Equation (1.2) is nontrivial only when  $u_i \in HH^0(A, A)$ . In this case the required condition is automatically satisfied since  $u_i$ 's are elements of the center  $Z(A)$  of  $A$ .  $\square$

In this paper we prove the Swiss Cheese version of Deligne's conjecture which extends Proposition 1.1 to the level of cochains.

To formulate this version of Deligne's conjecture we, first, construct a 2-colored DG operad  $\Lambda$  of natural operations on the pair  $(C^\bullet(A, A); A)$ . Roughly speaking, this operad is generated by the insertions of a cochain into a cochain, the cup-product of cochains and the insertions of elements of the algebra  $A$  into a cochain. The precise description of  $\Lambda$  is given in Section 2.

The main result of this paper is the following theorem

**Theorem 1.2.** *The 2-colored DG operad  $\Lambda$  of natural operations on the pair  $(C^\bullet(A, A), A)$  is quasi-isomorphic to the singular chain operad of Voronov's Swiss Cheese operad  $SC_2$ . The induced action of the homology operad  $H_{-\bullet}(SC_2)$  on the pair  $(HH^\bullet(A), A)$  recovers the one from Proposition 1.1.*

We prove this theorem using ideas from [32] and Batanin's theorem [2] which identifies the homotopy type of Voronov's Swiss Cheese operad with that of the symmetrization of a contractible cofibrant Swiss Cheese type 2-operad. The required facts about 2-operads are reviewed in Sections 4,5

**1.1. Remarks on higher dimensional versions.** Voronov's Swiss Cheese operad admits the obvious higher dimensional analogue  $SC_d$  ( $d \geq 2$ ). This operad extends the operad of  $d$ -cubes in the same way as the operad  $SC_2$  extends the operad of little disks. From this point of view, Theorem 1.2 is a 2-dimensional case of the following conjecture formulated by M. Kontsevich in [22]: *the DG operad of natural operations on the pair "a  $d$ -algebra<sup>2</sup> and its Hochschild complex" is quasi-isomorphic to the singular chain operad of  $SC_{d+1}$ .* In [22, Section 2.5] M. Kontsevich also conjectures that the Hochschild cochain complex of a  $d$ -algebra is a final object in an appropriate category of "Swiss Cheese algebras". In our paper, this question about universality is not addressed.

In [13] J.N.K. Francis showed that an appropriate deformation complex for a  $d$ -algebra  $A$  is an extension of its Hochschild complex by  $A$ . In the spirit of this result the above version of Deligne's conjecture can be reformulated as follows: *the DG operad of natural operations on the deformation complex of a  $d$ -algebra is quasi-isomorphic to the singular chain operad of  $SC_{d+1}$ .*

**Notation and conventions.** We denote by  $\mathbf{k}$  the ground field and by "(co)chain complexes" we mean (co)chain complexes of vector spaces over  $\mathbf{k}$ .  $A$  is a unital associative algebra over  $\mathbf{k}$  and  $C^\bullet(A, A)$  is the normalized Hochschild cochain complex of  $A$  with coefficients in  $A$

$$(1.3) \quad C^\bullet(A, A) = \text{hom}((A/\mathbf{k})^{\otimes \bullet}, A).$$

The abbreviation SMC stands for "symmetric monoidal category" and the notation  $\mathbf{1}$  is reserved for the unit of a symmetric monoidal category. We also use the abbreviation SC for "Swiss Cheese

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<sup>2</sup>Recall from [17] that a  $d$ -algebra is an algebra over the homology operad  $H_{-\bullet}(E_d)$  of the operad of little  $d$ -cubes  $E_d$ .

type” when we discuss the Swiss Cheese type symmetric operads, 2-operads, sets, ordinals, and 2-trees.

**1.2. Organization and layout of the paper.** All arguments of the paper can be restricted onto the setting when we only care about operations on  $C^\bullet(A, A)$  (and not on the pair  $(C^\bullet(A, A), A)$ ). Throughout the paper we use terms like ‘non SC part’ or ‘cochain part’ to indicate that we restrict to  $C^\bullet(A, A)$  only. The exposition is organized so that most of the constructions are first introduced in the non SC setting and then extended to the whole SC picture. As a rule, this SC extension is rather straightforward. In our exposition we tried to isolate the spots dealing with the SC setting; we hope that the reader interested in proving Deligne’s conjecture only will be able to easily recognize these spots and drop them without any harm to understanding.

Let us now go over the content of the paper. We start (Sec 2) with defining an operad  $\mathcal{O}$  of natural operations on the infinite collection of objects

$$(1.4) \quad C^n(A, A), n = 0, 1, 2, \dots; A.$$

Next, we explain how, using the functors of polysimplicial/cosimplicial totalization (which are called *condensation* in [4]), we can convert the operad  $\mathcal{O}$  into a dg operad  $|\mathcal{O}|$  which acts on the pair of complexes:  $C^\bullet(A, A)$  and  $A$ . The operad  $|\mathcal{O}|$  is the same as the operad  $\Lambda$  in Theorem 1.2.

In Sec 3 we give a combinatorial description of  $\mathcal{O}$  in terms of trees and then reformulate it in terms of sequences. The latter description is used in the rest of the paper.

Next, we invoke Batanin’s 2-operad theory: in Sec 4 we review the basic notions of the theory and in Sec 5 we discuss an SC version of these notions (also due to Batanin). This section is not needed for the cochain (Deilgne’s) part of the SC conjecture. In Section 6 we define a 2-operadic version **seq** of the operad  $\mathcal{O}$ .

In Section 6.3 we apply the totalization (=condensation) procedure to the operads **seq** and  $\mathfrak{s}\mathcal{O}$ . As a result we get an operad  $|\mathfrak{s}\mathcal{O}|$  acting on the complex  $C^\bullet(A, A)$  as well as its 2-operadic version  $|\mathbf{seq}|$ . At this moment the advantage of the 2-operadic approach can be seen: the 2-operad  $|\mathbf{seq}|$  turns out to be contractible, contrary to  $|\mathfrak{s}\mathcal{O}|$ .

We conclude the section with extending the above mentioned constructions to the SC setting. We obtain a contractible SC 2-operad  $|\mathbf{seq}^{\text{SC}}|$  which acts on the pair  $(C^\bullet(A, A), A)$ . If this operad satisfied a technical condition of being *reduced*, Batanin’s theory would imply an action of Voronov’s SC operad on  $(C^\bullet(A, A), A)$ . But  $|\mathbf{seq}^{\text{SC}}|$  happens to be non-reduced which causes us to find a reduced contractible sub-operad **br** of  $|\mathbf{seq}^{\text{SC}}|$ , see Sec 7. Using a similar approach we also construct a suboperad **braces** of  $|\mathfrak{s}\mathcal{O}|$ . The action of the operad  $|\mathbf{braces}|$  on  $C^\bullet(A, A)$  seems to be equivalent to the celebrated *brace structure* on  $C^\bullet(A, A)$  ([15], [16], [19]). Batanin’s theory can now be applied to  $|\mathbf{br}|$ ; we get an action on  $(C^\bullet(A, A), A)$  of a certain operad  $E$  which is homotopy equivalent to Voronov’s SC operad (the operad  $E$  is the symmetrization of a cofibrant resolution of **br**, i.e.  $E := \mathbf{sym} \mathcal{R}\mathbf{br}$ , see (7.25)).

It also follows that this action passes through the action of **braces** that is we have a map of operads  $E \rightarrow \mathbf{braces}$ . We prove that this map is a weak equivalence, see Theorem 7.9; the proof of this theorem occupies the whole Sec 8. We are now ready for proving the SC conjecture (Sec. 9). There is an Appendix which contains a certain contractibility statement needed for proving Lemma 7.2.

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2. THE OPERAD  $\mathcal{O}$  OF NATURAL OPERATIONS ON THE OBJECTS  $C^n(A, A), n = 0, 1, 2, \dots; A$

Let  $A$  be a unital monoid in some tensor (not necessarily symmetric) category (for example, in the category of complexes over a field). Consider the full nonsymmetric endomorphism operad of  $A$

$$C_A(n) := \text{hom}(A^{\otimes n}; A), \quad n \geq 0.$$

It is clear that  $A$  is naturally a  $C_A$ -algebra. The associative unital structure on  $A$  gives rise to a map of nonsymmetric operads  $\mathbf{assoc} \rightarrow C_A$ , where  $\mathbf{assoc}$  is the nonsymmetric operad of sets controlling unital monoids; each space  $\mathbf{assoc}(n), n \geq 0$ , is a point.

We fix a set of colors  $X_\rho := \mathbb{N} \sqcup \{\mathbf{a}\}$  and define a  $X_\rho$ -colored symmetric operad  $\mathcal{O}$  in the category of sets as an operad whose algebra structure on an  $X_\rho$ -family of objects  $(C(n), n \in \mathbb{N}; A)$  is:

- a nonsymmetric operad structure on the collection of objects  $C(n)$ ;
- a map of nonsymmetric operads  $\mathbf{assoc} \rightarrow C$ ;
- a  $C$ -algebra structure on  $A$ .

The operad  $\mathcal{O}$  has the following sets of operations:

$$\text{— } \mathcal{O}(k)_{n_1, n_2, \dots, n_k}^n := \mathcal{O}((n_1, n_2, \dots, n_k) \mapsto n)$$

where all the entries are in  $\mathbb{N}$ ;

$$\text{— } \mathcal{O}(k, N)_{n_1, n_2, \dots, n_k} := \mathcal{O}((n_1, n_2, \dots, n_k, \underbrace{\mathbf{a}, \mathbf{a}, \dots, \mathbf{a}}_N) \mapsto \mathbf{a}), \quad N \geq 0.$$

The operadic sets for other colorings are empty.

The sets  $\mathcal{O}(k)_{n_1, n_2, \dots, n_k}^n$  form a  $\mathbb{N}$ -colored operad in the obvious way. Call this operad *the cochain part of  $\mathcal{O}$* . An algebra over this operad is a non-symmetric operad  $C$  equipped with a map (of non-symmetric operads)

$$\mathbf{assoc} \rightarrow C.$$

Later on (see 3.2) an explicit combinatorial description of the operad  $\mathcal{O}$  will be given.

2.0.1. The unary operations in the colored operad  $\mathcal{O}$  endow the set of colors with the following category structure:

- $\text{hom}(n, \mathbf{a}) = \emptyset$  for all  $n > 0$ ,  $\text{hom}(0, \mathbf{a})$  is a one-point set;
- $\text{hom}(\mathbf{a}, n) = \emptyset$  for all  $n \in \mathbb{N}$ ;
- $\text{hom}(n, m) = \text{hom}_\Delta([n], [m])$  for all  $n, m \in \mathbb{N}$ ;
- $\text{hom}(\mathbf{a}, \mathbf{a}) = \{\text{Id}\}$ .

This implies that the operadic sets of our colored operad  $\mathcal{O}$  have a natural polysimplicial/cosimplicial structure, namely:

the collection of sets

$$\mathcal{O}(k)_{n_1, n_2, \dots, n_k}^n,$$

as  $n_i, n$  run through  $\mathbb{N}$ , is a functor

$$\mathcal{O}(k) : (\Delta^{\text{op}})^k \times \Delta \rightarrow \mathbf{Sets},$$

(the functor is simplicial in each of the arguments  $n_1, n_2, \dots, n_k$  and cosimplicial in  $n$ );

likewise, for each  $N$ , the collection of sets

$$\mathcal{O}(k, N)_{n_1, n_2, \dots, n_k}$$

forms a functor  $\mathcal{O}(k, N) : (\Delta^{\text{op}})^k \rightarrow \mathbf{Sets}$ .

2.0.2. Let  $\mathcal{S}$  be a cosimplicial complex given by

$$\mathcal{S}([n])^\bullet := \overline{C_{-\bullet}}(\Delta^n, \mathbf{k}),$$

where the complex on the right hand side is the normalized chain complex of the simplex  $\Delta^n$  put in the non-positive degrees.

Using this complex, we can convert polysimplicial/cosimplicial sets into complexes.

Namely, let  $F : (\Delta^{\text{op}})^k \rightarrow \mathbf{Sets}$  be a functor. Set

$$|F| := \mathbf{k}[F] \otimes_{(\Delta^{\text{op}})^k} \mathcal{S}^{\boxtimes k},$$

where  $\mathcal{S}^{\boxtimes k} : \Delta^k \rightarrow \mathbf{complexes}$  :

$$\mathcal{S}^{\boxtimes k}([n_1], [n_2], \dots, [n_k]) := \bigotimes_{i=1}^k \mathcal{S}([n_i]).$$

Given a functor

$$G : (\Delta^{\text{op}})^k \times \Delta \rightarrow \mathbf{Sets},$$

denote by  $G^n$  the evaluation at  $[n] \in \Delta$  so that

$$G^n : (\Delta^{\text{op}})^k \rightarrow \mathbf{Sets}$$

and

$$n \mapsto G^n$$

is a functor from  $\Delta$  to the category of  $k$ -simplicial sets. Set

$$|G| := \text{hom}_\Delta(\mathcal{S}^\bullet, |G^\bullet|).$$

2.0.3. Set

$$|\mathcal{O}|(k) := |\mathcal{O}(k)|;$$

$$|\mathcal{O}|(k, N) := |\mathcal{O}(k, N)|.$$

We see that these spaces form a 2-colored DG operad. Denote this two-colored operad by  $|\mathcal{O}|$ .

Now let  $A$  be a unital associative algebra over the field  $\mathbf{k}$ . It is easy to see that the normalized Hochschild cochain complex  $C^\bullet(A, A)$  (1.3) can be written as

$$C^\bullet(A, A) := \text{hom}_\Delta(S^*, C_A(*)).$$

Therefore the DG operad  $|\mathcal{O}|$  acts on the pair  $(C^\bullet(A, A), A)$ . This two-colored DG operad  $|\mathcal{O}|$  is the desired operad  $\Lambda$  of natural operations on the pair  $(C^\bullet(A, A), A)$  and our Theorem 1.2 can be reformulated as

**Theorem 2.1.** *The operad  $|\mathcal{O}|$  is weakly equivalent to the singular chain operad of Voronov's Swiss Cheese operad  $SC_2$ . The induced action of the homology operad  $H_{-\bullet}(SC_2)$  on the pair  $(HH^\bullet(A), A)$  recovers the one from Proposition 1.1.*

We prove this theorem in Section 9.

**Remark.** Our method also works in the topological setting: one can apply the topological realization functors to the polysimplicial/cosimplicial sets from 2.0.1 so as to get a topological colored operad  $|\mathcal{O}|_{\text{top}}$ . This operad can be proven to be weakly equivalent to Voronov's Swiss Cheese operad.

### 3. TWO COMBINATORIAL DESCRIPTIONS OF THE OPERAD $\mathcal{O}$

Our first description will be in terms of planar trees. Next, we will explain a transition from the tree description to another one, in terms of sequences.

Each construction will be first introduced for the cochain part of  $\mathcal{O}$  and then extended to the whole operad. These extensions for both constructions are rather straightforward.

We will start with fixing a more convenient language.

**3.1. Finite ordinals instead of natural numbers.** Recall that our set of colors is  $\mathbb{N} \sqcup \{\mathfrak{a}\}$ , and that an  $\mathcal{O}$ -algebra structure on the collection of spaces  $(C(n), n \in \mathbb{N}; A)$  is the same as a nonsymmetric operad structure on the collection of spaces  $C(n), n \in \mathbb{N}$ , a map of operads  $\mathbf{assoc} \rightarrow C$ , and a  $C$ -algebra structure on  $A$ . The definition of a nonsymmetric operad implies that we have a total order on the set of arguments so that it is better to replace the natural numbers with isomorphism classes of finite ordinals: the number  $n$  gets replaced with the ordinal  $\langle n \rangle = \{1 < 2 < \dots < n\}$ .

Given finite sets  $S, S_c$ , an  $S$ -family  $\{I_s\}_{s \in S}$  and an  $S_c$ -family  $\{I_s\}_{s \in S_c}$  of finite (possibly empty) ordinals, an ordinal  $J$ , and a set  $S_a$ , we then have the following operadic sets:

$$(3.1) \quad \mathcal{O}(S)_{\{I_s\}_{s \in S}}^J;$$

$$(3.2) \quad \mathcal{O}(S_c, S_a)_{\{I_s\}_{s \in S_c}},$$

where in (3.1) the set of arguments is  $S$  and the coloring of  $s \in S$  is  $I_s$ , the result has the color  $J$ . In (3.2), the set of arguments is  $S_c \sqcup S_a$  the argument  $s \in S_c$  has color  $I_s$  and all arguments from  $S_a$  have color  $\mathfrak{a}$ . The result also has color  $\mathfrak{a}$ .

### 3.2. Planar trees.

**3.2.1. The cochain part of  $\mathcal{O}$  via planar trees.** For a finite set  $S$  and ordinals  $I_s, s \in S; J$ , we describe

$$\mathcal{O}(S)_{\{I_s\}_{s \in S}}^J$$

as the set of equivalence classes of planar trees  $T$  with the following structure:

- a subset of the set of vertices of a tree  $T$  is identified with  $S \sqcup J$  in such a way that with elements of  $J$  we may only identify the terminal vertices of  $T$ . We call the vertices identified with elements of  $S \sqcup J$  marked.

- the ordered set of edges originating at the vertex marked by  $s \in S$  is identified with  $I_s$ .

Notice that, the subset of vertices identified with  $J$  acquires from  $J$  a natural linear order. We require that this linear order coincides with the order which is obtained by going around the tree in the clockwise direction starting from the root vertex.

The equivalence relation is the finest one in which two such trees are equivalent if one of them can be obtained from the other by either:

- the contraction of an edge with unmarked ends

or: removing an unmarked vertex with only one edge originating from it and joining the two edges adjacent to this vertex into one edge.

**Example.** The planar tree  $T$  in figure 1 represents an element in  $\mathcal{O}(S)_{\{I_s\}_{s \in S}}^J$  with  $S = \{s_1, s_2\}$ ,  $J = \{j_1, j_2, j_3\}$ ,  $I_{s_1} = \emptyset$ , and  $I_{s_2} = \langle 3 \rangle$ . In all the figures we use circles to denote the vertices

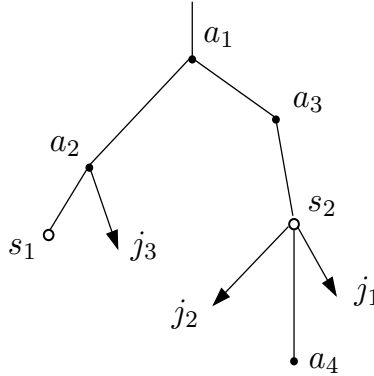


FIGURE 1. Tree  $T$

marked by elements of  $S$  and arrows to denote vertices marked by elements of  $J$ . Thus, in figure 1 the vertices  $a_1$ ,  $a_2$ ,  $a_3$ , and  $a_4$  are unmarked. The vertices  $a_1$  and  $a_2$  correspond to the product in  $\mathbf{assoc}(2)$ ,  $a_3$  corresponds to the identity operation in  $\mathbf{assoc}(1)$ , and  $a_4$  corresponds to the unit in  $\mathbf{assoc}(0)$ .

In figures 2 and 3 we depict the trees  $T_1$  and  $T_2$  which are equivalent to the original tree  $T$ .

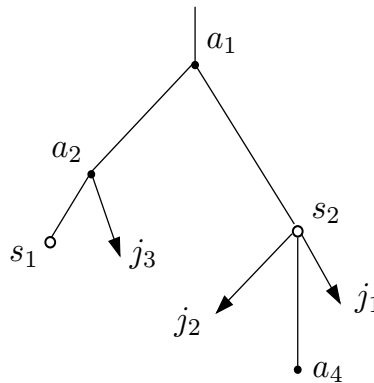


FIGURE 2. Tree  $T_1$

The tree  $T_1$  is obtained from  $T$  by removing the unmarked vertex  $a_3$  and joining the two edges adjacent to this vertex into one edge. The tree  $T_2$  is obtained from  $T$  by contracting the edge with the unmarked ends  $a_1$  and  $a_2$ . The unmarked vertex  $a$  of the tree  $T_2$  (figure 3) corresponds to the unique element of  $\mathbf{assoc}(3)$ .

Applying both of the equivalence operations to the tree  $T$  in figure 1 we obtain the tree  $T_3$  depicted in figure 4. Although the tree  $T_3$  has unmarked vertices  $a$  and  $a_4$ , it is no longer possible to apply any equivalence operation to  $T_3$ . We call such trees *minimal*. It is obvious that every equivalence class of  $\mathcal{O}(S)_{\{I_s\}_{s \in S}}^J$  contains at least one minimal tree.

The equivalence class containing all these planar trees  $T$ ,  $T_1$ ,  $T_2$ , and  $T_3$  corresponds to the operation which sends a Hochschild cochain  $P_1 \in C_A(0)$  and a Hochschild cochain  $P_2 \in C_A(3)$  to



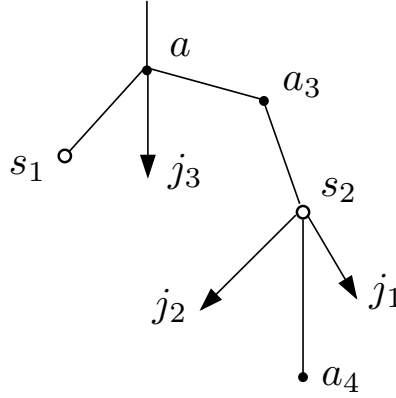


FIGURE 3. Tree  $T_2$

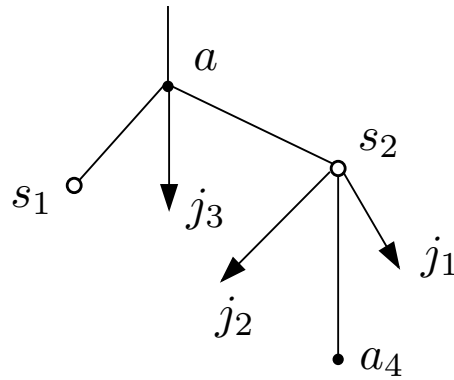


FIGURE 4. Tree  $T_3$

the Hochschild cochain  $Q \in C_A(3)$  defined by the formula

$$Q(b_1, b_2, b_3) = P_2(b_1, 1, b_2) b_3 P_1, \\ b_1, b_2, b_3 \in A.$$

3.2.2. *The whole operad  $\mathcal{O}$  in terms of planar trees.* Let us now describe the set

$$\mathcal{O}(S_c, S_a)_{\{I_s\}_{s \in S_c}},$$

where we use the same notation as above.

Each element of this set can be represented by a planar tree  $T$  with the following additional structure:

— a subset of the set of vertices of  $T$  is identified with  $S_c \sqcup S_a$  in such a way that with elements of  $S_a$  we may only identify the terminal vertices of  $T$ . We call the vertices identified with elements of  $S_c \sqcup S_a$  marked;

— the ordered set of edges originating at the vertex marked by  $s \in S_c$  is identified with  $I_s$ .

The equivalence relation on the set of isomorphism classes of such trees is defined in the same way as in the previous section.

This description implies the following identification:

$$\mathcal{O}(S_c, S_a)_{\{I_s\}_{s \in S_c}} = \bigsqcup_{> \in \mathbf{ord}(S_a)} \mathcal{O}(S_c)_{\{I_s\}_{s \in S_c}}^{S_a, >},$$

where  $\mathbf{ord}(S_a)$  is the set of all total orders on  $S_a$ .

Let us also describe the degenerate cases. In the case  $S$  is the empty set  $\emptyset$  we have

$$\mathcal{O}(\emptyset)^J = \mathbf{assoc}(J).$$

If  $S_a = \emptyset$  then

$$\mathcal{O}(S_c, \emptyset)_{\{I_s\}_{s \in S_c}} = \mathcal{O}(S_c)_{\{I_s\}_{s \in S_c}}^\emptyset.$$

Finally, if  $S_c$  is empty then

$$\mathcal{O}(\emptyset, S_a) = \bigsqcup_{> \in \mathbf{ord}(S_a)} \mathbf{assoc}(S_a, >).$$

**3.3. Replacing trees with sequences.** We will put into a correspondence to any planar tree from the previous subsection a certain sequence which will lead to another description of  $\mathcal{O}$ . We start with the cochain part of  $\mathcal{O}$ .

**3.3.1. Cochain part of  $\mathcal{O}$  in terms of sequences, I.** We need the following notation. Given a vertex  $v$  of a planar tree marked by an element  $s \in S$ , let us draw a little circle centered at this vertex. This circle gets split into sectors, the set of these sectors is totally ordered in the clockwise order. Denote this ordered set by  $I'_s$ . The set of edges originating at  $v$  is naturally identified with  $\vec{I}'_s$ , where  $\vec{I}'_s$  is the set of pairs  $\vec{i}_1 i_2$ , where  $i_2$  is an immediate successor of  $i_1$  and  $i_1, i_2 \in I'_s$ . We see that  $I'_s$  is the next ordinal after  $I_s$ . Below, given an ordinal  $K$ , we denote by  $K'$  its next ordinal.

Given a planar tree  $T$  which defines an element  $\bar{T} \in \mathcal{O}(S)_{\{I_s\}_{s \in S}}^J$ , let us consider its small tubular neighborhood and let us walk along its boundary starting from the root vertex of our tree in the clockwise direction. On our way, we will meet the vertices marked by elements of  $S$  and vertices marked by elements of  $J$ . (The latter ones are terminal according to our requirement.) Every time we approach a vertex  $v$  marked by  $s \in S$ , we are at a certain sector from  $I'_s$ . Thus, given a planar tree  $T$  representing an element  $\bar{T} \in \mathcal{O}(S)_{\{I_s\}_{s \in S}}^J$ , we obtain a total order  $>_T$  on the set

$$\bigsqcup_{s \in S} I'_s \sqcup J.$$

**Example.** Let us show how we obtain the order for the tree  $T_3$  given in figure 4. This tree represents an element in  $\mathcal{O}(\{s_1, s_2\})_{I_{s_1}, I_{s_2}}^{\{j_1, j_2, j_3\}}$  where  $I_{s_1}$  is empty and  $I_{s_2} = \langle 3 \rangle$ . This means that the vertex labeled by  $s_1$  (see figure 5) is surrounded by a single sector  $s_1^1$ , while the vertex labeled by  $s_2$  is surrounded by four sectors  $s_2^1, s_2^2, s_2^3, s_2^4$  which we number in the clockwise direction. Walking

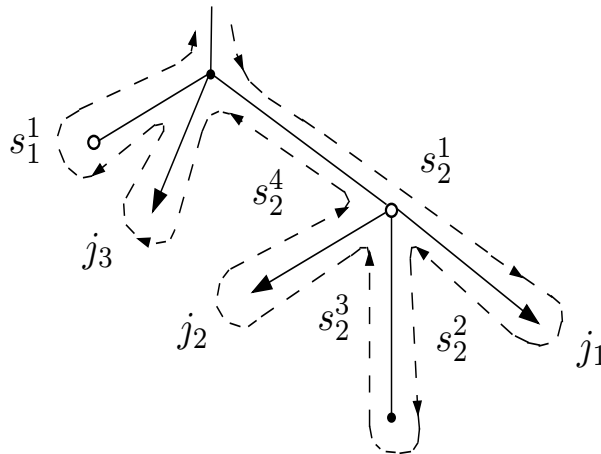


FIGURE 5. Tree  $T_3$

along the boundary of a small tubular neighborhood of  $T_3$ , as it is shown on figure 5, we get the following order on the set  $\{s_1^1, s_2^1, s_2^2, s_2^3, s_2^4, j_1, j_2, j_3\}$  :

$$s_2^1 < j_1 < s_2^2 < s_2^3 < j_2 < s_2^4 < j_3 < s_1^1 .$$

For every planar tree  $T$  representing an element  $\overline{T} \in \mathcal{O}(S)_{\{I_s\}_{s \in S}}^J$  the corresponding total order  $>_T$  satisfies:

- 1) let  $T_1, T_2$  be planar trees representing the same element

$$\overline{T}_1 = \overline{T}_2 \in \mathcal{O}(S)_{\{I_s\}_{s \in S}}^J$$

Then  $>_{T_1} = >_{T_2}$ . Hence, for each  $\overline{T} \in \mathcal{O}(S)_{\{I_s\}_{s \in S}}^J$  we have a well defined order, to be denoted by  $>_{\overline{T}}$ ;

- 2) The total order  $>_T = >_{\overline{T}}$  agrees with the existing orders on  $I'_s, J$ ;
- 3) given distinct  $s_1, s_2 \in S$  it is impossible to find  $i_1, j_1 \in I'_{s_1}; i_2, j_2 \in I'_{s_2}$  such that

$$i_1 <_T i_2 <_T j_1 <_T j_2 .$$

Let us denote the set of all total orders satisfying conditions 2) and 3) by  $\text{Ord}(S)_{\{I_s\}_{s \in S}}^J$ .

**Proposition 3.1.** *The set  $\text{Ord}(S)_{\{I_s\}_{s \in S}}^J$  is in 1-to-1 correspondence with  $\mathcal{O}(S)_{\{I_s\}_{s \in S}}^J$ .*

*Proof.* Let us describe an inductive construction which assigns to each total order  $\pi$  on

$$(3.3) \quad \bigsqcup_{s \in S} I'_s \sqcup J$$

satisfying conditions 2) and 3) a minimal tree  $T$  which recovers the order  $\pi$  by walking along a small tubular neighborhood of  $T$ .

The induction goes by the order  $|S|$  of the set  $S$ .

For  $S = \emptyset$  the set  $\text{Ord}(S)_{\{I_s\}_{s \in S}}^J$  consists of a single element. That is the given order on  $J$ . In this case it is very easy to find a minimal tree which recovers this order. It is also easy to see that such a tree is unique.

Let us suppose that we can construct a desired minimal tree for all elements of  $\text{Ord}(S_0)_{\{I_s\}_{s \in S_0}}^J$  if  $|S_0| < |S|$ . We need to present a construction for every  $\pi \in \text{Ord}(S)_{\{I_s\}_{s \in S}}^J$ .

Condition 3) implies that for an arbitrary pair  $s, \tilde{s} \in S$  exactly one of the following options realizes:

- (1) all elements of  $I'_{\tilde{s}}$  are smaller than elements of  $I'_s$ ,
- (2) all elements of  $I'_{\tilde{s}}$  are greater than elements of  $I'_s$ ,
- (3)  $I'_{\tilde{s}}$  splits into two non-empty subsets such that all elements of the first subset are smaller than all elements of  $I'_s$  while all the elements of the second subset are greater than elements of  $I'_s$
- (4) same as (3) with  $s$  and  $\tilde{s}$  interchanged.

If the third (resp. fourth) option realizes we say that  $s < \tilde{s}$  (resp.  $\tilde{s} < s$ ). Thus we get a partial order on the set  $S$ .

Since  $S$  is finite, it has at least one minimal element. Let us denote this element by  $s_{min}$  and introduce the interval  $\tilde{I}_{s_{min}}$  of the ordinal (3.3) between the minimal element of  $I'_{s_{min}}$  and the maximal element of  $I'_{s_{min}}$ . It is obvious that  $\tilde{I}_{s_{min}}$  consists of elements of  $I'_{s_{min}}$  and some elements of  $J$ .

Let us consider the set

$$(3.4) \quad \bigsqcup_{s \in S^{(1)}} I'_s \sqcup J^{(1)} ,$$

where  $S^{(1)} = S \setminus \{s_{min}\}$  and  $J^{(1)}$  is obtained from  $J$  by attaching the element  $s_{min}$  and removing those elements of  $J$  which belong to the interval  $\tilde{I}_{s_{min}}$ . In other words,

$$(3.5) \quad J^{(1)} = J \sqcup \{s_{min}\} \setminus (J \cap \tilde{I}_{s_{min}}).$$

Notice that, the set (3.4) is obtained from (3.3) by replacing the interval  $\tilde{I}_{s_{min}}$  by a single element  $s_{min}$ . Hence, (3.4) acquires a natural total order. Let us denote this order by  $\pi^{(1)}$ .

It is not hard to see that  $\pi^{(1)}$  satisfies conditions 2) and 3) and hence is an element of the set

$$\text{Ord}(S^{(1)})_{\{I_s\}_{s \in S^{(1)}}}^{J^{(1)}}.$$

Since  $|S^{(1)}| < |S|$  we can assign to  $\pi^{(1)}$  a minimal tree  $T^{(1)}$  which recovers the order  $\pi^{(1)}$  on (3.4).

To construct the desired tree  $T$  we observe that the element  $s_{min}$  is identified with an external vertex  $v$  of  $T^{(1)}$ . So, we draw from this vertex  $v$  edges labeled by elements  $\overrightarrow{i_1 i_2}$  of  $\overrightarrow{I'}_{s_{min}}$ . Recall that  $\overrightarrow{I'}_{s_{min}}$  consists of pairs  $\overrightarrow{i_1 i_2}$ , where  $i_2$  is an immediate successor of  $i_1$  and  $i_1, i_2 \in I'_{s_{min}}$ .

Let us denote by  $t^{i_1 i_2}$  the terminal vertex of the edge corresponding to  $\overrightarrow{i_1 i_2}$ .

If there are no elements of  $J$  between  $i_1$  and  $i_2$  then we leave  $t^{i_1 i_2}$  as an unmarked terminal vertex of the tree  $T$ .

If there is only one element  $j$  of  $J$  between  $i_1$  and  $i_2$  we leave  $t^{i_1 i_2}$  as a terminal vertex of  $T$  and mark it by  $j$ .

Finally, if we have elements  $j_1, \dots, j_m \in J$  ( $m > 1$ ) between  $i_1$  and  $i_2$ , then we draw from the vertex  $t^{i_1 i_2}$  exactly  $m$  terminal edges. We leave  $t^{i_1 i_2}$  unmarked and mark the corresponding terminal vertices by  $j_1, \dots, j_m$  in the clockwise direction.

Let us denote the resulting tree by  $T$ . It is not hard to see that, since  $T^{(1)}$  recovers the order  $\pi^{(1)}$  on (3.4) the tree  $T$  recovers the order  $\pi$  on (3.3). It is also obvious that, since the tree  $T^{(1)}$  is minimal, so is  $T$ .

We already have a map from the set  $\mathcal{O}(S)_{\{I_s\}_{s \in S}}^J$  to the set  $\text{Ord}(S)_{\{I_s\}_{s \in S}}^J$  which is defined by assigning the total order to a tree. Let us denote this map by  $\nu_{\text{ord}}$

$$\nu_{\text{ord}} : \mathcal{O}(S)_{\{I_s\}_{s \in S}}^J \rightarrow \text{Ord}(S)_{\{I_s\}_{s \in S}}^J.$$

The above construction provides us with the map in the opposite direction:

$$\nu_{\text{tree}} : \text{Ord}(S)_{\{I_s\}_{s \in S}}^J \rightarrow \mathcal{O}(S)_{\{I_s\}_{s \in S}}^J.$$

It is clear from the construction that the composition  $\nu_{\text{ord}} \circ \nu_{\text{tree}}$  is the identity on  $\text{Ord}(S)_{\{I_s\}_{s \in S}}^J$ .

It is not hard to verify that if we start with a minimal tree  $T$  representing an element  $\overline{T} \in \mathcal{O}(S)_{\{I_s\}_{s \in S}}^J$ , and assign to  $T$  the total order  $\pi$  from  $\text{Ord}(S)_{\{I_s\}_{s \in S}}^J$ , then the above construction gives us back exactly the same minimal tree  $T$ . This implies that the composition  $\nu_{\text{tree}} \circ \nu_{\text{ord}}$  is the identity on the set  $\mathcal{O}(S)_{\{I_s\}_{s \in S}}^J$  and the proposition follows<sup>3</sup>.  $\square$

**Remark.** The construction presented in the proof is reminiscent of Kontsevich-Soibelman pairs of complementary orders [23].

<sup>3</sup>In particular, it implies that in each equivalence class of trees there is exactly one minimal tree.

3.3.2. *Cochain part of  $\mathcal{O}$  via sequences, II: modification.* Given a total order as above, we can construct a map

$$Q : \bigsqcup_{s \in S} I'_s \rightarrow J'$$

as follows. We identify  $J' = J \sqcup \{M\}$ , where  $M > J$ . Set  $Q(x) = j$  if  $j$  is the minimal element from  $J$  such that  $j > x$ ; if there is no such  $j$ , set  $Q(x) = M$ .

Thus, given a total order as in the previous subsection, we obtain the following data:

— a total order on the set

$$\mathcal{I} := \bigsqcup_{s \in S} I'_s;$$

a non-decreasing map

$$\mathcal{I} \rightarrow J'.$$

These data should satisfy:

- i) the order on  $\mathcal{I}$  agrees with those on each  $I'_s$ ;
- ii) same as condition 3) from Sec 3.3.

Denote the set of such objects by

$$\mathbf{s}\mathcal{O}(S)_{\{I'_s\}_{s \in S}}^{J'}.$$

This set is in 1-to-1 correspondence with the set of total orders from the previous subsection, hence we have a bijection with the set  $\mathcal{O}(S)_{\{I_s\}_{s \in S}}^J$ :

$$(3.6) \quad \mathbf{s}\mathcal{O}(S)_{\{I'_s\}_{s \in S}}^{J'} \rightarrow \mathcal{O}(S)_{\{I_s\}_{s \in S}}^J.$$

3.4. **The whole operad  $\mathcal{O}$  in terms of sequences.** Likewise, one identifies the set  $\mathcal{O}(S_c, S_a)_{\{I_s\}_{s \in S_c}}$  with the set of total orders on

$$\bigsqcup_{s \in S_c} I'_s \sqcup S_a$$

satisfying:

- the total order agrees with those on each  $I'_s$ ;
- same as condition 3) from Sec. 3.3.

Denote the set of such total orders by  $\mathbf{s}\mathcal{O}(S_c, S_a)_{\{I_s\}_{s \in S_c}}$ .

The construction of the 1-to-1 correspondence

$$(3.7) \quad \mathbf{s}\mathcal{O}(S_c, S_a)_{\{I'_s\}_{s \in S_c}} \rightarrow \mathcal{O}(S_c, S_a)_{\{I_s\}_{s \in S_c}}$$

is the same as in the previous subsection.

3.5. **Operadic structure on  $\mathbf{s}\mathcal{O}$ .** Let  $\mathbb{N}'$  be the set of isomorphism classes of non-empty finite ordinals. The identifications (3.6), (3.7) imply that the colored operad structure on  $\mathcal{O}$  induces a colored operad structure on the collection of spaces  $\mathbf{s}\mathcal{O}$ . It turns out that this operadic structure can be naturally formulated in terms of  $\mathbf{s}\mathcal{O}$ .

**Warning.** We will not use the symbol  $'$  anymore when talking about ordinals from  $\mathbb{N}'$ . The reason is that in the sequel, instead of the operad  $\mathcal{O}$ , the isomorphic operad  $\mathbf{s}\mathcal{O}$  will be used.

3.5.1. *Operadic structure on the cochain part of  $\mathbf{s}\mathcal{O}$ .* Let  $T$  be a finite set and let  $S_t$  be a  $T$ -family of finite sets. Let  $S := \sqcup_t S_t$  and  $p : S \rightarrow T$  be the map which sends  $S_t$  to  $t$ .

Suppose we are given ordinals  $I_s, s \in S; J_t, t \in T$ , and  $J$ .

Describe the operadic composition

$$\mathbf{s}\mathcal{O}(T)_{\{J_t\}_{t \in T}}^J \times \prod_{t \in T} \mathbf{s}\mathcal{O}(S_t)_{\{I_s\}_{s \in S_t}}^{J_t} \rightarrow \mathbf{s}\mathcal{O}(S)_{\{I_s\}_{s \in S}}^J.$$

Let  $u \in \mathbf{s}\mathcal{O}(T)_{\{J_t\}_{t \in T}}^J$  and  $u_t \in \mathbf{s}\mathcal{O}(S_t)_{\{I_s\}_{s \in S_t}}^{J_t}$ . Let us describe the composition  $v$  of these elements.

- 1) the total order  $>_v$  is defined as a unique one
- which agrees with the orders  $>_{u_t}$  on

$$\sqcup_{s \in S_t} I_s \subset \sqcup_{s \in S} I_s$$

for each  $t \in T$ ;

- for which the map

$$(3.8) \quad \sqcup_{t \in T} F_{u_t} : \sqcup_{s \in S} I_s \rightarrow (\sqcup_{t \in T} J_t, >_u)$$

is non-decreasing.

- 2) the map  $F_v$  is just the composition of (3.8) with the map  $F_u$ .

**Remark.** The non-SC part of the operad  $\mathbf{s}\mathcal{O}$  as well as its totalization was considered in earlier papers on Deligne's conjecture and its variations. Thus, the non-SC part of  $\mathbf{s}\mathcal{O}$  is isomorphic to the second filtration stage of the lattice path operad introduced by M. Batanin and C. Berger in [4]. The non-SC part of the totalization of  $\mathbf{s}\mathcal{O}$  was considered in papers [28] and [29] by J. E. McClure and J. H. Smith.

3.5.2. *Operadic structure on the whole operad  $\mathbf{s}\mathcal{O}$ .* To describe the remaining composition maps we consider sets  $S_c, S_a, T_c, T_a$  and let  $P : S_c \sqcup S_a \rightarrow T_c \sqcup T_a$  be a map such that  $P^{-1}T_c \subset S_c$ . For  $t \in T_a$  we set  $(P^{-1}t)_a := P^{-1}t \cap S_a$ ;  $(P^{-1}t)_c := P^{-1}t \cap S_c$ .

Let  $\{I_s\}_{s \in S_c}; \{J_t\}_{t \in T_c}$ ; be non-empty ordinals. We need to define the following composition map:

$$\begin{aligned} \mathbf{s}\mathcal{O}(T_c, T_a)_{\{J_t\}_{t \in T_c}} \times \prod_{t \in T_c} \mathbf{s}\mathcal{O}(P^{-1}t)_{\{I_s\}_{s \in P^{-1}t}}^{J_t} \times \prod_{t \in T_a} \mathbf{s}\mathcal{O}((P^{-1}t)_c, (P^{-1}t)_a)_{\{I_s\}_{s \in (P^{-1}t)_c}} \\ \rightarrow \mathbf{s}\mathcal{O}(S_c, S_a)_{\{I_s\}_{s \in S_c}}. \end{aligned}$$

Choose elements

$$\begin{aligned} v &\in \mathbf{s}\mathcal{O}(T_c, T_a)_{\{J_t\}_{t \in T_c}}; \\ u_t &\in \mathbf{s}\mathcal{O}(P^{-1}t)_{\{I_s\}_{s \in P^{-1}t}}^{J_t}; \quad t \in T_c, \\ u_t &\in \mathbf{s}\mathcal{O}((P^{-1}t)_c, (P^{-1}t)_a)_{\{I_s\}_{s \in (P^{-1}t)_c}}; \quad t \in T_a \end{aligned}$$

and denote their composition by  $w$ .

Let us set

$$\mathcal{I}_w := \bigsqcup_{s \in S_c} I_s \sqcup S_a.$$

and define a map

$$F : \mathcal{I}_w \rightarrow \mathcal{I}_v,$$

where

$$\mathcal{I}_v = \bigsqcup_{t \in T_c} J_t \sqcup T_a,$$

as follows:

— If  $t \in T_c$  then the restriction of  $F$  to the subset

$$\mathcal{I}_{u_t} := \bigsqcup_{s \in P^{-1}t} I_s,$$

should coincide with the map  $F_{u_t} : \mathcal{I}_{u_t} \rightarrow J_t$ ;

— if  $t \in T_a$  then the restriction of  $F$  to the subset

$$\mathcal{I}_{u_t} := \bigsqcup_{s \in (P^{-1}t)_c} I_s \sqcup (P^{-1}t)_a$$

should send every element to  $t$ .

We define the order  $>_w$  as the unique one for which the map  $F$  is non-decreasing and which agrees with the orders  $>_{u_t}$  on  $\mathcal{I}_{u_t}$ ,  $t \in T_c \sqcup T_a$ .

#### 4. REVIEW OF 2-OPERADS

We are going to remind the basic definitions from Batanin's theory of 2-operads which will be used below. Next, we review Batanin's definition of SC 2-operad.

An *ordinal* is a finite totally ordered set. Another name for ordinals is a 1-tree.

A *2-tree*  $\mathbf{t}$  is a pair of ordinals  $S, T$  along with an order preserving map  $\mathbf{t} : S \rightarrow T$ .

A 2-tree is called *pruned* if the map  $\mathbf{t}$  is surjective.

A *map of 2-trees*

$$P : (\mathbf{t} : S \rightarrow T) \rightarrow (\mathbf{t}_1 : S_1 \rightarrow T_1)$$

is a pair of maps  $P_S : S \rightarrow S_1$ ;  $P_T : T \rightarrow T_1$  such that  $\mathbf{t}_1 P_S = P_T \mathbf{t}$ ;  $P_T$  is order preserving;  $P_S$  preserves the order on each set  $\mathbf{t}^{-1}t$ ,  $t \in T$ .

This way, 2-trees form a category **2-trees**.

Given  $s_1 \in S_1$ , we define a 2-tree  $P^{-1}s_1$  as follows:

$$\mathbf{t} \Big|_{(P_S)^{-1}s_1} : (P_S)^{-1}s_1 \rightarrow (P_T)^{-1}\mathbf{t}_1(s_1).$$

A *2-operad* in a symmetric monoidal category (SMC)  $\mathcal{C}$  is defined as:

— a functor  $\mathcal{O} : \mathbf{2-trees}^\times \rightarrow \mathcal{C}$ , where  $\mathbf{2-trees}^\times$  is the groupoid of isomorphisms of **2-trees** (note that every object in this groupoid has the trivial automorphism group);

— for every map of 2-trees  $P : \mathbf{t} \rightarrow \mathbf{t}_1$ , where  $\mathbf{t} : S \rightarrow T$ ;  $\mathbf{t}_1 : S_1 \rightarrow T_1$ , there should be given a map

$$\mathcal{O}(\mathbf{t}_1) \otimes \bigotimes_{s_1 \in S_1} \mathcal{O}(P^{-1}s_1) \rightarrow \mathcal{O}(\mathbf{t})$$

called *the operadic composition map*.

These maps should satisfy a certain associativity property. In order to formulate it let us define the objects  $\mathcal{O}(P)$ , where  $P : \mathbf{t} \rightarrow \mathbf{t}_1$  is a map of 2-trees as follows:

$$\mathcal{O}(P) := \bigotimes_{s_1 \in S_1} \mathcal{O}(P^{-1}s_1).$$

The operadic insertion maps can be rewritten as

$$\mathcal{O}(\mathbf{t}_1) \otimes \mathcal{O}(P) \rightarrow \mathcal{O}(\mathbf{t}).$$

Given a chain of maps of 2-trees

$$\mathbf{t} \xrightarrow{P} \mathbf{t}_1 \xrightarrow{Q} \mathbf{t}_2,$$

the operadic insertion maps naturally give rise to a map

$$(4.1) \quad \mathcal{O}(Q) \otimes \mathcal{O}(P) \rightarrow \mathcal{O}(QP).$$

Indeed, for every  $s_2 \in S_2$ , where  $t_2 : S_2 \rightarrow T_2$ , the map  $P$  naturally restricts to a map of 2-trees

$$P_{s_2} : (QP)^{-1}s_2 \rightarrow Q^{-1}s_2$$

and we have

$$\mathcal{O}(P) \cong \bigotimes_{s_2 \in S_2} \mathcal{O}(P_{s_2}).$$

We then define the map (4.1) as follows:

$$\mathcal{O}(Q) \otimes \mathcal{O}(P) \cong \bigotimes_{s_2 \in S_2} \mathcal{O}(Q^{-1}s_2) \otimes \mathcal{O}(P_{s_2}) \rightarrow \bigotimes_{s_2 \in S_2} \mathcal{O}((QP)^{-1}s_2) = \mathcal{O}(QP).$$

The associativity axiom requires that the map (4.1) be associative: the following maps should coincide:

$$\mathcal{O}(R) \otimes \mathcal{O}(Q) \otimes \mathcal{O}(P) \rightarrow \mathcal{O}(RQ) \otimes \mathcal{O}(P) \rightarrow \mathcal{O}(RQP)$$

and

$$\mathcal{O}(R) \otimes \mathcal{O}(Q) \otimes \mathcal{O}(P) \rightarrow \mathcal{O}(R) \otimes \mathcal{O}(QP) \rightarrow \mathcal{O}(RQP).$$

**Remark.** 1-trees are simply ordinals and the definition of a 1-operad based on 1-trees coincides with the definition of a nonsymmetric operad.

#### 4.1. Colored 2-operads.

4.1.1. *Colored 2-trees.* Fix a set of colors  $X\rho$ . Define a colored 2-tree  $\tau$  as:

- a 2-tree  $\mathbf{t}_\tau : S_\tau \rightarrow T_\tau$ ;
- a map  $\chi_\tau : S_\tau \rightarrow X\rho$ ;
- an element  $c_\tau \in X\rho$ .

4.1.2. Given colored 2-trees  $\tau_1, \tau_2$  we define their map  $P : \tau_1 \rightarrow \tau_2$  as follows:

- if  $c_{\tau_1} = c_{\tau_2}$ , then it is just a map  $P : \mathbf{t}_{\tau_1} \rightarrow \mathbf{t}_{\tau_2}$  of the underlying 2-trees;
- if  $c_{\tau_1} \neq c_{\tau_2}$ , then we declare that there are no maps  $\tau_1 \rightarrow \tau_2$ .

This way colored 2-trees form a category.

Given such a map and  $s_2 \in S_{\tau_2}$  the 2-tree  $P^{-1}s_2$  naturally receives a coloring as follows.

Recall that the 2-tree  $P^{-1}s_2$  is defined as

$$\mathbf{t}_{\tau_1} \Big|_{(P_S)^{-1}s_2} : (P_S)^{-1}s_2 \rightarrow (P_T)^{-1}\mathbf{t}_{\tau_2}s_2.$$

We then define

$$\chi_{P^{-1}s_2} : P_S^{-1}s_2 \rightarrow X\rho$$

as the restriction of  $\chi_{\tau_1}$  and set

$$c_{P^{-1}s_2} := \chi_{\tau_2}(s_2).$$

4.1.3. We then define a colored 2-operad in a SMC  $\mathcal{C}$  as:

- a functor  $\mathcal{O}$  from the isomorphism groupoid of the category of colored 2-trees to the category  $\mathcal{C}$ ;
- for every map  $P : \tau_1 \rightarrow \tau_2$  of colored 2-trees there should be given the operadic composition map

$$\mathcal{O}(\tau_2) \otimes \bigotimes_{s_2 \in S_{\tau_2}} \mathcal{O}(P^{-1}s_2) \rightarrow \mathcal{O}(\tau_1).$$

Next, given a map  $P : \tau_1 \rightarrow \tau_2$  we define

$$\mathcal{O}(P) := \bigotimes_{s_2 \in S_{\tau_2}} \mathcal{O}(P^{-1}s_2)$$



and observe that the operadic composition maps naturally produce maps

$$\mathcal{O}(Q) \otimes \mathcal{O}(P) \rightarrow \mathcal{O}(QP),$$

where  $P : \tau_1 \rightarrow \tau_2$ ,  $Q : \tau_2 \rightarrow \tau_3$ .

Lastly we require the associativity of this map in the same way as for the non-colored 2-operads.

## 4.2. Unital colored 2-operads.

4.2.1. Given a color  $c \in X\rho$ , consider a special 2-tree  $\mathbf{u}_c : \mathbf{pt} \rightarrow \mathbf{pt}$  such that  $\chi_{\mathbf{u}_c}$  sends  $\mathbf{pt}$  to  $c$  and  $c_{\mathbf{u}_c} := c$ .

For every isomorphism  $P : \tau_1 \rightarrow \tau_2$  of colored 2-trees every pre-image  $P^{-1}s_2$ ,  $s_2 \in S_{\tau_2}$ , is isomorphic (canonically) to  $\mathbf{u}_c$ , where  $c = \chi_{\tau_2}(s_2)$ . Furthermore, for every colored 2-tree  $\tau$  the pre-image  $Q^{-1}\mathbf{pt}$  of the point for a unique map  $Q : \tau \rightarrow \mathbf{u}_{c_\tau}$  is equal to  $\tau$ .

Let  $\mathbf{1}$  be the unit of the underlying symmetric monoidal category. Define a *unital 2-operad* as a colored 2-operad  $\mathcal{O}$  along with maps

$$(4.2) \quad \mathbf{1} \rightarrow \mathcal{O}(\mathbf{u}_c)$$

for each  $c \in X\rho$  satisfying:

— for every isomorphism  $P : \tau_1 \rightarrow \tau_2$ , the map

$$\mathcal{O}(\tau_2) \cong \mathcal{O}(\tau_2) \otimes \mathbf{1}^{\otimes S_{\tau_2}} \rightarrow \mathcal{O}(\tau_2) \otimes \bigotimes_{s_2 \in S_{\tau_2}} \mathcal{O}(P^{-1}s_2) \rightarrow \mathcal{O}(\tau_1)$$

coincides with the map  $\mathcal{O}(\tau_2) \rightarrow \mathcal{O}(\tau_1)$  induced by  $P^{-1}$  from the definition of  $\mathcal{O}$  as a functor from the isomorphism groupoid of the category of colored 2-trees.

— for every colored 2-tree  $\tau$  the composition

$$\mathcal{O}(\tau) \cong \mathbf{1} \otimes \mathcal{O}(\tau) \rightarrow \mathcal{O}(\mathbf{u}_{c_\tau}) \otimes \mathcal{O}(\tau) \rightarrow \mathcal{O}(\tau)$$

is the identity on  $\mathcal{O}(\tau)$ .

4.2.2. *Pruned colored 2-operads.* Let

$$P : \tau_1 \rightarrow \tau_2$$

be a map of colored 2-trees. According to M. Batanin [2]  $P$  is called a *full injection* if  $P_S : S_{\tau_1} \rightarrow S_{\tau_2}$  is a color-preserving isomorphism and  $P_T : T_{\tau_1} \rightarrow T_{\tau_2}$  is an injection. Next, let  $\tau$  be a colored 2-tree with its underlying 2-tree  $\mathbf{t} : S \rightarrow T$ . We say that  $\tau$  is *pruned* if the map  $\mathbf{t} : S \rightarrow T$  is surjective.

Let  $\mathcal{O}$  be a unital colored 2-operad. Consider the composition map associated with  $P$ :

$$\mathcal{O}(\tau_2) \otimes \bigotimes_{s_2 \in S_2} \mathcal{O}(P^{-1}s_2) \rightarrow \mathcal{O}(\tau_1).$$

It is clear that each  $P^{-1}s_2$  is a 2-tree of the form  $\mathbf{u}_c$ ,  $c \in X\rho$ . Hence we have unital maps  $\mathbf{1} \rightarrow \mathcal{O}(P^{-1}s_2)$ . Pre-composition with these maps gives rise to a map

$$(4.3) \quad \mathcal{O}(\tau_2) \rightarrow \mathcal{O}(\tau_1).$$

**Definition 4.1.** *We call  $\mathcal{O}$  a pruned 2-operad if for every full injection  $P$  the map (4.3) is an isomorphism.*

For every colored 2-tree  $\tau$  there exists a unique (up-to an isomorphism) pruned colored 2-tree  $\tau'$  together with a full injection  $\tau' \rightarrow \tau$ . Thus, a pruned 2-operad is completely determined by prescribing its spaces for each pruned 2-tree.

4.2.3. *Reduced 2-operads.* In this section all 2-operads are non-colored. Let us first define the trivial 2-operad **triv** by setting

$$\mathbf{triv}(\tau) = \mathbf{1}$$

for all 2-trees  $\tau$ . Here  $\mathbf{1}$  is the unit of the SMC and the operadic composition maps are the canonical maps sending tensor products of  $\mathbf{1}$  to  $\mathbf{1}$ .

We say that a pruned 2-operad  $\mathcal{O}$  is *reduced* if

- all the unit maps  $\mathbf{1} \rightarrow \mathcal{O}(\mathbf{u})$ ,  $\mathbf{1} \rightarrow \mathcal{O}(\mathbf{u})$  are isomorphisms;
- $\mathcal{O}(\tau) = \mathbf{1}$  whenever  $|S_\tau| \leq 1$  so that we have an identification  $\mathcal{O}(\tau) = \mathbf{triv}(\tau)$  for all such  $\tau$ ;
- for every map  $P : \tau_1 \rightarrow \tau_2$  where  $|S_{\tau_1}|, |S_{\tau_2}| \leq 1$  the corresponding operadic composition law coincides with that of **triv**.

Note that, equivalently, one can only require that the conditions are the case for pruned 2-trees  $\tau$ .

4.2.4. *Desymmetrization.* Given a colored symmetric operad  $\mathcal{O}$ , one can define a colored 2-operad **des**  $\mathcal{O}$  by setting

$$\mathbf{des} \mathcal{O}(\tau) = \mathcal{O}(S_\tau),$$

where the coloring on the right hand side is determined by that of  $\tau$ , and the operadic composition maps are inherited from those of  $\mathcal{O}$ .

4.3. **Symmetrization.** If the SMC  $\mathcal{C}$  has small colimits then the functor **des** has a left adjoint **sym**. For many categories of higher operads the functor **sym** can be elegantly expressed using colimits [2]. Here we recall from [2] a description of the functor **sym** for the category of reduced 2-operads.

For every set  $S$  we define a category  $\mathcal{J}(S)$ .

The objects of  $\mathcal{J}(S)$  are pruned 2-trees of the form

$$\mathbf{t} : S \rightarrow T.$$

Morphisms are the maps between 2-trees which induce the identity map on  $S$ .

Notice that, although elements of a set  $S$  are not ordered, choosing an object of the category  $\mathcal{J}(S)$  we equip  $S$  with a total order.

**Remark.** It is not hard to show that for every set  $S$  the category  $\mathcal{J}(S)$  is a poset whose opposite is called the Milgram poset [1].

Let  $\mathcal{O}$  be a reduced 2-operad.

For every set  $S$  the 2-operad  $\mathcal{O}$  gives us an obvious (contravariant) functor from the category  $\mathcal{J}(S)$  to the underlying SMC  $\mathcal{C}$ . We denote this functor by  $\mathcal{O}_S$ .

According to Theorem 4.3 from [2] we have

$$(4.4) \quad \mathbf{sym} \mathcal{O}(S) = \mathbf{colim}_{\mathcal{J}(S)} \mathcal{O}_S.$$

The operadic multiplications of **sym**  $\mathcal{O}$  can be easily obtained from those of  $\mathcal{O}$  using the properties of colimits.

4.3.1. *Model structure.* Let us consider the category of reduced 2-operads in the category of complexes over the ground field  $\mathbf{k}$  (i.e. a dg 2-operad). According to Theorem 5.3 from [2] this category has a closed model structure uniquely determined by the conditions that the class of fibrations (resp. weak equivalences) should consist of all maps  $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$  satisfying: given any 2-tree  $\mathbf{t}$ , the induced map of complexes  $f : \mathcal{O}_1(\mathbf{t}) \rightarrow \mathcal{O}_2(\mathbf{t})$  is component-wise surjective (resp. is quasi-isomorphism). Same Theorem 5.3 from [2] implies a model structure in the category of topological reduced 2-operads.

Let now  $\mathcal{J}(S)$  be the category of contravariant functors from  $\mathcal{J}(S)$  to the category of complexes over  $\mathbf{k}$ . One has a model structure on  $\mathcal{J}(S)$  which is defined in a similar way: Let  $F_1, F_2 \in \mathcal{J}(S)$ . The class of fibrations (resp. weak equivalences) by definition consists of all maps  $f : F_1 \rightarrow F_2$  satisfying: given any 2-tree  $\mathbf{t} \in \mathcal{J}(S)$ , the induced map of complexes  $f : F_1(\mathbf{t}) \rightarrow F_2(\mathbf{t})$  is component-wise surjective (resp. is quasi-isomorphism).

A functor  $F \in \mathcal{J}(S)$  is called *cofibrant* if the natural map from the initial object  $0 \rightarrow F$  is cofibrant.

**Lemma 4.2.** *For every dg reduced 2-operad  $\mathcal{O}$  there exists a cofibrant operad  $\mathcal{R}\mathcal{O}$  and a weak equivalence  $f : \mathcal{R}\mathcal{O} \rightarrow \mathcal{O}$  such that for every finite set, the functor  $\mathcal{R}\mathcal{O}_S \in \mathcal{J}(S)$  is cofibrant.*

*Proof.* See the proof of Theorem 7.1 in [3]. □

4.3.2. *Algebras over colored 2-operads.* Given an  $X\rho$ -colored 2-operad  $\mathcal{O}$  and an  $X\rho$ -family of objects  $X_c \in \mathcal{C}$ ,  $c \in X\rho$ , we define an  $\mathcal{O}$ -algebra structure on  $\{X_c\}_{c \in X\rho}$  as a map

$$f : \mathcal{O} \rightarrow \mathbf{des\ full}(\{X_c\}_{c \in X\rho}),$$

where  $\mathbf{full}(X)$  is the full colored symmetric endomorphism operad of  $X$ . If  $\mathcal{O}$  is unital, then we additionally require that  $f$  matches the units.

4.4. **Batanin's theorem.** We observe that the operad  $\mathbf{triv}$  is reduced and set  $\mathcal{R}\mathbf{triv} \rightarrow \mathbf{triv}$  to be its cofibrant resolution in the category of reduced 2-operads.

**Theorem 4.3** (Theorem 7.2, 7.3, [2]). *The symmetric operad  $\mathbf{sym}\mathcal{R}\mathbf{triv}$  is weakly equivalent to the operad of little discs if  $\mathcal{C}$  is the category of topological spaces, and to the singular chain operad of little discs if  $\mathcal{C}$  is the category of chain complexes of  $\mathbf{k}$ -vector spaces.*

Let us sketch its proof for  $\mathcal{C}$  being the category of topological spaces.

First, we observe, that the 2-operad  $\mathcal{R}\mathbf{triv}$  can be replaced with any weakly equivalent one. Batanin uses the Getzler-Jones 2-operad  $\mathbf{GJ}$ .

This 2-operad is constructed in [2] as a sub 2-operad of the desymmetrization  $\mathbf{des}(\mathbf{FM})$  of the Fulton-MacPherson version  $\mathbf{FM}$  of little discs operad

Then, since the desymmetrization functor  $\mathbf{des}$  admits the left adjoint  $\mathbf{sym}$ , the inclusion

$$\mathbf{GJ} \hookrightarrow \mathbf{des}(\mathbf{FM})$$

produces the following map

$$\mathbf{sym}(\mathbf{GJ}) \rightarrow \mathbf{FM}$$

which can be shown to be an isomorphism, hence a weak equivalence. This completes the proof. The case when  $\mathcal{C}$  is the category of chain complexes is treated by applying the singular chain functor.

## 5. SWISS CHEESE (SC) OPERADS

In this section we discuss SC-modifications of the notions of colored operad and colored 2-operad. We conclude with formulating the SC version of Batanin's theorem on the symmetrization of the trivial 2-operad.

**5.1. Symmetric Swiss Cheese type operads.** In this subsection we recall from [2] the notion of the symmetric Swiss Cheese type operads. Here, we call them symmetric SC operads for short.

Let  $X\rho := \{\mathfrak{a}, \mathfrak{c}\}$  be the set of colors. An SC-set is a collection of the following data:

- a finite set  $S$ ;
- a map  $\chi_S : S \rightarrow X\rho$ ;
- an element  $c_S \in X\rho$ .

These data should satisfy:

- if  $\mathfrak{a} \in \chi(S)$ , then  $c_S = \mathfrak{a}$ .

A map of SC-sets is a usual map  $P : S_1 \rightarrow S_2$  satisfying: if  $s_1 \in S_1$  is such that  $\chi_{S_1}(s_1) = \mathfrak{a}$ , then  $\chi_{S_2}(P(s_1)) = \mathfrak{a}$ . Given such  $P$  and  $s_2 \in S_2$ ,  $P^{-1}s_2$  is naturally an SC-set:  $\chi_{P^{-1}s_2}$  is the restriction of  $\chi_{S_1}$ ;  $c_{P^{-1}s_2} = \chi_{S_2}(s_2)$ .

An SC-operad in a symmetric monoidal category  $\mathcal{C}$  is a functor  $\mathcal{O}$  from the groupoid of SC-sets and their color preserving bijections to  $\mathcal{C}$ .

For every map  $P : S_1 \rightarrow S_2$  of SC-sets, there should be given a composition map

$$\mathcal{O}(S_2) \otimes \bigotimes_{s_2 \in S_2} \mathcal{O}(P^{-1}s_2) \rightarrow \mathcal{O}(S_1).$$

These compositions should satisfy the associativity law which is similar to that for usual operads. It is clear how to define *unital* symmetric SC operads. In this paper all our symmetric SC operads are unital.

**5.1.1. Reduced symmetric SC operads.** We say that a unital symmetric SC operad  $\mathcal{O}$  is *reduced* if for every SC set  $S$  with at most one element

$$\mathcal{O}(S) \cong \mathbf{1}.$$

For the one element SC sets these isomorphisms should coincide with the unit maps. Furthermore, the operadic compositions of zero-ary and unary operations send products of  $\mathbf{1}$  to  $\mathbf{1}$  via the corresponding isomorphism of the symmetric monoidal category.

**5.1.2. Colored symmetric SC-operads.** Fix two sets of colors  $X\rho_c$  and  $X\rho_a$ . A colored SC-set  $S$  is a map  $\chi_S : S \rightarrow X\rho_c \sqcup X\rho_a$  and an element  $c_S \in X\rho_c \sqcup X\rho_a$  satisfying: if  $\chi_S^{-1}X\rho_a$  is non-empty, then  $c_S \in X\rho_a$ .

We declare that there are no maps between colored SC-sets  $S_1$  and  $S_2$  if  $c_{S_1} \neq c_{S_2}$ . On the other hand if  $c_{S_1} = c_{S_2}$  then a map from  $S_1$  to  $S_2$  is a map of sets  $P : S_1 \rightarrow S_2$  satisfying the property: for any  $s_2 \in S_2$ , the set  $P^{-1}s_2$  along with the map  $\chi_{S_1}|_{P^{-1}s_2} : P^{-1}s_2 \rightarrow X\rho_c \sqcup X\rho_a$  and the element  $c_{P^{-1}s_2} := \chi_{S_2}(s_2)$  is a colored SC-set. Thus, given a map of colored SC-sets  $P : S_1 \rightarrow S_2$  and  $s_2 \in S_2$ , we have a colored SC-set  $P^{-1}s_2$ .

A colored SC-operad  $\mathcal{O}$  in a SMC  $\mathcal{C}$  is a functor  $\mathcal{O}$  from the isomorphism groupoid of colored SC-sets to  $\mathcal{C}$  along with the composition maps: given a map  $P : S_1 \rightarrow S_2$  of colored sets, one should have a map

$$\mathcal{O}(S_2) \otimes \bigotimes_{s_2 \in S_2} \mathcal{O}(P^{-1}s_2) \rightarrow \mathcal{O}(S_1)$$

satisfying the associativity property as above.

The operad  $\mathfrak{s}\mathcal{O}$  is an example of colored SC operad. Indeed, let  $X\rho_c := \mathbb{N}$  and  $X\rho_a := \{\mathfrak{a}\}$  and let  $S$  be a colored SC-set. Let  $S_c := \chi^{-1}X\rho_c$  and  $S_a := \chi^{-1}X\rho_a$ .

In the case  $c_S \in X\rho_c$ , set

$$\mathfrak{s}\mathcal{O}(S) := \mathfrak{s}\mathcal{O}(S)_{\{\chi(s)\}_{s \in S}}^{c_S};$$

if  $c_S = \mathfrak{a}$ , we set

$$\mathfrak{s}\mathcal{O}(S) := \mathfrak{s}\mathcal{O}(S_c, S_a)_{\{\chi(s)\}_{s \in S_c}}.$$

5.2. **SC 2-operads.** Let us review Batanin's definition of a Swiss Cheese type (or simply SC) 2-operad from [2]. This notion is obtained via modifying the definition of a usual 2-operad as follows:

- 1) An *SC-ordinal* is any non-empty ordinal; its minimum is considered to be marked.
- 2) A *map of SC-ordinals* is a monotonous map preserving the minima.
- 3) An *SC 2-tree*  $\mathbf{t}$  is a monotonous map  $\mathbf{t} : S \rightarrow T$  where  $S$  is a usual ordinal and  $T$  is an SC-ordinal. A map of SC 2-trees

$$(\mathbf{t} : S \rightarrow T) \rightarrow (\mathbf{t}_1 : S_1 \rightarrow T_1)$$

is a map of sets  $P_S : S \rightarrow S_1$  as well as a map of SC-ordinals  $P_T : T \rightarrow T_1$  such that  $\mathbf{t}_1 P_S = P_T \mathbf{t}$ ; the map  $P_S$  must preserve the order on each set  $\mathbf{t}^{-1}t$ ,  $t \in T$ . Given  $s_1 \in S_1$  such that  $\mathbf{t}_1(s_1)$  is not the minimum of  $T_1$ , we define a usual 2-tree  $P^{-1}s_1$  in the same way as for the usual 2-trees (see the beginning of Sec. 4); in the case  $\mathbf{t}_1(s_1)$  is the minimum of  $T_1$ , we naturally get an SC 2-tree  $P_S^{-1}s_1$ .

- 4) We define an *SC 2-operad in a symmetric monoidal category*  $\mathcal{C}$  as:

— a functor

$$\mathcal{O} : \mathbf{2-trees}^\times \sqcup \mathbf{SC\ 2-trees}^\times \rightarrow \mathcal{C};$$

— for every map of 2-trees or SC 2-trees  $P : \mathbf{t} \rightarrow \mathbf{t}_1$ , there should be given a map

$$\mathcal{O}(\mathbf{t}_1) \otimes \bigotimes_{s_1 \in S_1} \mathcal{O}(P^{-1}s_1) \rightarrow \mathcal{O}(\mathbf{t}).$$

These maps should satisfy the associativity property which is similar to that for usual 2-operads.

5.2.1. *Unital SC 2-operads.* In this paper all SC 2-operads are assumed to be unital.

To introduce the notion of unital SC 2-operads we define  $\mathbf{u}_c$  to be the ordinary 2-tree  $\mathbf{pt} \rightarrow \mathbf{pt}$ . We also define  $\mathbf{u}_a$  to be an SC 2-tree in which a 1-element ordinal is mapped into a one-element SC ordinal.

For every isomorphism  $P : \mathbf{t}_1 \rightarrow \mathbf{t}_2$  of 2-trees or SC 2-trees every pre-image  $P^{-1}s_2$ ,  $s_2 \in S_{\mathbf{t}_2}$ , is either  $\mathbf{u}_c$  or  $\mathbf{u}_a$ . For every 2-tree  $\mathbf{t}$  the pre-image  $Q_c^{-1}\mathbf{pt}$  of the point for a unique map  $Q_c : \mathbf{t} \rightarrow \mathbf{u}_c$  is equal to  $\mathbf{t}$ . Furthermore, for every SC 2-tree  $\mathbf{t}$  the pre-image  $Q_a^{-1}\mathbf{pt}$  of the point for a unique map  $Q_a : \mathbf{t} \rightarrow \mathbf{u}_a$  is also equal to  $\mathbf{t}$ .

Define a *unital SC 2-operad* as an SC 2-operad  $\mathcal{O}$  along with maps  $\mathbf{1} \rightarrow \mathcal{O}(\mathbf{u}_c)$  and  $\mathbf{1} \rightarrow \mathcal{O}(\mathbf{u}_a)$  satisfying:

— for every isomorphism  $P : \mathbf{t}_1 \rightarrow \mathbf{t}_2$ , of 2-trees or SC 2-trees the map

$$\mathcal{O}(\mathbf{t}_2) \cong \mathcal{O}(\mathbf{t}_2) \otimes \mathbf{1}^{\otimes S_{\mathbf{t}_2}} \rightarrow \mathcal{O}(\mathbf{t}_2) \otimes \bigotimes_{s_2 \in S_{\mathbf{t}_2}} \mathcal{O}(P^{-1}s_2) \rightarrow \mathcal{O}(\mathbf{t}_1)$$

coincides with the map  $\mathcal{O}(\mathbf{t}_2) \rightarrow \mathcal{O}(\mathbf{t}_1)$  induced by  $P^{-1}$  from the definition of  $\mathcal{O}$  as a functor from the corresponding groupoid.

— for every 2-tree  $\mathbf{t}$  the composition

$$\mathcal{O}(\mathbf{t}) \cong \mathbf{1} \otimes \mathcal{O}(\mathbf{t}) \rightarrow \mathcal{O}(\mathbf{u}_c) \otimes \mathcal{O}(\mathbf{t}) \rightarrow \mathcal{O}(\mathbf{t})$$

is the identity on  $\mathcal{O}(\mathbf{t})$ .

— for every SC 2-tree  $\mathbf{t}$  the composition

$$\mathcal{O}(\mathbf{t}) \cong \mathbf{1} \otimes \mathcal{O}(\mathbf{t}) \rightarrow \mathcal{O}(\mathbf{u}_a) \otimes \mathcal{O}(\mathbf{t}) \rightarrow \mathcal{O}(\mathbf{t})$$

is the identity on  $\mathcal{O}(\mathbf{t})$ .

5.2.2. We define the trivial SC 2-operad  $\mathbf{triv}$  by setting

$$\mathbf{triv}(\mathbf{t}) = \mathbf{1},$$

for all 2-trees and SC 2-trees  $\mathbf{t}$ . Here  $\mathbf{1}$  is the unit of the SMC and the operadic multiplications are the canonical maps sending tensor products of  $\mathbf{1}$  to  $\mathbf{1}$ .

5.2.3. *Colored SC 2-operads.* We define a colored SC 2-operad as follows. Fix 2 sets of colors:  $X\rho_c$  and  $X\rho_a$ . Define a coloring of an SC 2-tree  $\mathbf{t}_\tau : S_\tau \rightarrow T_\tau$  as follows.

First decompose  $S_\tau = S_{\tau,a} \sqcup S_{\tau,c}$ , where  $S_{\tau,a}$  is the  $\mathbf{t}_\tau$ -preimage of the minimum of  $T_\tau$ , and  $S_{\tau,c}$  is the complement.

A  $(X\rho_c, X\rho_a)$ -coloring of  $\tau$  is a prescription of maps  $\chi_{\tau,c} : S_{\tau,c} \rightarrow X\rho_c$ ;  $\chi_{\tau,a} : S_{\tau,a} \rightarrow X\rho_a$  and an element  $c_\tau \in X\rho_a$ .

As well as for ordinary colored 2-trees we declare that there are no maps between colored SC 2-trees  $\tau$  and  $\tau_1$  if  $c_\tau \neq c_{\tau_1}$ . On the other hand, if  $c_\tau = c_{\tau_1}$  then a map  $P : \tau \rightarrow \tau_1$  is just the map of the underlying SC 2-trees. Then it is clear that for every  $s_1 \in S_{\tau_1}$  such that  $\mathbf{t}_{\tau_1}s_1$  is the minimum, the SC 2-tree  $P^{-1}s_1$  is naturally  $(X\rho_c, X\rho_a)$ -colored. Furthermore, for every  $s_1 \in S_{\tau_1}$  such that  $\mathbf{t}_{\tau_1}s_1$  is not the minimum, the 2-tree  $P^{-1}s_1$  is naturally  $X\rho_c$ -colored.

We define a  $(X\rho_c, X\rho_a)$ -colored SC 2-operad as a functor  $\mathcal{O}$  from the disjoint union of the groupoid of  $X\rho_c$ -colored 2-trees and the groupoid of  $(X\rho_c, X\rho_a)$ -colored SC 2-trees to  $\mathcal{C}$ . Given a map  $P : \tau \rightarrow \tau_1$  of  $X\rho_c$ -colored 2-trees or  $(X\rho_c, X\rho_a)$ -colored SC 2-trees there should be given a map

$$\mathcal{O}(\tau_1) \otimes \bigotimes_{s_1 \in S_1} \mathcal{O}(P^{-1}s_1) \rightarrow \mathcal{O}(\tau).$$

The associativity axiom should be satisfied.

5.2.4. *Unital colored SC 2-operads.* As well as SC 2-operads all colored SC 2-operads are assumed to be unital.

To introduce the notion of unital colored SC 2-operads we define  $\mathbf{u}_c$ ,  $c \in X\rho_c$ , be the colored 2-tree  $\mathbf{pt} \rightarrow \mathbf{pt}$  for which the point  $\mathbf{pt}$  has the color  $c$  and  $c_{\mathbf{u}_c} = c$ . Similarly, we define  $\mathbf{u}_a$ ,  $a \in X\rho_a$  to be the colored SC 2-tree in which the one-element ordinal is mapped into the one-element SC ordinal and all colorings are  $a$ .

For every isomorphism  $P : \tau_1 \rightarrow \tau_2$  of  $X\rho_c$ -colored 2-trees or SC 2-trees every pre-image  $P^{-1}s_2$ ,  $s_2 \in S_{\tau_2}$ , is either  $\mathbf{u}_c$  or  $\mathbf{u}_a$ . For every colored 2-tree or colored SC 2-tree  $\tau$  the pre-image  $Q_\tau^{-1}\mathbf{pt}$  of the point for a unique map  $Q_\tau : \tau \rightarrow \mathbf{u}_{c_\tau}$  is equal to  $\tau$ .

Define a *unital colored SC 2-operad* as a colored SC 2-operad  $\mathcal{O}$  along with maps  $\mathbf{1} \rightarrow \mathcal{O}(\mathbf{u}_c)$  and  $\mathbf{1} \rightarrow \mathcal{O}(\mathbf{u}_a)$  for all  $c \in X\rho_c$  and  $a \in X\rho_a$  satisfying:

— for every isomorphism  $P : \tau_1 \rightarrow \tau_2$ , of colored 2-trees or colored SC 2-trees the map

$$\mathcal{O}(\tau_2) \cong \mathcal{O}(\tau_2) \otimes \mathbf{1}^{\otimes S_{\tau_2}} \rightarrow \mathcal{O}(\tau_2) \otimes \bigotimes_{s_2 \in S_{\tau_2}} \mathcal{O}(P^{-1}s_2) \rightarrow \mathcal{O}(\tau_1)$$

coincides with the map  $\mathcal{O}(\tau_2) \rightarrow \mathcal{O}(\tau_1)$  induced by  $P^{-1}$  from the definition of  $\mathcal{O}$  as a functor from the corresponding groupoid.

— for every colored 2-tree or colored SC 2-tree  $\tau$  the composition

$$\mathcal{O}(\tau) \cong \mathbf{1} \otimes \mathcal{O}(\tau) \rightarrow \mathcal{O}(\mathbf{u}_{c_\tau}) \otimes \mathcal{O}(\tau) \rightarrow \mathcal{O}(\tau)$$

is the identity on  $\mathcal{O}(\tau)$ .

5.2.5. *Pruned SC 2-operads.* A (colored) SC 2-tree  $\tau$  is called pruned if  $\text{Im}(\mathbf{t}_\tau) \supset T_\tau \setminus m_{T_\tau}$ , where  $m_{T_\tau}$  is the marked minimum of  $T_\tau$ .

For every colored SC 2-tree  $\tau$  there exists a unique up to isomorphism pruned colored SC 2-tree  $\tau'$  and a map  $P : \tau' \rightarrow \tau$  such that  $P_S : S_{\tau'} \rightarrow S_\tau$  is a bijection;  $P_T$  is injective, and  $P$  induces an isomorphism of colorings. For every such  $P$  the pre-images  $P^{-1}s$  are of the form  $\mathbf{u}_c$  or  $\mathbf{u}_a$ , therefore, given a unital operad  $\mathcal{O}$ , we have a map

$$(5.1) \quad \mathcal{O}(\tau') \rightarrow \mathcal{O}(\tau).$$

By analogy with ordinary 2-operads (see Subsection 4.2.2)  $\mathcal{O}$  is called *pruned* if all such maps (5.1) are isomorphisms.

5.2.6. *Reduced SC 2-operads.* We say that a pruned (non-colored) SC 2-operad  $\mathcal{O}$  is *reduced* if

- all the unit maps  $\mathbf{1} \rightarrow \mathcal{O}(\mathbf{u}_c)$ ,  $\mathbf{1} \rightarrow \mathcal{O}(\mathbf{u}_a)$  are isomorphisms;
- $\mathcal{O}(\tau) = \mathbf{1}$  whenever  $|S_\tau| \leq 1$  so that we have an identification  $\mathcal{O}(\tau) = \mathbf{triv}(\tau)$  for all such  $\tau$ ;
- for every map  $P : \tau_1 \rightarrow \tau_2$  where  $|S_{\tau_1}|, |S_{\tau_2}| \leq 1$  the corresponding operadic composition law coincides with that of  $\mathbf{triv}$ .

Note that, equivalently, one can only require that the conditions are the case for pruned 2-trees  $\tau$ .

5.3. **Desymmetrization.** Given a symmetric SC-operad  $Q$ , Batanin defines its desymmetrization  $\mathbf{des} Q$  by setting  $\mathbf{des} Q(\mathbf{t}) := Q(S_\mathbf{t})$  for all 2-trees and SC 2-trees  $\mathbf{t}$ . Here  $S_\mathbf{t}$  is treated as an SC-set as follows:

- if  $\mathbf{t}$  is a usual 2-tree then we define all the colorings to be  $\mathbf{c}$ ;
- if  $\mathbf{t}$  is an SC 2-tree, we set  $\mathbf{c}_{S_\mathbf{t}} := \mathbf{a}$  and we give the preimage of marked element of  $T_\mathbf{t}$  the color  $\mathbf{a}$ , the remaining elements of  $S_\mathbf{t}$  receive color  $\mathbf{c}$ .

Given a reduced symmetric SC-operad  $Q$ , its desymmetrization  $\mathbf{des} Q$  is a reduced SC 2-operad so that  $\mathbf{des}$  is a functor from the category of reduced symmetric SC operads to that of reduced SC 2-operads.

5.3.1. In the same spirit, one defines the desymmetrization of a colored symmetric SC-operad  $\mathcal{O}$ . Let  $\tau$  be a colored SC 2-tree; we then see that  $S_\tau$  is a colored SC-set in the natural way: the map  $\chi_{S_\tau} \Big|_{S_{\tau,c}} := \chi_{\tau,c}$  and  $\chi_{S_\tau} \Big|_{S_{\tau,a}} := \chi_{\tau,a}$ . Finally,  $c_{S_\tau} := c_\tau$ . We then set

$$\mathbf{des}(\mathcal{O})(\tau) := \mathcal{O}(S_\tau)$$

with the composition law determined by that in  $\mathcal{O}$ .

5.4. **Symmetrization.** The content of this section is a straightforward SC generalization of Sec 4.3.

Under an assumption that SMC  $\mathcal{C}$  has small colimits, the functor  $\mathbf{des}$  has a left adjoint  $\mathbf{sym}$ . We have a description of the functor  $\mathbf{sym}$  for the category of reduced SC 2-operads which is similar to that for reduced 2-operads (see. Sec 4.3).

For every SC set  $S$  we define a category  $\mathcal{J}(S)$ .

If  $c_S = \mathbf{c}$ , then the category  $\mathcal{J}(S)$  is the same as in Sec 4.3: the objects of  $\mathcal{J}(S)$  are pruned 2-trees of the form

$$\mathbf{t} : S \rightarrow T.$$

Morphisms are the maps between 2-trees which induce the identity map on  $S$

If  $c_S = \mathbf{a}$  then objects of  $\mathcal{J}(S)$  are pruned SC 2-trees  $\mathbf{t} : S \rightarrow T$  such that the preimage of the minimal element of  $T$  coincides with  $S_\mathbf{a} = \chi^{-1}(\mathbf{a})$ . Morphisms are the maps between SC 2-trees which induce the identity map on  $S$ .

As in the non SC case, all categories  $\mathcal{J}(S)$  are in fact posets.

Let  $\mathcal{O}$  be a reduced SC 2-operad. For every SC set  $S$  the operad  $\mathcal{O}$  gives us an obvious (contravariant) functor from the category  $\mathcal{J}(S)$  to the underlying SMC  $\mathcal{C}$ . We denote this functor by  $\mathcal{O}_S$ .

According to Theorem 9.1 from [2] we have

$$(5.2) \quad \mathbf{sym} \mathcal{O}(S) = \mathbf{colim}_{\mathcal{J}(S)} \mathcal{O}_S.$$

The operadic multiplications of  $\mathbf{sym} \mathcal{O}$  can be easily obtained from those of  $\mathcal{O}$  using the properties of colimits.

5.4.1. *Model structure.* The category of dg pruned SC 2-operads has a model structure which is defined in the same way as in Sec 4.3.1 Same is true for the model structure on the category  $\mathcal{J}(S)$  of contravariant functors from  $\mathcal{J}(S)$  to the category of complexes over  $\mathbf{k}$ .

Lemma 4.2 holds true in the SC context.

**Lemma 5.1.** *For every dg reduced SC 2-operad  $\mathcal{O}$  there exists a cofibrant dg reduced SC 2-operad  $\mathcal{R}\mathcal{O}$  and a quasi-isomorphism  $f : \mathcal{R}\mathcal{O} \rightarrow \mathcal{O}$  such that the functors  $\mathcal{R}\mathcal{O}_S \in \mathcal{J}(S)$  are cofibrant for any SC set  $S$ .*

*Proof.* Similar to the proof of Lemma 4.2. □

5.4.2. *Batanin's theorem.*

**Theorem 5.2** (Theorem 9.2, 9.4, [2]). *The symmetric SC operad  $\mathbf{sym} \mathcal{R}\mathbf{triv}$  is weakly equivalent to Voronov's Swiss Cheese operad if  $\mathcal{C}$  is the category of topological spaces, and to the singular chain operad of Voronov's Swiss Cheese operad if  $\mathcal{C}$  is the category of chain complexes of  $\mathbf{k}$ -vector spaces.*

Batanin's proof goes along the same lines as his proof of Theorem 4.3.

## 6. LINKING THE OPERAD $\mathbf{s}\mathcal{O}$ WITH 2 OPERADS: A 2-OPERAD $\mathbf{seq}$

In this section we will define a 2-sub-operad  $\mathbf{seq} \subset \mathbf{des} \mathbf{s}\mathcal{O}$ . Next, we define the SC-version of  $\mathbf{seq}$ .

6.1. **2-operad  $\mathbf{seq}$  (cochain part).** Let us first recall the definition of the cochain part (i.e. 'non-SC part') of the  $\mathbb{N}$ -colored operad  $\mathbf{s}\mathcal{O}$  (Sec 3.3.2) with the notation slightly changed.

Let  $S$  be a finite set and  $J; I_s, s \in S$  be non-empty finite ordinals. Each element  $u$  of the operadic space  $\mathbf{s}\mathcal{O}(S)_{\{I_s\}_{s \in S}}^J$  is defined by means of the following data:

— a total order  $>_u$  on the set

$$\mathcal{I} := \bigsqcup_{s \in S} I_s;$$

a non-decreasing map

$$Q_u : \mathcal{I} \rightarrow J.$$

These data should satisfy:

- i) the order  $>_u$  on  $\mathcal{I}$  agrees with those on each  $I_s$ ;
- ii) same as condition 3) from Sec 3.3.

The composition law for the operad  $\mathbf{s}\mathcal{O}$  was defined in Sec 3.5.

Let us now define a colored sub -2-operad  $\mathbf{seq}$  of  $\mathbf{des} \mathbf{s}\mathcal{O}$ . Let  $\tau$  be a  $\mathbb{N}$ -colored 2-tree, which is defined by means of a 2-tree  $\mathbf{t} : S \rightarrow T$  and its  $\mathbb{N}$ -coloring such that an  $s \in S$  has color  $I_s$ , where  $I_s$  is a non-empty finite ordinal, and the color of the result is  $J$ . More formally,  $\chi_\tau(s) = I_s$ ;  $c_\tau = J$ .

Let us define subsets

$$\mathbf{seq}(\tau) := \mathbf{seq}(\mathbf{t})_{\{I_s\}_{s \in S}}^J \subset \mathbf{s}\mathcal{O}(S)_{\{I_s\}_{s \in S}}^J$$



which consists of all elements  $u \in \mathbf{s}\mathcal{O}(S)_{\{I_s\}, s \in S}^J$  satisfying:

- if  $i, k \in I_{s_1}, j \in I_{s_2}, s_1 \neq s_2$  and  $i <_u j <_u k$ , then  $\mathbf{t}(s_2) < \mathbf{t}(s_1)$ ;
- if  $s_1, s_2 \in S, s_1 < s_2$ , and  $\mathbf{t}(s_1) = \mathbf{t}(s_2)$ , then  $I_{s_1} <_u I_{s_2}$ .

One can check that thus defined subspaces are closed under all 2-operadic composition maps so that  $\mathbf{seq} \subset \mathbf{des}\mathbf{s}\mathcal{O}$  is a colored sub-2-operad. Thus defined colored sub-2-operad coincides with the colored 2-operad  $\mathbf{seq}$  as defined in Sec 6.1 of [32]. The check that  $\mathbf{seq}$  is closed under the 2-operadic compositions follows from the observation that the 2-operadic composition maps inherited from  $\mathbf{s}\mathcal{O}$  are the same as in loc. cit.

**6.2. SC version of seq.** Let us define a colored SC 2-operad  $\mathbf{seq}^{\mathbf{SC}}$  by modifying the definition of  $\mathbf{seq}$  as follows.

First of all we fix the sets of colors:

- the set  $X\rho_c$  is the same as the set of colors of  $\mathbf{seq}$ , i.e.  $\mathbb{N}$ ;
- the set  $X\rho_a$  is the one element set  $\{\mathbf{a}\}$ ; we identify a unique element of  $X\rho_a$  with the ordinal consisting of 1 element.
- Given a usual colored 2-tree  $\tau$  we set  $\mathbf{seq}^{\mathbf{SC}}(\tau) := \mathbf{seq}(\tau)$ ;
- given a colored SC 2-tree  $\tau$ , let us construct a usual colored 2-tree  $\tau'$  with the underlying 2-tree  $\mathbf{t}' = \mathbf{t} : S \rightarrow T$ . Define a map  $\chi_{\tau'} : S \rightarrow X\rho_c = \mathbb{N}$  by setting
  - a) if  $s \in S$  and  $\mathbf{t}(s)$  is the minimum of  $T$ , then we set  $\chi_{\tau'}(s)$  to be the one-element ordinal;
  - b) if  $s \in S$  and  $\mathbf{t}(s)$  is not the minimum of  $T$ , then we set  $\chi_{\tau'}(s) = \chi_{\tau,c}(s)$ , where  $\chi_{\tau,c}$  is a defining map of the coloring for  $\tau$  (see Sec. 5.2.3).

Lastly, we set  $c_{\tau'}$  to be the one-element ordinal.

We then define  $\mathbf{seq}^{\mathbf{SC}}(\tau) := \mathbf{seq}(\tau')$ .

Note that we have natural inclusions

$$\mathbf{seq}^{\mathbf{SC}}(\tau) \subset \mathbf{s}\mathcal{O}(S_\tau) = (\mathbf{des}\mathbf{s}\mathcal{O})(\tau),$$

where  $S_\tau$  is the colored SC-set corresponding to the colored 2-tree or the colored SC 2-tree  $\tau$  as defined in Sec 5.3. Thus  $\mathbf{seq}^{\mathbf{SC}}$  is a colored SC 2-suboperad of  $\mathbf{des}\mathbf{s}\mathcal{O}$ .

**6.3. Totalization: A dg 2-operad  $|\mathbf{seq}|$  and a dg operad  $|\mathbf{s}\mathcal{O}|$ .** Using the functor of (co)-simplicial totalization we will convert a colored operad  $\mathbf{s}\mathcal{O}$  and a colored 2-operad  $\mathbf{seq}$  into differential graded operads. The SC versions will be covered in the next section 6.4.

The spaces of unary operations in  $\mathbf{seq}$  and  $\mathbf{s}\mathcal{O}$  give a category structure on  $\mathbb{N}$

$$\mathrm{hom}(I_1, I_2) = \mathbf{seq}(t_0)_{I_1}^{I_2} = \mathbf{s}\mathcal{O}_{I_1}^{I_2},$$

where  $t_0 : \mathbf{pt} \rightarrow \mathbf{pt}$ . This category is isomorphic to the simplicial category  $\Delta$ .

The action of these unary operations defines a polysimplicial/cosimplicial structure on the collection of operadic sets.

Given a 2-tree  $\mathbf{t} : S \rightarrow T$ , the collection of sets

$$\mathbf{seq}(\mathbf{t})_{\{I_s\}_{s \in S}}^J,$$

where  $I_s, J$  are non-empty final ordinals, forms a functor

$$\mathbf{seq}(\mathbf{t}) : \Delta \times (\Delta^{\mathrm{op}})^S \rightarrow \mathbf{Sets},$$

where  $J \in \Delta$  and  $I_s \in \Delta^{\mathrm{op}}$ .

Using the functor  $\mathcal{S} : \Delta \rightarrow \mathbf{complexes}$  we can take the total complexes of these polysimplicial (cosimplicial) sets, in the same way as in Subsection 2.0.2. We set

$$|\mathbf{seq}|(\mathbf{t}) := |\mathbf{seq}(\mathbf{t})|;$$

The complexes  $|\mathbf{seq}|(\mathbf{t})$  automatically form a dg-operad.

Similarly, the sets  $\mathbf{s}\mathcal{O}(S)_{\{I_s\}_{s \in S}}^J$  form a functor

$$\mathbf{s}\mathcal{O}(S) : \Delta \times (\Delta^{\text{op}})^S \rightarrow \mathbf{Sets}$$

so that we can define

$$|\mathbf{s}\mathcal{O}|(S) := |\mathbf{s}\mathcal{O}(S)|.$$

We have a dg-operad structure on  $|\mathbf{s}\mathcal{O}|$ . The embedding  $\mathbf{seq} \subset \mathbf{des}\mathbf{s}\mathcal{O}$  induces a map

$$|\mathbf{seq}| \rightarrow \mathbf{des}|\mathbf{s}\mathcal{O}|.$$

**6.4. Extension to the SC-setting: a dg SC 2-operad  $\overset{\text{SC}}{\mathbf{seq}}$  and an SC operad  $|\mathbf{s}\mathcal{O}|$ .** Let us now extend the construction of  $|\mathbf{seq}|$  and  $|\mathbf{s}\mathcal{O}|$  to the SC case.

Given an SC 2-tree  $\mathbf{t} : S \rightarrow S_1$ , let us decompose  $S = S_c \sqcup S_a$ , where  $S_a$  is the pre-image of the minimum of  $S_1$ . Suppose we are given ordinals  $I_s, s \in S_c$ . Using these data, we naturally get a colored SC 2-tree  $\tau := \tau(\mathbf{t}, \{I_s\}_{s \in S_c})$ , where the coloring sets are  $X_{\rho_c} = \mathbb{N}$  and  $X_{\rho_a} = \{\mathbf{a}\}$ . Each element of  $s \in S_c$  receives color  $I_s$ ; each element of  $S_a$  gets colored in  $\mathbf{a}$ ; we set  $c_\tau = \mathbf{a}$ .

We set

$$\overset{\text{SC}}{\mathbf{seq}}(\mathbf{t})_{\{I_s\}_{s \in S_c}} := \overset{\text{SC}}{\mathbf{seq}}(\tau).$$

Thus, given an SC 2-tree  $\mathbf{t} : S \rightarrow S_1$ , we get a polysimplicial set

$$\overset{\text{SC}}{\mathbf{seq}}(\mathbf{t}) : (\Delta^{\text{op}})^{S_c} \rightarrow \mathbf{Sets}.$$

Set  $|\overset{\text{SC}}{\mathbf{seq}}|(\mathbf{t}) := |\overset{\text{SC}}{\mathbf{seq}}(\mathbf{t})|$ . For  $\mathbf{t}$  being a 2-tree we set  $|\overset{\text{SC}}{\mathbf{seq}}|(\mathbf{t}) := |\mathbf{seq}|(\mathbf{t})$ . This way the dg 2-operad  $|\mathbf{seq}|$  extends to an SC 2-operad  $|\overset{\text{SC}}{\mathbf{seq}}|$ .

Given an SC set  $S = S_c \sqcup S_a$  and ordinals  $I_s, s \in S_c$  we get a  $\mathbb{N}$ -colored SC-set which determines the operadic set

$$\mathbf{s}\mathcal{O}(S_c, S_a)_{\{I_s\}_{s \in S_c}}.$$

These sets form a functor

$$\mathbf{s}\mathcal{O}(S) : (\Delta^{\text{op}})^{S_c} \rightarrow \mathbf{Sets}.$$

and we can set  $|\mathbf{s}\mathcal{O}|(S) := |\mathbf{s}\mathcal{O}(S)|$  thereby getting an SC symmetric operad  $|\mathbf{s}\mathcal{O}|$  which is an SC extension of the symmetric operad  $|\mathbf{s}\mathcal{O}|$  from the previous section 6.3.

The map  $\overset{\text{SC}}{\mathbf{seq}} \rightarrow \mathbf{des}\mathbf{s}\mathcal{O}$  of SC 2-operads induces a map

$$(6.1) \quad |\overset{\text{SC}}{\mathbf{seq}}| \rightarrow \mathbf{des}|\mathbf{s}\mathcal{O}|$$

of dg SC 2-operads.

Since the operad  $\mathbf{s}\mathcal{O}$  is isomorphic to  $\mathcal{O}$  the DG operad  $|\mathbf{s}\mathcal{O}|$  is isomorphic to the DG operad  $\Lambda = |\mathcal{O}|$  of natural operations on the pair  $(C^\bullet(A, A); A)$ . Thus we have a map from the SC 2-operad  $|\overset{\text{SC}}{\mathbf{seq}}|$  to  $\mathbf{des}\Lambda$ .

7. THE SC 2-OPERAD **br** AND THE SC OPERAD **braces**

It is not hard to see that the SC 2-operad  $|\mathbf{seq}^{\mathbf{SC}}|$  is pruned. However, neither  $|\mathbf{seq}^{\mathbf{SC}}|$  nor  $|\mathbf{sO}|$  is reduced. In this section we construct a reduced SC 2-operad **br** which is quasi-isomorphic to the SC 2-operad  $|\mathbf{seq}^{\mathbf{SC}}|$ . Similarly, we construct a reduced SC operad **braces** which is quasi-isomorphic to the SC operad  $|\mathbf{sO}|$ . Both **br** and **braces** are obtained as suboperads of  $|\mathbf{seq}^{\mathbf{SC}}|$  and  $|\mathbf{sO}|$ , respectively.

As usual we will first make all definition for the non-SC part and then extend them to the SC-setting

7.1. **An increasing filtration on the colored 2-operad seq.** Let **t** be a 2-tree  $\mathbf{t} : S \rightarrow T$  and

$$v \in \mathbf{seq}(\mathbf{t})_{\{I_s\}_{s \in S}}^J.$$

Consider the order  $>_v$  on

$$(7.1) \quad \mathcal{I} := \mathcal{I}_S := \bigsqcup_{s \in S} I_s.$$

Call two elements  $i_1, i_2 \in \mathcal{I}_S$  *elementary equivalent* if  $i_1, i_2 \in I_s$  for some  $s \in S$  and for every  $i \in \mathcal{I}_S$  between  $i_1$  and  $i_2$  with respect to the order  $<_v$  the element  $i$  belongs to  $I_s$ . In this way we get an equivalence relation on  $\mathcal{I}_S$ . Denote by  $|v|$  the number of equivalence classes with respect to this relation.

Let  $F_N \mathbf{seq}(\mathbf{t})_{\{I_s\}_{s \in S}}^J$  be the subset consisting of all elements  $v$  with  $|v| \leq N + |S|$ . Roughly speaking, the difference  $|v| - |S|$  counts how many times the order  $<_v$  cuts the ordinals  $I_s$ ,  $s \in S$  into subordinals.

7.2. **Extension of the filtration onto  $\mathbf{seq}^{\mathbf{SC}}$ .** Let **t** :  $S \rightarrow T$  be an SC 2-tree. As above, we set  $S_a$  to be the pre-image of the minimum of  $T$  and  $S_c := S \setminus S_a$ .

Recall that an element

$$v \in \mathbf{seq}^{\mathbf{SC}}(\mathbf{t})_{\{I_s\}_{s \in S_c}}$$

is nothing else but a total order  $>_v$  on

$$(7.2) \quad \bigsqcup_{s \in S_c} I_s \sqcup S_a$$

subject to certain conditions.

In order to define the elementary equivalence relation on (7.2) we replace (7.2) by the isomorphic set

$$(7.3) \quad \mathcal{I}_{S_c \sqcup S_a} = \bigsqcup_{s \in S} I_s,$$

where  $I_s$  is the one element ordinal for every  $s \in S_a$ .

Using the total order  $>_v$  on  $\mathcal{I}_{S_c \sqcup S_a}$  and the construction from the previous subsection we get the elementary equivalence relation on the set  $\mathcal{I}_{S_c \sqcup S_a}$  and hence on (7.2).

On the set (7.2) the elementary equivalence relation can be described as follows. The restriction of this relation onto  $S_a$  coincides with the identity relation, there is no element of  $S_a$  which is equivalent to an element

$$i \in \bigsqcup_{s \in S_c} I_s.$$

Finally we call two elements

$$i_1, i_2 \in \bigsqcup_{s \in S_c} I_s$$

elementary equivalent iff

- $i_1, i_2 \in I_s$  for some  $s \in S_c$ ,
- for every element  $i$  of the set (7.2) between  $i_1$  and  $i_2$  with respect to the order  $<_v$  we have  $i \in I_s$ .

We denote the number of equivalence classes in (7.3)  $\mathcal{I}_{S_c \sqcup S_a}$  by  $|v|$  and define the subset

$$F_N^{\text{SC}} \mathbf{seq}(\mathbf{t})_{\{I_s\}_{s \in S_c}} \subset \mathbf{seq}(\mathbf{t})_{\{I_s\}_{s \in S_c}}$$

to consist of all elements  $v$  with  $|v| \leq N + |S|$ .

### 7.3. Compatibility of the filtration with the operadic structure.

**Lemma 7.1.** *The filtration  $F$  is compatible with operadic compositions on  $\mathbf{seq}^{\text{SC}}$ .*

*Proof.* Let us first prove Lemma for the filtration on 2-operad  $\mathbf{seq}$  (i.e. the 'non-SC'-part). Consider operadic compositions of the following type:

Let  $P : \mathbf{t}_1 \rightarrow \mathbf{t}_2$  be a map of 2-trees, where  $\mathbf{t}_1 : S_1 \rightarrow T_1$  and  $\mathbf{t}_2 : S_2 \rightarrow T_2$ . Let  $P_S : S_1 \rightarrow S_2$  be the induced map. For every  $s_2 \in S_2$ , we have a pre-image  $P_S^{-1}(s_2) \subset S_1$ .

Let  $I_{s_1}, s_1 \in S_1; J_{s_2}, s_2 \in S_2; J$  be non-empty ordinals.

Let

$$\begin{aligned} w &\in \mathbf{seq}(\mathbf{t}_2)_{\{J_{s_2}\}_{s_2 \in S_2}}^J; \\ v_{s_2} &\in \mathbf{seq}(P^{-1}s_2)_{\{I_{s_1}\}_{s_1 \in P_S^{-1}(s_2)}}^{J_{s_2}}. \end{aligned}$$

Let us denote by  $z$  the composition of these elements and estimate  $|z|$ . Suppose that

$$J_\sigma \subset \left( \bigsqcup_{s_2 \in S_2} J_{s_2}, >_w \right)$$

for  $\sigma \in S_2$  is split into  $|\sigma|$  equivalence classes.

Consider the map

$$\mathcal{I}_\sigma := \bigsqcup_{s_1 \in P^{-1}\sigma} I_{s_1} \rightarrow J_\sigma.$$

It is clear that the number of equivalence classes of

$$\mathcal{I}_\sigma \subset \left( \bigsqcup_{s_1 \in S_1} I_{s_1}, >_z \right)$$

does not exceed  $|v_\sigma| + |\sigma| - 1$ . Therefore

$$|z| \leq \sum_{\sigma \in S_2} (|v_\sigma| + |\sigma| - 1) = |w| - |S_2| + \sum_{\sigma} |v_\sigma|.$$

Hence,

$$|z| - |S_1| \leq |w| - |S_2| + \sum_{\sigma} (|v_\sigma| - |P^{-1}\sigma|)$$

which means that this composition is compatible with the filtration  $F$ . This concludes the proof for  $\mathbf{seq}$ . The extension to  $\mathbf{seq}^{\text{SC}}$  is straightforward.  $\square$

This Lemma, in particular implies that the polysimplicial/cosimplicial structure on  $\mathbf{seq}^{\text{SC}}$  is compatible with the filtration  $F$ . Therefore, the filtration  $F$  descends onto the level of total complexes so that we have an increasing filtration on each operadic complex  $|\mathbf{seq}^{\text{SC}}|(\mathbf{t})$ :  $F_N |\mathbf{seq}^{\text{SC}}|(\mathbf{t}) \subset |\mathbf{seq}^{\text{SC}}|(\mathbf{t})$ .

**Lemma 7.2.** *The filtration  $F$  on  $|\mathbf{seq}^{\text{SC}}|$  satisfies the following properties:*

- (1) *The operadic compositions in  $|\mathbf{seq}^{\text{SC}}|$  are compatible with the filtration.*
- (2) *The complex  $F_N |\mathbf{seq}^{\text{SC}}|(\mathbf{t})$  is concentrated in the degrees  $\geq -N$ .*

(3) *The quotient  $F_N|\mathbf{seq}^{\mathbf{SC}}|(\mathbf{t}) / F_{N-1}|\mathbf{seq}^{\mathbf{SC}}|(\mathbf{t})$  only has cohomology concentrated in degree  $-N$ .*

*Proof.*

(1) Follows from the previous lemma.

(2) Let us first consider the non-SC part of the statement (that is, we will prove the statement for  $\mathbf{seq}$ ). Let  $\mathbf{t} : S \rightarrow T$  be a 2-tree. Let us consider the simplicial realization with respect to the lower indices for

$$(7.4) \quad \mathbf{seq}(\mathbf{t})_{\{I_s\}_{s \in S}}^J.$$

Let

$$v \in \mathbf{seq}(\mathbf{t})_{\{I_s\}_{s \in S}}^J.$$

According to Subsection 7.1 the order  $>_v$  on

$$(7.5) \quad \mathcal{I}_S := \bigsqcup_{s \in S} I_s$$

defines on  $\mathcal{I}_S$  an equivalence relation.

If

$$(7.6) \quad |v| + |J| - 1 < \sum_{s \in S} |I_s|.$$

then there exist two different but equivalent elements of  $\mathcal{I}_S$  which go to the same element in  $J$ . In this case the element  $v$  is obtained from another element by applying a degeneracy.

Thus if inequality (7.6) holds for  $v$  then  $v$  does not contribute to the realization of (7.4).

Therefore if  $v$  contributes to the realization then

$$|J| - 1 - \sum_{s \in S} (|I_s| - 1) \geq -|v| + |S|$$

and hence the complex

$$F_N|\mathbf{seq}|(\mathbf{t})$$

is concentrated in degrees

$$\geq -N.$$

This finishes the proof for  $|\mathbf{seq}|$ .

The general SC-case is similar. Let  $\mathbf{t} : S \rightarrow T$  be an SC 2-tree. Let  $S_{\mathbf{a}}$  be the pre-image of the minimal element of  $T$  and  $S_{\mathbf{c}} = S \setminus S_{\mathbf{a}}$ . An element  $v$  of

$$(7.7) \quad \mathbf{seq}^{\mathbf{SC}}(\mathbf{t})_{\{I_s\}_{s \in S_{\mathbf{c}}}}$$

is a total order  $>_v$  on

$$(7.8) \quad \bigsqcup_{s \in S_{\mathbf{c}}} I_s \sqcup S_{\mathbf{a}}$$

subject to certain conditions.

According to Subsection 7.2 the order  $>_v$  gives us the elementary equivalence relation on the set (7.8).

If at least one equivalence class in (7.8) contains more than 1 element then the corresponding element  $v$  in (7.7) is obtained from another element by applying a degeneracy. Indeed, only the equivalence classes in

$$\bigsqcup_{s \in S_{\mathbf{c}}} I_s$$

may contain more than one element. And if at least one class contains more than 1 element then there are distinct elements  $i_1, i_2 \in I_s$  for some  $s \in S_c$  such that one of them goes right after another in the ordinal (7.8).

Therefore, if  $v$  contributes to the realization of (7.7) then

$$\sum_{s \in S_c} |I_s| + |S_a| = |v|.$$

Hence

$$\sum_{s \in S_c} (|I_s| - 1) = |v| - |S_a| - |S_c|$$

or equivalently

$$- \sum_{s \in S_c} (|I_s| - 1) = |S| - |v|.$$

If  $v \in F_N^{\text{SC}} \mathbf{seq}(\mathbf{t})_{\{I_s\}_{s \in S_c}}$  then the right hand side of the latter equation is  $\geq -N$ . Thus the complex

$$F_N |^{\text{SC}} \mathbf{seq}|(\mathbf{t})$$

is concentrated in degrees  $\geq -N$ .

(3) Let us first consider the cochain complex

$$(7.9) \quad F_N |^{\text{SC}} \mathbf{seq}|(\mathbf{t}) / F_{N-1} |^{\text{SC}} \mathbf{seq}|(\mathbf{t})$$

in the case when  $\mathbf{t} : S \rightarrow T$  is a usual 2-tree.

If an element  $v$  in (7.4) represents a non-zero vector in (7.9) then the set  $\mathcal{I}_S$  (7.5) has exactly  $N + |S|$  equivalence classes. The total order on  $\mathcal{I}_S$  gives a total order on the set of these equivalence classes. Hence the set of equivalence classes in  $\mathcal{I}_S$  can be identified with the ordinal  $\{1, 2, \dots, N + |S|\}$ . Furthermore each equivalence class is a subset of  $I_s$  for some  $s \in S$ .

Thus to every such element  $v$  in (7.4) we assign a surjection

$$(7.10) \quad \sigma : \{1, 2, \dots, N + |S|\} \rightarrow S$$

from the ordinal  $\{1, 2, \dots, N + |S|\}$  to the set<sup>4</sup>  $S$ .

Not all such surjections can be gotten from the elements of (7.4) representing non-zero vectors in (7.9). The 2-tree  $\mathbf{t} : S \rightarrow T$ , the definition of  $|^{\text{SC}} \mathbf{seq}|$ , and the definition of the elementary equivalence relation impose the following conditions on the possible surjections (7.10):

- A**  $\sigma(i) \neq \sigma(i+1) \forall i = 1, 2, \dots, N + |S| - 1$ ,
- B** if  $s \neq \tilde{s}$  and  $j_1 < i < j_2$  for  $i \in \sigma^{-1}(s)$  and  $j_1, j_2 \in \sigma^{-1}(\tilde{s})$  then  $\mathbf{t}(s) < \mathbf{t}(\tilde{s})$  in  $T$ ,
- C** if  $\mathbf{t}(s) = \mathbf{t}(\tilde{s})$  and  $s < \tilde{s}$  then all elements of  $\sigma^{-1}(s)$  are smaller than all elements of  $\sigma^{-1}(\tilde{s})$ .

Let us denote by  $D(\mathbf{t}, N)$  the set of all surjections (7.10) satisfying above conditions **A**, **B**, and **C**.

It is not hard to see that the elements of (7.4) representing non-zero vectors in (7.9) and corresponding to the same surjection (7.10) span a subcomplex of (7.9). Furthermore for every map (7.10) this subcomplex is isomorphic to the cochain complex  $|\Xi_{N+|S|}|^{\bullet+N}$ , where  $|\Xi_k|^\bullet$  are the complexes described in the Appendix.

Thus (7.9) is isomorphic to the direct sum of identical cochain complexes

$$(7.11) \quad F_N |^{\text{SC}} \mathbf{seq}|(\mathbf{t}) / F_{N-1} |^{\text{SC}} \mathbf{seq}|(\mathbf{t}) \cong \bigoplus_{\sigma \in D(\mathbf{t}, N)} |\Xi_{N+|S|}|^{\bullet+N}.$$

<sup>4</sup>Recall that  $S$  is also equipped with a total order but in general  $\sigma$  is not a map of ordinals.

Therefore, due to Proposition 9.2 from the Appendix we have,

$$(7.12) \quad H^\bullet \left( F_N |^{\text{SC}} \text{seq}|(\mathbf{t}) / F_{N-1} |^{\text{SC}} \text{seq}|(\mathbf{t}) \right) = \begin{cases} \bigoplus_{\sigma \in D(\mathbf{t}, N)} \mathbf{k}, & \text{if } \bullet = -N, \\ 0, & \text{otherwise.} \end{cases}$$

Let us now consider the cochain complex

$$(7.13) \quad F_N |^{\text{SC}} \text{seq}|(\mathbf{t}) / F_{N-1} |^{\text{SC}} \text{seq}|(\mathbf{t})$$

in the case when  $\mathbf{t} : S \rightarrow T$  is an SC 2-tree.

As above  $S_a$  is the pre-image of the minimal element of  $T$  and  $S_c = S \setminus S_a$ .

If an element  $v$  of (7.7) represents a non-zero vector in (7.13) then the set

$$(7.14) \quad \mathcal{I}_{S_c \sqcup S_a} = \bigsqcup_{s \in S_c} I_s \sqcup S_a$$

has exactly  $N + |S|$  equivalence classes. The total order on  $\mathcal{I}_{S_c \sqcup S_a}$  gives us a total order on the set of its equivalence classes. Hence the set of the equivalence classes can be identified with the standard ordinal  $\{1, 2, \dots, N + |S|\}$ . Furthermore, each equivalence class is either a subset of  $I_s$  for some  $s \in S_c$  or a one element subset of  $S_a$ . Thus we get a surjection

$$(7.15) \quad \sigma : \{1, 2, \dots, N + |S|\} \rightarrow S$$

from the ordinal  $\{1, 2, \dots, N + |S|\}$  to the set  $S$ .

As well as in the case of the usual 2-tree this surjection satisfies above conditions **A**, **B**, and **C**.

Let us remark that, since  $S_a$  is the pre-image of the minimal element of  $T$ , conditions **A**, **B**, and **C** imposed on the surjection (7.15) imply that for every  $s \in S_a$  the pre-image  $\sigma^{-1}(s)$  is a one element set.

As above we denote by  $D(\mathbf{t}, N)$  the set of all surjections (7.15) satisfying above conditions **A**, **B**, and **C**.

Similarly to the case of a usual 2-tree the set of elements of (7.7) representing non-zero vectors in (7.13) splits into the disjoint union of subsets, corresponding surjections  $\sigma \in D(\mathbf{t}, N)$ . And similarly the elements of (7.7) representing non-zero vectors in (7.13) and corresponding to the same map (7.15) span a subcomplex of (7.13). These subcomplexes are all isomorphic to the cochain complex

$$|\Xi_{N+|S_c}|^{\bullet+N, 0},$$

where the bicomplexes  $|\Xi_k|^{\bullet, \bullet}$  are described in the Appendix.

It is not hard to see that the complex  $|\Xi_{N+|S_c}|^{\bullet, 0}$  consists of the field  $\mathbf{k}$  placed in degree 0.

Thus for an SC 2-tree  $\mathbf{t}$  we have

$$(7.16) \quad \left( F_N |^{\text{SC}} \text{seq}|(\mathbf{t}) / F_{N-1} |^{\text{SC}} \text{seq}|(\mathbf{t}) \right)^\bullet \cong \begin{cases} \bigoplus_{\sigma \in D(\mathbf{t}, N)} \mathbf{k}, & \text{if } \bullet = -N, \\ 0, & \text{otherwise} \end{cases}$$

and statement (3) holds in this case too. □

**7.4. Definition of the (SC) 2-operad  $\mathbf{br}$ .** Using this filtration we give the following definition.

**Definition 7.3.** We define the dg (SC) 2-operad  $\mathbf{br}$  as a suboperad of  $|^{\text{SC}} \text{seq}|$  with

$$(7.17) \quad \mathbf{br}(\mathbf{t}) = \bigoplus_{N \geq 0} G^N |^{\text{SC}} \text{seq}|(\mathbf{t}),$$

where

$$G^N |\mathbf{seq}|^{\mathbf{SC}}(\mathbf{t}) = \{v \in F_N |\mathbf{seq}|^{\mathbf{SC}}(\mathbf{t})^{-N} \mid dv \in F_{N-1} |\mathbf{seq}|^{\mathbf{SC}}(\mathbf{t})\},$$

and  $\mathbf{t}$  is either a 2-tree or an SC 2-tree.

Lemma 7.2 implies that the inclusion

$$\mathbf{br} \hookrightarrow |\mathbf{seq}|^{\mathbf{SC}}$$

is a quasi-isomorphism. Furthermore,

**Proposition 7.4.** *The SC 2-operad  $\mathbf{br}$  is reduced.*

*Proof.* Let  $\mathbf{t} : S \rightarrow T$  be a 2-tree or an SC 2-tree with  $|S| \leq 1$ . The condition  $|S| \leq 1$  implies that the filtration  $F$  on  $|\mathbf{seq}|^{\mathbf{SC}}(\mathbf{t})$  is trivial:  $F_{-1} |\mathbf{seq}|^{\mathbf{SC}}(\mathbf{t}) = 0$  and  $F_N |\mathbf{seq}|^{\mathbf{SC}}(\mathbf{t}) = |\mathbf{seq}|^{\mathbf{SC}}(\mathbf{t})$  for all  $N \geq 0$ . Therefore,  $\mathbf{br}(\mathbf{t})$  is simply the vector space of degree 0 cocycles in  $|\mathbf{seq}|^{\mathbf{SC}}(\mathbf{t})$

$$(7.18) \quad \mathbf{br}(\mathbf{t}) = |\mathbf{seq}|^{\mathbf{SC}}(\mathbf{t})^0 \cap \ker d.$$

Due to Lemma 7.2 the complex  $|\mathbf{seq}|^{\mathbf{SC}}(\mathbf{t})$  is concentrated in nonnegative degrees. Hence

$$H^0(|\mathbf{seq}|^{\mathbf{SC}}(\mathbf{t})) = |\mathbf{seq}|^{\mathbf{SC}}(\mathbf{t})^0 \cap \ker d.$$

On the other hand, equations (7.12) and (7.16) imply that

$$H^0(|\mathbf{seq}|^{\mathbf{SC}}(\mathbf{t})) = \mathbf{k}[D(\mathbf{t}, 0)]$$

and it is easy to see that if  $|S| \leq 1$  then  $D(\mathbf{t}, 0)$  is a one element set.

Thus  $\mathbf{br}(\mathbf{t})$  is indeed isomorphic to  $\mathbf{k}$ .

It is not hard to check that the isomorphisms  $\mathbf{k} \cong \mathbf{br}(\mathbf{u}_c)$  and  $\mathbf{k} \cong \mathbf{br}(\mathbf{u}_a)$  are given by the unit maps.  $\square$

**7.5. An increasing filtration on  $\mathbf{sO}$ .** We will now define an analogue of the filtration  $F$  from the previous subsection for the (SC) operad  $\mathbf{sO}$ .

Let us first consider the non-SC case.

Let  $S$  be a finite set and  $J, I_s, s \in S$  be non-empty finite ordinals. Every element

$$u \in \mathbf{sO}_{\{I_s\}_{s \in S}}^J$$

gives us a total order  $>_u$  on the set

$$\mathcal{I}_S := \bigsqcup_{s \in S} I_s.$$

Following Subsection 7.1 this order gives us the elementary equivalence relation on  $\mathcal{I}_S$ . We denote the number of equivalence classes in  $\mathcal{I}_S$  by  $|u|$  and define  $F_N \mathbf{sO}_{\{I_s\}_{s \in S}}^J$  as the subset consisting of all elements  $u \in \mathbf{sO}_{\{I_s\}_{s \in S}}^J$  with  $|u| \leq N + |S|$ .

Let us now extend this definition for the SC-case. Let  $S$  be an SC with  $c_S = \mathbf{a}$  (the case  $c_S = \mathbf{c}$  corresponds to the non-SC part and has just been considered). We split  $S$  as  $S = S_{\mathbf{c}} \sqcup S_{\mathbf{a}}$  where  $S_{\mathbf{c}} = \chi^{-1}(\mathbf{c})$  and  $S_{\mathbf{a}} = \chi^{-1}(\mathbf{a})$ .

By definition an element

$$u \in \mathbf{sO}(S_{\mathbf{c}}, S_{\mathbf{a}})_{\{I_s\}_{s \in S_{\mathbf{c}}}}$$

is a total order on the set

$$\mathcal{I}_{S_{\mathbf{c}} \sqcup S_{\mathbf{a}}} = \bigsqcup_{s \in S_{\mathbf{c}}} I_s \sqcup S_{\mathbf{a}}$$

subject to certain conditions.

Following Subsection 7.2 this order gives us the elementary equivalence relation on  $\mathcal{I}_{S_{\mathbf{c}} \sqcup S_{\mathbf{a}}}$ .



Let us denote the number of the equivalence classes in  $\mathcal{I}_{S_c \sqcup S_a}$  by  $|u|$  and define

$$F_N \mathbf{s}\mathcal{O}(S)_{\{I_s\}_{s \in S_c}}$$

as the set of all elements  $u \in \mathbf{s}\mathcal{O}(S)_{\{I_s\}_{s \in S_c}}$  with  $|u| \leq N + |S|$ .

We claim that

**Lemma 7.5.** *The filtration  $F$  is compatible with operadic compositions on  $\mathbf{s}\mathcal{O}$ .*

*Proof.* Similar to proof of Lemma 7.1. □

This lemma implies that the filtration  $F$  on  $\mathbf{s}\mathcal{O}$  is compatible with the polysimplicial/cosimplicial structure. Therefore, the formula

$$F_N |\mathbf{s}\mathcal{O}|(S) = |F_N(\mathbf{s}\mathcal{O})(S)|$$

defines an increasing filtration on the dg SC operad  $|\mathbf{s}\mathcal{O}|$ .

**Lemma 7.6.** *The filtration  $F$  on  $|\mathbf{s}\mathcal{O}|$  satisfies the following properties:*

- (1) *The operadic compositions in  $|\mathbf{s}\mathcal{O}|$  are compatible with the filtration  $F$ .*
- (2) *The complexes  $F_N |\mathbf{s}\mathcal{O}|(S)$  are concentrated in the degrees  $\geq -N$ .*
- (3) *The cohomology of the quotient  $F_N |\mathbf{s}\mathcal{O}|(S)/F_{N-1} |\mathbf{s}\mathcal{O}|(S)$  is concentrated in the degree  $-N$ .*

*Proof.* Since the proof is very similar to that of Lemma 7.2 we will only briefly outline the proof of (3).

Let us first treat the non-SC part. Let  $S$  be a finite set. As well as for the 2-operad  $|\mathbf{seq}|$  the cochain complex  $F_N |\mathbf{s}\mathcal{O}|(S)/F_{N-1} |\mathbf{s}\mathcal{O}|(S)$  is isomorphic to a direct sum of identical complexes

$$(7.19) \quad F_N |\mathbf{s}\mathcal{O}|(S) / F_{N-1} |\mathbf{s}\mathcal{O}|(S) \cong \bigoplus_{\sigma \in D(S, N)} |\Xi_{N+|S|}|^{\bullet+N},$$

where the complexes  $|\Xi_k|$  are described in the Appendix and  $D(S, N)$  is the set of surjections

$$(7.20) \quad \sigma : \{1, 2, \dots, N + |S|\} \rightarrow S$$

satisfying the following conditions

- I**  $\sigma(i) \neq \sigma(i+1) \forall i = 1, 2, \dots, N + |S| - 1$ ,
- II** if  $s \neq \tilde{s} \in S$  then it is impossible to have  $i_1, i_2 \in \sigma^{-1}(s)$ , and  $j_1, j_2 \in \sigma^{-1}(\tilde{s})$  such that  $i_1 < j_1 < i_2 < j_2$ .

Thus Proposition 9.2 implies statement (3) in the case  $c_S = \mathbf{c}$ .

Let us now consider the SC-case. If  $S$  is an SC set with  $c_S = \mathbf{a}$ ,  $S_c = \chi^{-1}(\mathbf{c})$  and  $S_a = \chi^{-1}(\mathbf{a})$  then the complex  $F_N |\mathbf{s}\mathcal{O}|(S)/F_{N-1} |\mathbf{s}\mathcal{O}|(S)$  is isomorphic to a direct sum of identical complexes

$$(7.21) \quad F_N |\mathbf{s}\mathcal{O}|(S) / F_{N-1} |\mathbf{s}\mathcal{O}|(S) \cong \bigoplus_{\sigma \in D(S, N)} |\Xi_{N+|S_c|}|^{\bullet+N, 0},$$

where the bicomplexes  $|\Xi_k|^{\bullet, \bullet}$  are described in the Appendix and  $D(S, N)$  is the set of surjections (7.20) satisfying above conditions **I**, **II** and the additional condition:

- III** if  $s \in S_a$  then  $\sigma^{-1}(s)$  consists of exactly one element.

Since the complex  $|\Xi_{N+|S_c|}|^{\bullet, 0}$  consists of the field  $\mathbf{k}$  placed in degree 0, statement (3) follows in this case too. □

We would like to remark that condition **B** in the proof of Lemma 7.2 implies condition **II** in the proof of Lemma 7.6. Therefore, for every 2-tree  $\mathbf{t} : S \rightarrow T$  we have the inclusion

$$(7.22) \quad D(\mathbf{t}, N) \subset D(S, N).$$

Similarly, if  $\mathbf{t}$  is an SC 2-tree then conditions **A**, **B**, and **C** imply conditions **I**, **II**, and **III**. Therefore, we have the inclusion (7.22) for SC 2-trees  $\mathbf{t}$  as well. We will use this inclusion later.

**7.6. Definition of the (SC) operad braces.** We now define a useful suboperad of  $|\mathbf{s}\mathcal{O}|$

**Definition 7.7.** *We define the dg SC operad **braces** as a suboperad of  $|\mathbf{s}\mathcal{O}|$  with*

$$(7.23) \quad \mathbf{braces}(S) = \bigoplus_{N \geq 0} G^N |\mathbf{s}\mathcal{O}|(S),$$

where

$$G^N |\mathbf{s}\mathcal{O}|(S) = \{v \in F_N |\mathbf{s}\mathcal{O}|(S)^{-N} \mid dv \in F_{N-1} |\mathbf{s}\mathcal{O}|(S)\},$$

and  $S$  is an SC set.

Lemma 7.6 implies that the inclusion

$$\mathbf{braces} \hookrightarrow |\mathbf{s}\mathcal{O}|$$

is a quasi-isomorphism.

**Proposition 7.8.** *The dg SC operad **braces** is reduced.*

*Proof.* Let  $S$  be an SC set with  $|S| \leq 1$ . It is not hard to construct a pruned 2-tree or a pruned SC 2-tree  $\mathbf{t} : S \rightarrow T$  with  $S$  being the source ordinal.

It is easy to see that if  $|S| < 1$  then

$$\mathbf{br}(\mathbf{t}) = \mathbf{braces}(S)$$

as cochain complexes.

Thus the desired statement follows immediately from Proposition 7.4.  $\square$

Let us now consider a cofibrant resolution  $\mathcal{R}\mathbf{br} \rightarrow \mathbf{br}$  of  $\mathbf{br}$  in the closed model category of reduced dg (SC) 2-operads.

It is clear from the definitions of  $\mathbf{br}$  and  $\mathbf{braces}$  that we have the embedding of dg (SC) 2-operads

$$\mathbf{br} \hookrightarrow \mathbf{des} \mathbf{braces}.$$

Since  $\mathbf{sym}$  is the left adjoint functor for  $\mathbf{des}$  this embedding produces the map

$$(7.24) \quad \mathbf{sym} \mathbf{br} \rightarrow \mathbf{braces}.$$

Composing (7.24) with the map

$$\mathbf{sym} \mathcal{R}\mathbf{br} \rightarrow \mathbf{sym} \mathbf{br}$$

we get the map

$$(7.25) \quad \mathbf{sym} \mathcal{R}\mathbf{br} \rightarrow \mathbf{braces}.$$

We claim that

**Theorem 7.9.** *The map (7.25) is a quasi-isomorphism of dg (SC) 2-operads.*

This theorem plays a crucial role in proving our main result (Theorem 2.1). We devote the next section to the proof of this theorem.

## 8. PROOF OF THEOREM 7.9

We need to show that for every (SC) set  $S$  the map

$$(8.1) \quad (\mathbf{sym} \mathcal{R}\mathbf{br})(S) \rightarrow \mathbf{braces}(S)$$

is a quasi-isomorphism of cochain complexes.

Due to the symmetrization formula (see equation (5.2))

$$\mathbf{sym} \mathcal{R}\mathbf{br}(S) = \mathbf{colim}_{\mathcal{J}(S)} \mathcal{R}\mathbf{br}_S.$$

As  $\mathcal{R}\mathbf{br}$  is a cofibrant resolution of  $\mathbf{br}$ , Lemma 5.1 implies that the functor  $\mathcal{R}\mathbf{br}_S$  is cofibrant. Hence, the natural map

$$\mathbf{hocolim}_{\mathcal{J}(S)} \mathcal{R}\mathbf{br}_S \rightarrow \mathbf{colim}_{\mathcal{J}(S)} \mathcal{R}\mathbf{br}_S$$

is a weak equivalence. Hence we have a zig-zag weak equivalence

$$\mathbf{sym} \mathcal{R}\mathbf{br}(S) \xrightarrow{\sim} \mathbf{hocolim}_{\mathcal{J}(S)} \mathbf{br}_S.$$

Thus we need to show that the map

$$\mathbf{hocolim}_{\mathcal{J}(S)} \mathbf{br}_S \rightarrow \mathbf{braces}(S)$$

is a quasi-isomorphism of cochain complexes.

For this, it suffices to show that so is the map

$$(8.2) \quad \mathbf{hocolim}_{\mathcal{J}(S)} (F_N \mathbf{br} / F_{N-1} \mathbf{br})_S \rightarrow F_N \mathbf{braces}(S) / F_{N-1} \mathbf{braces}(S)$$

for every  $N$ .

Equations (7.12), (7.16) and statement (2) of Lemma 7.2 imply that for every 2-tree or SC 2-tree  $\mathbf{t}$ ,

$$(8.3) \quad F_N \mathbf{br}(\mathbf{t}) / F_{N-1} \mathbf{br}(\mathbf{t}) = \mathbf{k}[D(\mathbf{t}, N)][N],$$

where  $\mathbf{k}[D(\mathbf{t}, N)][N]$  is considered as a cochain complex with the zero differential.

Similarly, equations (7.19), (7.21) and statement (2) of Lemma 7.6 imply that for every SC set  $S$

$$(8.4) \quad F_N \mathbf{braces}(S) / F_{N-1} \mathbf{braces}(S) = \mathbf{k}[D(S, N)][N],$$

where  $\mathbf{k}[D(S, N)][N]$  is considered as a cochain complex with the zero differential.

Let us recall that for every (SC) set  $S$  and for every  $\mathbf{t} \in \mathcal{J}(S)$  we have the inclusion

$$D(\mathbf{t}, N) \subset D(S, N).$$

Let  $S$  be a finite (SC) set. Then for  $\sigma \in D(S, N)$  we set  $\mathcal{J}(\sigma) \subset \mathcal{J}(S)$  to be the full subcategory of all 2-trees  $\mathbf{t}$  such that

$$\sigma \in D(\mathbf{t}, N).$$

Recall that for every (SC) set  $S$  the category  $\mathcal{J}(S)$  is a poset. It is not hard to see that for every morphism

$$P : \mathbf{t} \rightarrow \tilde{\mathbf{t}}$$

in the category  $\mathcal{J}(S)$  we have the inclusion

$$(8.5) \quad D(\tilde{\mathbf{t}}, N) \subset D(\mathbf{t}, N).$$

Furthermore, the morphism

$$F_N \mathbf{br}(\tilde{\mathbf{t}}) / F_{N-1} \mathbf{br}(\tilde{\mathbf{t}}) \rightarrow F_N \mathbf{br}(\mathbf{t}) / F_{N-1} \mathbf{br}(\mathbf{t})$$

corresponding to  $P : \mathbf{t} \rightarrow \tilde{\mathbf{t}}$  is given by this inclusion.

Combining this observation with equations (8.3) and (8.4) we conclude that

$$(8.6) \quad \mathbf{hocolim}_{\mathcal{J}(S)}(F_N \mathbf{br}/F_{N-1} \mathbf{br})_S = \bigoplus_{\sigma \in D(S,N)} \mathbf{hocolim}_{\mathcal{J}(\sigma)} \mathbf{k},$$

$$(F_N/F_{N-1}) \mathbf{braces}(S)(N) = \bigoplus_{\sigma \in D(S,N)} \mathbf{k},$$

and (8.2) is induced by the natural maps

$$(8.7) \quad \mathbf{hocolim}_{\mathcal{J}(\sigma)} \mathbf{k} \rightarrow \mathbf{k},$$

where, by abuse of notation,  $\mathbf{k}$  denotes both the underlying field and the functor which assigns  $\mathbf{k}$  to every object of  $\mathcal{J}(\sigma)$ .

Thus it suffices to show that the map (8.7) is a quasi-isomorphism for every  $\sigma \in D(S, N)$ .

The obvious topological counterpart of this statement can be formulated as

**Proposition 8.1.** *For every (SC) set  $S$  and every element  $\sigma \in D(S, N)$  the natural map*

$$(8.8) \quad \mathbf{hocolim}_{\mathcal{J}(\sigma)} \mathbf{pt} \rightarrow \mathbf{pt}$$

*is a weak equivalence.*

In what follows, by abuse of notation, we denote a constant functor from  $\mathcal{J}(\sigma)$  to another category by the underlying object. For example, in (8.8)  $\mathbf{pt}$  denotes both the one-point space and the functor from  $\mathcal{J}(\sigma)$  to the category of topological spaces which assigns  $\mathbf{pt}$  to every object of  $\mathcal{J}(\sigma)$ .

Let us postpone the proof of Proposition 8.1 to the end of the section and show that this proposition indeed implies that (8.7) is a quasi-isomorphism.

We, first, use the adjunction

$$(8.9) \quad |_{\mathbf{top}} : \mathbf{sSets} \longleftrightarrow \mathbf{Top} : C_*^{sing}$$

between the category  $\mathbf{Top}$  of topological spaces and the category  $\mathbf{sSets}$  of simplicial sets. Here  $|_{\mathbf{top}}$  denotes the realization functor and  $C_*^{sing}$  is the singular chain functor.

Using the fact that the adjunction (8.9) gives a Quillen equivalence between  $\mathbf{Top}$  and  $\mathbf{sSets}$  it is not hard to deduce from Proposition 8.1 its counterpart for simplicial sets. Namely, Proposition 8.1 implies that for every  $\sigma \in D(S, N)$  the natural map

$$(8.10) \quad \mathbf{hocolim}_{\mathcal{J}(\sigma)} \Delta^0 \rightarrow \Delta^0$$

is a weak equivalence of simplicial sets, where

$$\Delta^0 = \mathbf{hom}_{\Delta}(\cdot, [0])$$

is the terminal object of the category  $\mathbf{sSets}$ .

Therefore, for the simplicial Abelian group  $\mathbb{Z}\Delta^0$ , the natural map

$$(8.11) \quad \mathbf{hocolim}_{\mathcal{J}(\sigma)} \mathbb{Z}\Delta^0 \rightarrow \mathbb{Z}\Delta^0$$

is a weak equivalence.

Notice that, via the Dold-Kan correspondence, (8.11) can be viewed as a map of cochain<sup>5</sup> complexes of Abelian groups. Furthermore, to say that (8.11) is a weak equivalence of simplicial Abelian groups is to say that (8.11) is a quasi-isomorphism of the corresponding cochain complexes.

Recall that the forgetful functor

$$\Psi : \mathbf{k} - \mathbf{Vect} \rightarrow \mathbf{Ab}$$

---

<sup>5</sup>Here we reverse the standard grading of the Dold-Kan correspondence.

from the category  $\mathbf{k} - \mathbf{Vect}$  of  $\mathbf{k}$ -vector spaces to the category  $\mathbf{Ab}$  of Abelian groups admits the left adjoint functor

$$\mathbf{k} \otimes_{\mathbb{Z}} : \mathbf{Ab} \rightarrow \mathbf{k} - \mathbf{Vect} .$$

Using this adjunction and the quasi-isomorphism (8.11) we deduce that the natural map

$$(8.12) \quad \mathbf{hocolim}_{\mathcal{J}(\sigma)} \mathbf{k}\Delta^0 \rightarrow \mathbf{k}\Delta^0$$

is a quasi-isomorphism of cochain complexes of  $\mathbf{k}$ -vector spaces. Here  $\mathbf{k}\Delta^0$  is the cochain complex

$$\dots \xrightarrow{\text{id}} \mathbf{k} \xrightarrow{0} \mathbf{k} \xrightarrow{\text{id}} \mathbf{k} \xrightarrow{0} \mathbf{k}$$

with the right most term placed in degree 0. This complex is obviously quasi-isomorphic to  $\mathbf{k}$  placed in degree 0. And hence the map (8.7) is indeed a quasi-isomorphism of cochain complexes.

In order to complete the proof of Theorem 7.9 it remains to prove Proposition 8.1.

**8.1. Proof of Proposition 8.1.** We need a cofibrant resolution of the trivial functor from the poset  $\mathcal{J}(\sigma)$  to the category of topological spaces. The closed model structure on the category of functors from  $\mathcal{J}(\sigma)$  to  $\mathbf{Top}$  is obtained from that on topological spaces using the transfer principle<sup>6</sup> of C. Berger and I. Moerdijk [9]. In other words, fibrations (resp. weak equivalences) between functors from  $\mathcal{J}(\sigma)$  are object-wise fibrations (resp. object-wise weak equivalences).

In order to construct the resolution, given a finite set  $S$ , we consider the configuration space  $\mathbf{Conf}(S)$  of distinct points on  $\mathbb{R}^2$  labeled by elements of  $S$ .

It is known that the space  $\mathbf{Conf}(S)$  admits a cellular subdivision into the Fox-Neuwirth cells [7], [17], [37]. Each Fox-Neuwirth cell  $\mathbf{FN}_{\mathbf{t}}$  corresponds to a pruned 2-tree  $\mathbf{t} : S \rightarrow T$  and it can be defined as the space of all injective maps from the 2-tree  $\mathbf{t}$  to the generalized 2-tree:

$$(x, y) \rightarrow x : \mathbb{R}^2 \rightarrow \mathbb{R} ,$$

where on  $\mathbb{R}^2$  we use the lexicographic order.

In other words, a configuration  $\{(x_s, y_s)\}_{s \in S}$  belongs to  $\mathbf{FN}_{\mathbf{t}}$  iff the following conditions are satisfied:

- if  $\mathbf{t}(s) = \mathbf{t}(\tilde{s})$  and  $s < \tilde{s}$  then  $x_s = x_{\tilde{s}}$  and  $y_s < y_{\tilde{s}}$ ,
- if  $\mathbf{t}(s) < \mathbf{t}(\tilde{s})$  then  $x_s < x_{\tilde{s}}$ .

An example of a configuration from  $\mathbf{FN}_{\mathbf{t}_1}$  for the 2-tree

$$\mathbf{t}_1 : \{1, 2, 3, 4, 5\} \rightarrow \{1, 2, 3\}$$

$$\mathbf{t}_1(1) = \mathbf{t}_1(2) = 1, \quad \mathbf{t}_1(3) = \mathbf{t}_1(4) = 2, \quad \mathbf{t}_1(5) = 3$$

is depicted in figure 6

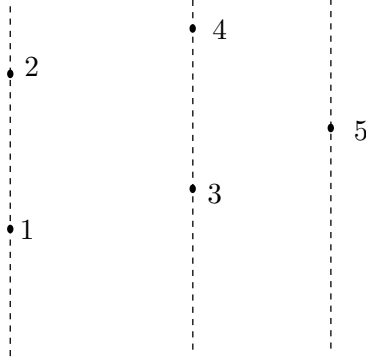
This construction can be easily generalized to pruned SC 2-trees. Namely, if  $\mathbf{t} : S \rightarrow T$  is a pruned SC 2-tree with  $S_{\mathbf{a}}$  being the preimage of the minimal element of  $T$  and  $S_{\mathbf{c}} = S \setminus S_{\mathbf{a}}$  then  $\mathbf{FN}_{\mathbf{t}}$  consists of configurations  $\{(x_s, y_s)\}_{s \in S}$  satisfying the following conditions:

- if  $s \in S_{\mathbf{a}}$  then  $x_s = 0$ ; if  $s \in S_{\mathbf{c}}$  then  $x_s > 0$ ,
- if  $\mathbf{t}(s) = \mathbf{t}(\tilde{s})$  and  $s < \tilde{s}$  then  $x_s = x_{\tilde{s}}$  and  $y_s < y_{\tilde{s}}$ ,
- if  $\mathbf{t}(s) < \mathbf{t}(\tilde{s})$  then  $x_s < x_{\tilde{s}}$ .

Recall that for pruned SC 2-trees the range  $\mathbf{t}(S)$  does not in general include the minimal element. In other words, the subset  $S_{\mathbf{a}}$  may be empty. In this case we still require that  $x_s > 0$  for  $s \in S_{\mathbf{c}}$ .

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<sup>6</sup>The transfer principle can be applied in this case because  $\mathcal{J}(\sigma)$  is a finite poset and the Quillen's path-object argument obviously works for topological spaces.

FIGURE 6. A typical point of  $\mathbf{FN}_{\mathbf{t}_1}$ 

If  $S$  is an (SC) set then for every map  $P : \mathbf{t} \rightarrow \tilde{\mathbf{t}}$  of pruned (SC) 2-trees in the category  $\mathcal{J}(S)$  we have the obvious inclusion

$$(8.13) \quad \mathbf{FN}_{\tilde{\mathbf{t}}} \hookrightarrow \partial \mathbf{FN}_{\mathbf{t}},$$

where  $\partial \mathbf{FN}_{\mathbf{t}}$  denotes the boundary of the Fox-Neuwirth cell  $\mathbf{FN}_{\mathbf{t}}$ .

For example, we may consider the 2-tree

$$\mathbf{t}_2 : \{1, 2, 3, 4, 5\} \rightarrow \{1, 2\}$$

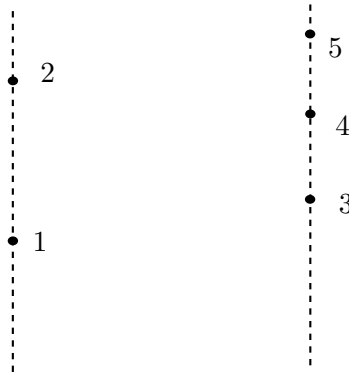
$$\mathbf{t}_2(1) = \mathbf{t}_2(2) = 1, \quad \mathbf{t}_2(3) = \mathbf{t}_2(4) = \mathbf{t}_2(5) = 2$$

with a (unique) map in  $\mathcal{J}(\{1, 2, 3, 4, 5\})$

$$P : \mathbf{t}_1 \rightarrow \mathbf{t}_2,$$

$$P_S = id, \quad P_T(1) = 1, \quad P_T(2) = P_T(3) = 2.$$

A configurations from  $\mathbf{FN}_{\mathbf{t}_2}$  consists of a pair of distinct vertical lines; the left line carries points 1 and 2 such that 1 is below 2; the right line carries points 3, 4, 5 which are put in the order from the bottom to the top. (See figure 7.) It is clear that  $\mathbf{FN}_{\mathbf{t}_2}$  belongs to the boundary of  $\mathbf{FN}_{\mathbf{t}_1}$ .

FIGURE 7. A typical point of  $\mathbf{FN}_{\mathbf{t}_2}$ 

Let  $\sigma \in D(S, N)$  and  $\mathcal{J}(\sigma)$  be the sub-poset of  $\mathcal{J}(S)$  defined above. Using the inclusion (8.13) we upgrade the correspondence

$$(8.14) \quad \mathbf{t} \rightarrow \Phi_\sigma(\mathbf{t}) = \bigcup_{\tilde{\mathbf{t}} \in \mathcal{J}(\sigma); \mathbf{t} \rightarrow \tilde{\mathbf{t}}} \mathbf{FN}_{\tilde{\mathbf{t}}}$$

to the functor

$$\Phi_\sigma : \mathcal{J}(\sigma) \rightarrow \mathbf{Top}.$$

The union in (8.14) is taken over all the pruned (SC) 2-trees  $\tilde{\mathbf{t}} \in \mathcal{J}(\sigma)$  for which we have a map from  $\mathbf{t}$  to  $\tilde{\mathbf{t}}$ .

**Example 8.2.** We consider  $S = \{\alpha, \beta, \gamma, \delta\}$  with  $c_S = \mathbf{a}$ ,  $\chi(\alpha) = \chi(\gamma) = \chi(\delta) = \mathbf{c}$ ,  $\chi(\beta) = \mathbf{a}$ , and  $\sigma$  being the following map

$$\begin{aligned} \sigma : \{1, 2, 3, 4, 5, 6\} &\rightarrow S \\ \sigma(1) = \alpha, \quad \sigma(2) = \delta, \quad \sigma(3) = \gamma, \quad \sigma(4) = \beta, \quad \sigma(5) = \gamma, \quad \sigma(6) = \delta. \end{aligned}$$

The map  $\sigma$  is an element of  $D(S, 2)$  and the SC 2-tree  $\mathbf{t} : \{\beta < \gamma < \alpha < \delta\} \rightarrow \{1, 2, 3, 4\}$

$$\mathbf{t}(\beta) = 1, \quad \mathbf{t}(\gamma) = 2, \quad \mathbf{t}(\alpha) = 3, \quad \mathbf{t}(\delta) = 4$$

is an object of  $\mathcal{J}(\sigma)$ .

There are exactly three pruned SC 2-trees  $\tilde{\mathbf{t}} \in \mathcal{J}(\sigma)$  for which there is a map  $\mathbf{t} \rightarrow \tilde{\mathbf{t}}$ . The first one is  $\tilde{\mathbf{t}}_1 = \mathbf{t}$  and the second one is  $\tilde{\mathbf{t}}_2 : \{\beta < \gamma < \alpha < \delta\} \rightarrow \{1, 2, 3\}$

$$\tilde{\mathbf{t}}_2(\beta) = 1, \quad \tilde{\mathbf{t}}_2(\gamma) = 2, \quad \tilde{\mathbf{t}}_2(\alpha) = \tilde{\mathbf{t}}_2(\delta) = 3.$$

The third SC 2-tree  $\tilde{\mathbf{t}}_3 : \{\beta < \alpha < \gamma < \delta\} \rightarrow \{1, 2, 3\}$

$$\tilde{\mathbf{t}}_3(\beta) = 1, \quad \tilde{\mathbf{t}}_3(\alpha) = \tilde{\mathbf{t}}_3(\gamma) = 2, \quad \tilde{\mathbf{t}}_3(\delta) = 3.$$

So the space  $\Phi_\sigma(\mathbf{t})$  consists of configurations  $\{(x_s, y_s)\}_{s \in \{\alpha, \beta, \gamma, \delta\}}$  satisfying the following conditions:

- $x_\beta = 0 < x_\gamma \leq x_\alpha \leq x_\delta$ , and  $x_\gamma < x_\delta$ ,
- if  $x_\alpha = x_\gamma$  then  $y_\alpha < y_\gamma$ ,
- if  $x_\alpha = x_\delta$  then  $y_\alpha < y_\delta$ .

**Proposition 8.3.** *Let  $S$  be an (SC) set and  $\sigma \in D(S, N)$ . Then the functor  $\Phi_\sigma$  (8.14) is a cofibrant resolution of the trivial functor from  $\mathcal{J}(\sigma)$  to the category of topological spaces.*

*Proof.* Let  $S$  be an (SC) set and  $\sigma \in D(S, N)$ . Let us show that  $\Phi_\sigma(\mathbf{t})$  is contractible for every pruned (SC) 2-tree  $\mathbf{t} : S \rightarrow T$  for which  $\sigma \in D(\mathbf{t}, N)$ .

We give a detailed proof of contractibility of  $\Phi_\sigma(\mathbf{t})$  in the case when  $c_S = \mathbf{c}$  (i.e.  $S$  is a usual, non-SC, set) and hence  $\mathbf{t}$  is a pruned (non-SC) 2-tree. The SC case  $c_S = \mathbf{a}$  is very similar.

The 2-tree  $\mathbf{t} : S \rightarrow T$  gives us a total order on the set  $S$ . So we identify  $S$  with the ordinal  $\{1, 2, 3, \dots, |S|\}$  and denote by  $(x_i, y_i)$  the coordinates of the point labeled by  $i \in \{1, 2, 3, \dots, |S|\}$ .

Next, we consider the following sequence of subspaces

$$\Phi_\sigma(\mathbf{t}) = F_0 \supset F_1 \supset F_2 \supset \dots \supset F_{|S|}$$

where  $F_k$  consists of configurations  $\{(x_i, y_i)\} \in \Phi_\sigma(\mathbf{t})$  with

$$y_i = i, \quad \forall i \leq k.$$

Let us show that  $F_{k+1}$  is a deformation retract of  $F_k$  for all  $k = 0, 1, 2, \dots, |S| - 1$ .

A deformation retraction  $f : F_k \times [0, 1] \rightarrow F_k$  of  $F_k$  onto  $F_{k+1}$  is given by the formula:

$$(8.15) \quad f(\{(x_i, y_i)\}, t) = \{(x_i, y_i(t))\},$$

where

$$y_i(t) = \begin{cases} i, & \text{if } i \leq k, \\ (1-t)y_i + t(k+1+y_i-y_{k+1}), & \text{if } i > k. \end{cases}$$

We need to show that for all  $t \in [0, 1]$  and for all configurations  $\{(x_i, y_i)\} \in F_k$  the point  $f(\{(x_i, y_i)\}, t)$  belongs to  $\Phi_\sigma(\mathbf{t})$ . More precisely, we need to check that if  $x_i = x_j$  and  $i < j$  then  $y_i(t) < y_j(t)$  for all  $t \in [0, 1]$ .

First, it is obvious that if  $i < j \leq k$  the  $y_i(t) < y_j(t)$  regardless of whether  $x_i$  equals  $x_j$  or not. Second, it is not hard to see that if  $k < i < j$  and  $y_i < y_j$  then  $y_i(t) < y_j(t)$  for all  $t \in [0, 1]$ . Finally, if  $i \leq k < j$  and  $x_i = x_j$  then the configuration  $\{(x_i, y_j)\}$  belongs to the Fox-Neuwirth cell  $\mathbf{FN}_{\tilde{\mathbf{t}}}$  corresponding to a pruned 2-tree  $\tilde{\mathbf{t}}$  for which

$$\tilde{\mathbf{t}}(i) = \tilde{\mathbf{t}}(j).$$

The latter implies that  $\tilde{\mathbf{t}}(i) = \tilde{\mathbf{t}}(i+1) = \dots = \tilde{\mathbf{t}}(j-1) = \tilde{\mathbf{t}}(j)$  and hence

$$x_i = x_{i+1} = \dots = x_{j-1} = x_j.$$

Therefore  $y_i < y_{i+1} < \dots < y_{j-1} < y_j$  and, in particular<sup>7</sup>,

$$y_j \geq y_{k+1} > y_k = k.$$

Using these inequalities we conclude that for all  $t \in [0, 1]$

$$y_j(t) = (1-t)y_j + t(k+1) + t(y_j - y_{k+1}) > k + t(y_j - y_{k+1}) \geq k.$$

On the other hand  $y_i(t) \leq k$ . Thus, if  $x_i = x_j$  and  $i < j$  then

$$y_j(t) > y_i(t)$$

for all  $t \in [0, 1]$ .

Furthermore, if  $y_i = i$  for all  $i \leq k+1$  then  $y_i(t) \equiv i$  for all  $i \leq k+1$ . Thus  $f$  is indeed a deformation retraction of  $F_k$  onto  $F_{k+1}$ .

Let us now identify  $T$  with the standard ordinal  $\{1, 2, 3, \dots, |T|\}$ . Next we note that if  $\tilde{\mathbf{t}} \in \mathcal{J}(\sigma)$  admits a map  $\mathbf{t} \rightarrow \tilde{\mathbf{t}}$  then equality  $\mathbf{t}(i) = \mathbf{t}(j)$  implies the equality  $\tilde{\mathbf{t}}(i) = \tilde{\mathbf{t}}(j)$ . Hence, if  $\mathbf{t}(i) = \mathbf{t}(j)$  then  $x_i = x_j$  for every configuration  $\{(x_i, y_i)\} \in \Phi_\sigma(\mathbf{t})$ .

Therefore the function  $i \rightarrow x_i$  factors through

$$\mathbf{t} : \{1, 2, 3, \dots, |S|\} \rightarrow \{1, 2, 3, \dots, |T|\}$$

and hence, we may describe configurations from  $\Phi_\sigma(\mathbf{t})$  using the collections of coordinates  $\{z_l, y_i\}$ ,  $z_l, y_i \in \mathbb{R}$  where  $l \in \{1, 2, 3, \dots, |T|\}$  and  $i \in \{1, 2, 3, \dots, |S|\}$ .

For every configuration  $\{z_l, y_i\}$  from  $\Phi_\sigma(\mathbf{t})$  we have

$$(8.16) \quad z_1 \leq z_2 \leq z_3 \leq \dots \leq z_{|T|}$$

and if  $z_l = z_m$  for  $l \neq m$  then the corresponding configuration belongs to the Fox-Neuwirth cell  $\mathbf{FN}_{\tilde{\mathbf{t}}}$  of a 2-tree  $\tilde{\mathbf{t}} \neq \mathbf{t}$ .

To show the contractibility of  $F_{|S|}$  we consider the following sequence of subspaces:

$$F_{|S|} = G_0 \supset G_1 \supset G_2 \supset \dots \supset G_{|T|} \cong \mathbf{pt}.$$

where  $G_k$  consists of configurations  $\{z_l, y_i\} \in F_{|S|}$  satisfying the following condition

$$z_l = l, \quad \forall \quad l \leq k.$$

In terms of the original coordinates  $(x_i, y_i)$  the latter condition reads

$$x_i = \mathbf{t}(i), \quad \text{if } \mathbf{t}(i) \leq k.$$

We show that for all  $k \leq |T| - 1$  the space  $G_{k+1}$  is a deformation retract of  $G_k$ .

<sup>7</sup>In this case  $y_j = y_{k+1}$  only if  $j = k+1$



The desired deformation retraction is defined by the formula

$$(8.17) \quad g(\{z_l, y_i\}, t) = \{z_l(t), y_i\},$$

where

$$z_l(t) = \begin{cases} l, & \text{if } l \leq k, \\ (1-t)z_l + t(k+1+z_l-z_{k+1}), & \text{if } l > k. \end{cases}$$

To prove that the configuration  $\{z_l(t), y_i\}$  belongs to  $F_{|S|}$  for all  $t \in [0, 1]$  we need to check that inequalities

$$(8.18) \quad z_1(t) \leq z_2(t) \leq z_3(t) \leq \dots \leq z_{|T|}(t)$$

hold for all  $t \in [0, 1]$ . Furthermore we need to check that if  $z_l < z_m$  then  $z_l(t) < z_m(t)$  for all  $t \in [0, 1]$ .

In the case  $l < m \leq k$  we simply have the inequality  $z_l(t) < z_m(t)$ . Also it is not hard to see that in the case  $k < l < m$  the inequality  $z_l(t) \leq z_m(t)$  (resp.  $z_l(t) < z_m(t)$ ) follows from  $z_l \leq z_m$  (resp.  $z_l < z_m$ ).

Thus it remains to consider the case  $l = k$  and  $m = k + 1$ .

In this case we have  $z_k(t) \equiv z_k = k$ . Furthermore, due to (8.16) we have  $z_{k+1} \geq k$  and hence

$$z_{k+1}(t) = (1-t)z_{k+1} + t(k+1) \geq (1-t)k + tk = k.$$

It is also obvious that if  $z_{k+1} > k$  then

$$z_{k+1}(t) = (1-t)z_{k+1} + t(k+1) > (1-t)k + tk = k = z_k(t)$$

for all  $t \in [0, 1]$ .

Finally, it is clear that if  $z_{k+1} = k + 1$  then

$$z_{k+1}(t) \equiv k + 1.$$

Thus  $g$  (8.17) is indeed a deformation retraction of  $G_k$  onto  $G_{k+1}$ .

Since  $G_{|T|}$  is a one-point space we conclude that  $F_{|S|}$ , and hence, the space  $\Phi_\sigma(\mathbf{t})$  is contractible.

The proof of the fact that  $\Phi_\sigma$  is a cofibrant object in the category of functors from  $\mathcal{J}(\sigma)$  to **Top** is very similar to the proof of Theorem 7.2 from [2].

Following the arguments of [2] we define the following sequence of functors

$$\Phi_\sigma^m, \quad m \in \mathbb{Z}, \quad m \geq 0.$$

On the level of objects the functor  $\Phi_\sigma^m$  operates as

$$(8.19) \quad \Phi_\sigma^m(\mathbf{t}) = \begin{cases} \Phi_\sigma(\mathbf{t}), & \text{if } |S| + |T| < m, \text{ and } \mathbf{t} \text{ is a 2-tree,} \\ \Phi_\sigma(\mathbf{t}), & \text{if } |S| + |T| - 1 < m, \text{ and } \mathbf{t} \text{ is an SC 2-tree,} \\ \emptyset, & \text{otherwise.} \end{cases}$$

We would like to remark that the number  $|S| + |T|$  (resp.  $|S| + |T| - 1$ ) for a 2-tree  $\mathbf{t} : S \rightarrow T$  (resp. for an SC 2-tree  $\mathbf{t} : S \rightarrow T$ ) is the dimension of the Fox-Neuwirth cell  $\mathbf{FN}_\mathbf{t}$ . Thus the collection  $\Phi_\sigma^m$  may be considered as a filtration of  $\Phi_\sigma$  by dimension.

We have the obvious sequence of natural transformations

$$\Phi_\sigma^0 \rightarrow \Phi_\sigma^1 \rightarrow \Phi_\sigma^2 \rightarrow \dots$$

and the functor  $\Phi_\sigma$  is the sequential colimit

$$\Phi_\sigma = \mathbf{colim}_m \Phi_\sigma^m.$$

Similarly to the proof of Theorem 7.2 from [2] we show that for every  $m$  the natural transformation

$$\Phi_\sigma^m \rightarrow \Phi_\sigma^{m+1}$$

is a cellular extension generated by a cofibration.

Thus  $\Phi_\sigma$  is indeed a cofibrant object in the category of functors from  $\mathcal{J}(\sigma)$  to the category of topological spaces.

This completes the proof of Proposition 8.3. □

Now that we have a cofibrant resolution  $\Phi_\sigma$  of the trivial functor from  $\mathcal{J}(\sigma)$  to **Top** we prove Proposition 8.1 by showing that the space

$$(8.20) \quad X_\sigma = \mathbf{colim}_{\mathcal{J}(\sigma)} \Phi_\sigma$$

is contractible for every surjection  $\sigma \in D(S, N)$ , where  $S$  is an (SC) set.

It is easy to see that

$$(8.21) \quad X_\sigma = \bigcup_{\mathbf{t} \in \mathcal{J}(\sigma)} \mathbf{FN}_{\mathbf{t}}.$$

To get a more explicit description of the space  $X_\sigma$  (8.21) we recall that the set  $D(S, N)$  consists of surjections

$$\sigma : \{1, 2, 3, \dots, |S| + N\} \rightarrow S$$

from the standard ordinal  $\{1, 2, 3, \dots, |S| + N\}$  to the set  $S$ ; the surjections  $\sigma$  should satisfy two conditions **I** and **II** from the proof of Lemma 7.6; if, in addition  $c_S = \mathbf{a}$ , then we should also impose on  $\sigma$  condition **III** from the proof of the same lemma.

Let us also recall that a 2-tree (an SC 2-tree)  $\mathbf{t} : S \rightarrow T$  belongs to  $\mathcal{J}(\sigma)$  iff the following conditions are met:

— if for  $s \neq \tilde{s}$  there exist  $i_1, i_2 \in \sigma^{-1}(\tilde{s})$  and  $i \in \sigma^{-1}(s)$  such that  $i_1 < i < i_2$  then  $\mathbf{t}(s) < \mathbf{t}(\tilde{s})$  in the (SC) ordinal  $T$ ,

— if  $\mathbf{t}(s) = \mathbf{t}(\tilde{s})$  and  $s <_{\mathbf{t}} \tilde{s}$  then all elements of  $\sigma^{-1}(s)$  are smaller than all elements of  $\sigma^{-1}(\tilde{s})$ .

Here  $<_{\mathbf{t}}$  is the total order on  $S$  coming from the structure of the (SC) 2-tree  $\mathbf{t}$ .

Thus the space  $X_\sigma$  (8.21) consists of the configurations  $\{(x_s, y_s)\}$  from **Conf**( $S$ ) satisfying the following conditions:

**C1** if  $\exists i_1, i_2 \in \sigma^{-1}(\tilde{s})$  and  $i \in \sigma^{-1}(s)$  such that  $i_1 < i < i_2$  then  $x_s < x_{\tilde{s}}$

**C2** if  $x_s = x_{\tilde{s}}$  and all elements of  $\sigma^{-1}(s)$  are smaller than all elements of  $\sigma^{-1}(\tilde{s})$  then  $y_s < y_{\tilde{s}}$ .

If  $c_S = \mathbf{a}$  then we have to impose on the configuration  $\{(x_s, y_s)\}$  the additional condition

**C3** if  $\chi(s) = \mathbf{a}$  then  $x_s = 0$  and if  $\chi(s) = \mathbf{c}$  then  $x_s > 0$ .

**Remark.** Let  $S$  be a usual (non-SC) set. It can be shown that every surjection  $\sigma \in D(S, N)$  gives us a pair of complementary orders on the set  $S$  in the sense of M. Kontsevich and Y. Soibelman [23]. (See also Section 2 in [2] about complementary orders and higher trees.) To a pair of complementary orders  $>_0$  and  $>_1$  M. Kontsevich and Y. Soibelman assign a subspace  $X_{>_0, >_1}$  [23] of the compactified configuration space of points on  $\mathbb{R}^2$  labeled by elements of  $S$ . Our space  $X_\sigma$  is an uncompactified version of the subspace considered by M. Kontsevich and Y. Soibelman in [23].

8.1.1. *Contractibility of  $X_\sigma$ .* We give a detailed proof of the contractibility of  $X_\sigma$  (8.21) in the SC case when the color  $c_S$  of the SC set  $S$  is  $\mathbf{a}$ . The non SC case ( $c_S = \mathbf{c}$ ) is very similar.

Although every SC 2-tree  $\mathbf{t} \in \mathcal{J}(\sigma)$  gives us a total order  $>_{\mathbf{t}}$  on  $S$ , we equip the set  $S$  with yet another total order which we denote by  $<_\sigma$ . Namely, we set  $s <_\sigma \tilde{s}$  iff

- either  $\exists i_1, i_2 \in \sigma^{-1}(\tilde{s})$  and  $i \in \sigma^{-1}(s)$  such that  $i_1 < i < i_2$  or
- all elements of  $\sigma^{-1}(s)$  are smaller than all elements of  $\sigma^{-1}(\tilde{s})$ .

**Warning.** In general, the order total  $>_{\mathbf{t}}$  on  $S$  coming from the structure of an SC 2-tree  $\mathbf{t} \in \mathcal{J}(\sigma)$  does not coincide with the order  $>_\sigma$ . Thus, in Example 8.2, the map  $\sigma$  induces on the SC set  $S$  the order

$$\alpha < \beta < \gamma < \delta.$$

On the other hand we have a pruned SC 2-tree  $\mathbf{t} : \{\beta < \gamma < \alpha < \delta\} \rightarrow \{1, 2, 3, 4\}$  which belongs to  $\mathcal{J}(\sigma)$ . A similar example can be found for an SC set  $S$  with  $c_S = \mathbf{c}$ .

Using the total order  $>_\sigma$  we identify  $S$  with the standard ordinal  $\{1 < 2 < 3 < \dots < |S|\}$ .

Next we define the following functions on  $\mathbf{Conf}(S)$

$$(8.22) \quad \mu_k(\{(x_s, y_s)\}) = \min(y_k, y_{k+1}, \dots, y_{|S|})$$

which are obviously continuous.

Then we introduce the sequence of subspaces

$$X_\sigma = Y_0 \supset Y_1 \supset \dots \supset Y_{|S|},$$

where  $Y_k$  consists of configurations  $\{(x_s, y_s)\} \in X_\sigma$  satisfying the properties

$$(8.23) \quad y_s = y_1 + s - 1, \quad \forall s \leq k,$$

$$(8.24) \quad \mu_{k+1}(\{(x_s, y_s)\}) = y_k + 1.$$

Let us show that  $Y_{k+1}$  is homotopy equivalent to  $Y_k$  for all  $k < |S|$ .

For this purpose we introduce an intermediate subspace  $Z_k$

$$Y_k \supset Z_k \supset Y_{k+1}.$$

This subspace consists of configurations  $\{(x_s, y_s)\} \in Y_k$  satisfying the property

$$(8.25) \quad y_{k+1} = \mu_{k+1}(\{(x_s, y_s)\}).$$

Let us consider the map  $h : Y_k \times [0, 1] \rightarrow Y_k$

$$(8.26) \quad h(\{(x_s, y_s)\}, t) = \{(x_s, y_s(t))\},$$

where

$$y_s(t) = \begin{cases} y_s, & \text{if } s \neq k+1, \\ (1-t)y_{k+1} + t\mu_{k+1}(\{(x_s, y_s)\}), & \text{if } s = k+1. \end{cases}$$

In order to show that  $h(\{(x_s, y_s)\}, t) \in Y_k$  we only need to check condition **C2** for all  $t \in [0, 1]$ .

It is clear that

$$(8.27) \quad y_{k+1} \geq y_{k+1}(t) \geq \mu_{k+1}(\{(x_s, y_s)\}), \quad \forall t \in [0, 1].$$

Since  $\{(x_s, y_s)\} \in Y_k$  we have

$$\mu_{k+1}(\{(x_s, y_s)\}) > y_s, \quad \forall s \leq k$$

and hence

$$y_{k+1}(t) > y_s, \quad \forall s \leq k, \quad t \in [0, 1].$$

Furthermore, since condition **C2** is satisfied for  $\{(x_s, y_s)\}$  we conclude that all points  $(x_s, y_s)$  with  $s > k + 1$  and  $x_s = x_{k+1}$  lie above the point  $(x_{k+1}, y_{k+1})$ . Combining this observation with inequality (8.27) we conclude that if  $s > k + 1$  and  $x_s = x_{k+1}$  then  $y_{k+1}(t) < y_s$  for all  $t \in [0, 1]$ .

It is clear that  $h(\{(x_s, y_s)\}, 1) \in Z_k$  and for all  $\{(x_s, y_s)\} \in Z_k$  we have

$$h(\{(x_s, y_s)\}, t) = \{(x_s, y_s)\}, \quad \forall t \in [0, 1].$$

Thus  $h$  is a deformation retraction of  $Y_k$  onto  $Z_k$ .

It is clear that the subspace  $Y_{k+1}$  consists of configurations  $\{(x_s, y_s)\} \in Z_k$  satisfying the additional property

$$\mu_{k+2}(\{(x_s, y_s)\}) = y_{k+1} + 1.$$

So we consider the map  $h_Z : Z_k \times [0, 1] \rightarrow Z_k$

$$(8.28) \quad h_Z(\{(x_s, y_s)\}, t) = \{(x_s, y_s(t))\},$$

where

$$y_s(t) = \begin{cases} y_s, & \text{if } s \leq k + 1, \\ y_s + t(y_{k+1} + 1 - \mu_{k+2}(\{(x_s, y_s)\})), & \text{if } s > k + 1. \end{cases}$$

In order to show that  $h_Z$  lands in  $Z_k$  we need to check condition **C2** and condition (8.25).

Since

$$\begin{aligned} & \min(y_{k+2}(t), y_{k+3}(t), \dots, y_{|S|}(t)) = \\ & \min(y_{k+2}, y_{k+3}, \dots, y_{|S|}) + t(y_{k+1} + 1 - \mu_{k+2}(\{(x_s, y_s)\})) = \\ & (1 - t)\mu_{k+2}(\{(x_s, y_s)\}) + t(y_{k+1} + 1) \geq \mu_{k+1}(\{(x_s, y_s)\}) \end{aligned}$$

we conclude that

$$\mu_{k+1}(\{(x_s, y_s(t))\})$$

does not depend on  $t$ . Thus condition (8.25) is satisfied.

Next, if  $s \geq k + 2$  then

$$\begin{aligned} y_s(t) & \geq \mu_{k+2}(\{(x_s, y_s)\}) + t(y_{k+1} + 1 - \mu_{k+2}(\{(x_s, y_s)\})) = \\ & (1 - t)\mu_{k+2}(\{(x_s, y_s)\}) + t(y_{k+1} + 1) > y_{k+1} \end{aligned}$$

for all  $t \in (0, 1]$  because  $\mu_{k+2}(\{(x_s, y_s)\}) \geq y_{k+1}$  and  $y_{k+1} + 1 > y_{k+1}$ . Hence

$$y_s(t) > y_{\tilde{s}}$$

for all  $s \geq k + 2$ ,  $\tilde{s} \leq k + 1$  and  $t \in (0, 1]$ .

Furthermore, if for  $s, \tilde{s} \geq k + 2$  we have  $y_s > y_{\tilde{s}}$  then obviously  $y_s(t) > y_{\tilde{s}}(t)$  for all  $t \in [0, 1]$ . Thus we conclude that condition **C2** is satisfied for every configuration  $h_Z(\{(x_s, y_s)\}, t)$ .

It is not hard to see that for all  $\{(x_s, y_s)\} \in Z_k$

$$h_Z(\{(x_s, y_s)\}, 1) \in Y_{k+1}$$

and for all  $t \in [0, 1]$  and  $\{(x_s, y_s)\} \in Y_{k+1}$

$$h_Z(\{(x_s, y_s)\}, t) = \{(x_s, y_s)\}.$$

Thus  $h_Z$  is a deformation retraction of  $Z_k$  onto  $Y_{k+1}$ .

We proved that  $X_\sigma$  is homotopy equivalent to the subspace  $Y_{|S|}$  which consists of configurations  $\{(x_s, y_s)\} \in X_\sigma$  satisfying the property

$$(8.29) \quad y_s = y_1 + s - 1, \quad \forall s \in S.$$

To show that  $Y_{|S|}$  is contractible we set, as above,  $S_{\mathbf{a}} = \chi^{-1}(\mathbf{a})$  and  $S_{\mathbf{c}} = \chi^{-1}(\mathbf{c})$ .

Due to Condition **C3**  $x_s = 0$  for all  $s \in S_{\mathbf{a}}$  and  $x_s > 0$  for all  $s \in S_{\mathbf{c}}$ .

Restricting the total order  $>_\sigma$  from  $S$  to  $S_c$  we get an isomorphism

$$\beta : S_c \rightarrow \{1 < 2 < 3 < \dots < |S_c|\}$$

from  $S_c$  to the standard ordinal  $\{1 < 2 < 3 < \dots < |S_c|\}$ .

Using this isomorphism we define the following map  $H : Y_{|S|} \times [0, 1] \rightarrow Y_{|S|}$

$$(8.30) \quad H(\{(x_s, y_s)\}, t) = \{(x_s(t), y_s)\},$$

where

$$x_s(t) = \begin{cases} 0, & \text{if } s \in S_a, \\ (1-t)x_s + t\beta(s), & \text{if } s \in S_c. \end{cases}$$

Let us show that  $H$  indeed lands in  $X_\sigma$ .

Since  $x_s(t) > 0$  for all  $s \in S_c$  and  $t \in [0, 1]$  we need to check Condition **C1** only for  $s, \tilde{s} \in S_c$ .

If  $s, \tilde{s} \in S_c$ ,  $s \neq \tilde{s}$  and there exists  $i_1, i_2 \in \sigma^{-1}(\tilde{s})$  and  $i \in \sigma^{-1}(s)$  such that  $i_1 < i < i_2$  then  $x_s < x_{\tilde{s}}$  and  $\beta(s) < \beta(\tilde{s})$  according to the definition of the total order  $<_\sigma$  on  $S$ . Hence

$$(1-t)x_s + t\beta(s) < (1-t)x_{\tilde{s}} + t\beta(\tilde{s}), \quad \forall t \in [0, 1].$$

Condition **C2** is satisfied automatically because for every configuration in  $Y_{|S|}$  we have (8.29).

Condition **C3** is also obviously satisfied.

It also follows from the construction that

$$H(\{(x_s, y_s)\}, t) \in Y_{|S|}$$

for all  $\{(x_s, y_s)\} \in Y_{|S|}$  and  $t \in [0, 1]$ .

Furthermore, it is clear that  $H$  is a deformation retraction of  $Y_{|S|}$  onto the subspace  $L$  of configurations  $\{(x_s, y_s)\} \in X_\sigma$  with

$$\begin{aligned} y_s &= y_1 + s - 1, & \forall s \in S, \\ x_s &= 0, & \forall s \in S_a, \end{aligned}$$

and

$$x_s = \beta(s), \quad \forall s \in S_c.$$

The subspace  $L$  is obviously homeomorphic to the real line  $\mathbb{R}$ .

Thus we conclude that  $Y_{|S|}$  and hence  $X_\sigma$  is contractible.

This completes the proof of Proposition 8.1 and hence the proof of Theorem 7.9.

**Example 8.4.** Let us illustrate the proof of contractibility for  $X_\sigma$  with the map

$$\sigma : \{1, 2, 3, 4, 5, 6\} \rightarrow \{\alpha, \beta, \gamma, \delta\}$$

from Example 8.2. Recall that  $c_S = \mathbf{a}$ ,  $\chi(\alpha) = \chi(\gamma) = \chi(\delta) = \mathbf{c}$ , and  $\chi(\beta) = \mathbf{a}$ .

The space  $X_\sigma$  consists of configurations from  $\mathbf{Conf}(\{\alpha, \beta, \gamma, \delta\})$  satisfying the following conditions:

- i)  $x_\beta = 0 < x_\gamma < x_\delta$ ,
- ii)  $x_\alpha > 0$ ,
- iii) if  $x_\alpha = x_\gamma$  then  $y_\alpha < y_\gamma$ ,
- iv) if  $x_\alpha = x_\delta$  then  $y_\alpha < y_\delta$ .

In the first step of the above proof we retract  $X_\sigma$  onto the subspace  $Z_0$  of configurations satisfying the property

$$y_\alpha = \min(y_\alpha, y_\beta, y_\gamma, y_\delta).$$

Second, we retract  $Z_0$  to the subspace  $Y_1$  of configurations satisfying in addition the property

$$\min(y_\beta, y_\gamma, y_\delta) = y_\alpha + 1.$$

Next, we retract  $Y_1$  to the subspace  $Z_1$  which consists of configurations  $\{(x_s, y_s)\} \in Y_1$  with

$$y_\beta = \min(y_\beta, y_\gamma, y_\delta).$$

We keep doing so until we get the subspace  $Y_4$  of configurations  $\{(x_s, y_s)\} \in X_\sigma$  with

$$(8.31) \quad y_\delta = y_\gamma + 1 = y_\beta + 2 = y_\alpha + 3.$$

Then we retract the resulting space  $Y_4$  to the subspace  $L$  of configurations  $\{(x_s, y_s)\} \in X_\sigma$  satisfying (8.31) and

$$x_\alpha = 1, \quad x_\beta = 0, \quad x_\gamma = 2, \quad x_\delta = 3.$$

Performing the latter retraction we may need to move horizontally the point labeled by  $\alpha$  through the vertical lines containing the points labeled by  $\gamma$  and  $\delta$ . In doing so we will not violate conditions *iii*) and *iv*) because the inequalities  $y_\alpha < y_\gamma$  and  $y_\alpha < y_\delta$  are already achieved at the previous steps.

The subspace  $L$  is obviously homeomorphic to the real line. Thus contractibility of  $X_\sigma$  follows.

## 9. PROOF OF THEOREM 2.1

Let us return to the dg (SC) 2-operad  $\mathbf{br}$  introduced in Definition 7.3 and show that

**Proposition 9.1.** *For every pruned 2-tree (pruned SC 2-tree)  $\mathbf{t}$*

- 1) *the cochain complex  $\mathbf{br}(\mathbf{t})$  is contractible;*
- 2) *there exist natural identifications*

$$H^0(\mathbf{br}(\mathbf{t})) = \mathbf{k}$$

*under which all operadic composition maps of the operad  $H^\bullet(\mathbf{br})$  evaluated on  $1 \in \mathbf{k}$  produce  $1 \in \mathbf{k}$ .*

*Proof.* Due to Lemma 7.2 the inclusion

$$\mathbf{br} \hookrightarrow |\mathbf{seq}|^{\mathbf{SC}}$$

is a quasi-isomorphism of dg SC 2-operads. We start with the non-SC case. We have to show that for every pruned 2-tree  $\mathbf{t} : S \rightarrow T$ , the cochain complex

$$|\mathbf{seq}|(\mathbf{t})$$

is contractible. This was proved in Proposition 6.4 in [32]. For the convenience of the reader we briefly recall the argument.

By definition,  $|\mathbf{seq}|(\mathbf{t})$  is the realization of the cosimplicial/polysimplicial set (see Section 6)

$$(9.1) \quad \{\{I_s\}_{s \in S}; J\} \rightarrow \mathbf{seq}(\mathbf{t})_{\{I_s\}_{s \in S}}^J$$

in the category of cochain complexes.

Thus we need to show that realizing (9.1) in the category of topological spaces we get a contractible space.

For this purpose we fix the ordinal  $J$  and consider the corresponding polysimplicial set

$$(9.2) \quad \{\{I_s\}_{s \in S}\} \rightarrow \mathbf{seq}(\mathbf{t})_{\{I_s\}_{s \in S}}^J.$$

It is shown in [32] that for every (non-empty) ordinal  $J$

$$(9.3) \quad |\mathbf{seq}(\mathbf{t})_{\bullet, \dots, \bullet}^J|_{\mathbf{top}} \cong |\mathbf{seq}(\mathbf{t})_{\bullet, \dots, \bullet}^{[0]}|_{\mathbf{top}} \times \Delta^J$$

and moreover the collection of homeomorphisms (9.3) gives an isomorphism of the corresponding cosimplicial topological spaces. Here  $[0]$  is the one element ordinal.

Thus, in order to prove contractibility of the realization of (9.1) we need to prove contractibility of the topological space

$$(9.4) \quad |\mathbf{seq}(\mathbf{t})_{\bullet, \dots, \bullet}^{[0]}|_{\mathbf{top}}.$$

This space admits the following explicit description. A point of  $|\mathbf{seq}(\mathbf{t})_{\bullet, \dots, \bullet}^{[0]}|_{\mathbf{top}}$  is given by an equivalence class of decompositions of the segment  $[0, |S|]$  into a number of subsegments labeled by elements of  $S$ . The labeling should satisfy the following conditions:

ℵ1) if  $s_1, s_2 \in S$  and a segment labeled by  $s_2$  lies between segments labeled by  $s_1$  then  $\mathbf{t}(s_1) > \mathbf{t}(s_2)$  in  $T$ ,

ℵ2) if for  $s_1, s_2 \in S$  we have  $\mathbf{t}(s_1) = \mathbf{t}(s_2)$  and  $s_1 < s_2$  then all segments labeled by  $s_1$  are on the left-hand side of all segments labeled by  $s_2$ ,

ℵ3) for every  $s \in S$  the total length of all segments labeled by  $s$  is 1.

Two such decompositions are equivalent if one is obtained from the other by a number of operations of the following two types:

a) adding into or deleting from our decomposition a number of labeled segments of length 0,

b) joining two neighboring segments of our decomposition labeled by an element  $s \in S$  into one segment labeled by  $s$ , or the inverse operation.

In [32] it was proved, by induction on  $|T|$ , that the space (9.4) is a product of simplices and hence (9.4) is contractible. Thus we deduce that so is the cochain complex  $|\mathbf{seq}|(\mathbf{t})$ .

We now pass to the SC-case. Let  $\mathbf{t} : S \rightarrow T$  be a pruned SC 2-tree with  $S = S_a \sqcup S_c$ , where  $S_a$  is the preimage of the minimal element of  $T$  and  $S_c = S \setminus S_a$ . The subset  $S_a$  may, in principle, be empty.

Recall that  $|\mathbf{seq}^{\mathbf{SC}}(\mathbf{t})|$  is the realization of the polysimplicial set

$$(9.5) \quad \{\{I_s\}_{s \in S_c}\} \rightarrow \mathbf{seq}^{\mathbf{SC}}(\mathbf{t})_{\{I_s\}_{s \in S_c}}$$

in the category of cochain complexes.

Each element  $u$  of  $\mathbf{seq}^{\mathbf{SC}}(\mathbf{t})_{\{I_s\}_{s \in S_c}}$  is a total order  $>_u$  on

$$\mathcal{I} = \bigsqcup_{s \in S_c} I_s \sqcup S_a$$

satisfying the following conditions:

— it agrees with the total order on each  $I_s$  and with the total order on  $S_a$ ,

— if  $i, k \in I_{s_1}$ ,  $j \in I_{s_2}$ ,  $s_1 \neq s_2$  and  $i <_u j <_u k$ , then  $\mathbf{t}(s_2) < \mathbf{t}(s_1)$ ,

— if  $s_1, s_2 \in S_c$ ,  $s_1 < s_2$ , and  $\mathbf{t}(s_1) = \mathbf{t}(s_2)$ , then all elements of  $I_{s_1}$  are strictly smaller than all elements of  $I_{s_2}$ .

As well as the space (9.4) the realization  $|\mathbf{seq}^{\mathbf{SC}}(\mathbf{t})|_{\mathbf{top}}$  of (9.5) has the following explicit description. A point of  $|\mathbf{seq}^{\mathbf{SC}}(\mathbf{t})|_{\mathbf{top}}$  is given by an equivalence class of decompositions of the segment  $[0, |S|]$  into a number of subsegments labeled by elements of  $S$ . The labeling should satisfy the following conditions:

ℵ0') for each  $s \in S_a$  there is exactly one segment labeled by  $s$  and its length is 1; if for  $s_1, s_2 \in S_a$  we have  $s_1 < s_2$  then the segment labeled by  $s_1$  is on the left-hand side of the segment labeled by  $s_2$ .

ℵ1') if  $s_1, s_2 \in S_c$  and a segment labeled by  $s_2$  lies between segments labeled by  $s_1$  then  $\mathbf{t}(s_1) > \mathbf{t}(s_2)$ ,

ℵ2') if for  $s_1, s_2 \in S_c$  we have  $\mathbf{t}(s_1) = \mathbf{t}(s_2)$  and  $s_1 < s_2$  then all segments labeled by  $s_1$  are on the left-hand side of all segments labeled by  $s_2$ ,

ℵ3') for every  $s \in S_c$  the total length of all segments labeled by  $s$  is 1.

Two such decompositions are equivalent if one is obtained from the other by a number of operations of the following two types:

- a) adding into or deleting from our decomposition a number of labeled segments of length 0,
- b) joining two neighboring segments of our decomposition labeled by an element  $s \in S_c$  into one segment labeled by  $s$ , or the inverse operation.

If we remove all elements of  $S_a$  from  $S$  and the minimal element  $t_{min}$  from  $T$  then we get a usual pruned (non-SC) 2-tree

$$(9.6) \quad \tilde{\mathbf{t}} = \mathbf{t} \Big|_{S_c} : S_c \rightarrow T \setminus \{t_{min}\}.$$

To this 2-tree we assign the following polysimplicial set

$$(9.7) \quad \{\{I_s\}_{s \in S_c}\} \rightarrow \mathbf{seq}(\tilde{\mathbf{t}})_{\{I_s\}_{s \in S_c}}^{[0]}$$

and the corresponding topological space

$$(9.8) \quad |\mathbf{seq}(\tilde{\mathbf{t}})_{\bullet, \dots, \bullet}^{[0]}|_{\mathbf{top}}$$

which was explicitly described above. (The space (9.8) is obtained from the space (9.4) via replacing  $\mathbf{t}$  by  $\tilde{\mathbf{t}}$ .)

We have the obvious projection

$$P : |\mathbf{seq}(\mathbf{t})|_{\mathbf{top}}^{\mathbf{SC}} \rightarrow |\mathbf{seq}(\tilde{\mathbf{t}})_{\bullet, \dots, \bullet}^{[0]}|_{\mathbf{top}}$$

which sends a point of  $|\mathbf{seq}(\mathbf{t})|_{\mathbf{top}}^{\mathbf{SC}}$  to a point of  $|\mathbf{seq}(\tilde{\mathbf{t}})_{\bullet, \dots, \bullet}^{[0]}|_{\mathbf{top}}$  by collapsing each segment labeled by an element of  $S_a$  to a point.

Conversely, given:

- i) a point  $x \in |\mathbf{seq}(\tilde{\mathbf{t}})_{\bullet, \dots, \bullet}^{[0]}|_{\mathbf{top}}$ , and
- ii) a monotonous map  $U : S_a \rightarrow [0, |S_c|]$

one can reconstruct a point in  $|\mathbf{seq}(\mathbf{t})|_{\mathbf{top}}$  by inserting unit segments labeled by  $s \in S_a$  in the place of the point  $U(s)$ .

Thus we conclude that

$$|\mathbf{seq}(\mathbf{t})|_{\mathbf{top}}^{\mathbf{SC}} \cong |\mathbf{seq}(\tilde{\mathbf{t}})_{\bullet, \dots, \bullet}^{[0]}|_{\mathbf{top}} \times \Delta^{|S_a|}.$$

Due to Proposition 6.4 from [32] the first component  $|\mathbf{seq}(\tilde{\mathbf{t}})_{\bullet, \dots, \bullet}^{[0]}|_{\mathbf{top}}$  is contractible. Hence so is  $|\mathbf{seq}(\mathbf{t})|_{\mathbf{top}}^{\mathbf{SC}}$ .

Thus we proved that  $|\mathbf{seq}(\mathbf{t})|_{\mathbf{top}}^{\mathbf{SC}}$  is contractible for every pruned SC 2-tree  $\mathbf{t}$ .

The identifications from Part 2) of this proposition come from the fact that the topological spaces

$$|\mathbf{seq}(\mathbf{t})_{\bullet, \dots, \bullet}^{\mathbf{SC}}|_{\mathbf{top}}$$

for pruned 2-trees  $\mathbf{t}$  and

$$|\mathbf{seq}(\mathbf{t})_{\bullet, \dots, \bullet}^{\mathbf{SC}}|_{\mathbf{top}}$$

for pruned SC 2-trees  $\mathbf{t}$  are contractible. These topological realizations inherit the operadic compositions, whence Part 2) of this proposition.  $\square$

Proposition 9.1 implies that the cofibrant resolution  $\mathcal{R}\mathbf{br}$  of  $\mathbf{br}$  is also a cofibrant resolution of the trivial (SC) 2-operad  $\mathbf{triv}$  in the category of reduced (SC) 2-operads over cochain complexes.

Therefore, due to Batanin's theorem (Theorem 5.2) the symmetrization  $\mathbf{sym} \mathcal{R}\mathbf{br}$  of  $\mathcal{R}\mathbf{br}$  is quasi-isomorphic to the singular chain operad of Voronov's Swiss Cheese operad  $\mathbf{SC}_2$  (in particular, the non-SC part of  $\mathbf{sym} \mathcal{R}\mathbf{br}$  is quasi-isomorphic to the singular chain operad of the little disc operad (Theorem 4.3)).



Due to Theorem 7.9 the SC operad  $\mathbf{sym} \mathcal{R}\mathbf{br}$  is quasi-isomorphic to  $\mathbf{braces}$  which is, in turn, quasi-isomorphic to the SC operad  $|\mathbf{s}\mathcal{O}|$  by Lemma 7.6.

Finally, by construction the SC operad  $|\mathbf{s}\mathcal{O}|$  is isomorphic to the operad  $|\mathcal{O}|$ .

Thus we conclude that the two-colored operad  $|\mathcal{O}|$  is quasi-isomorphic to the singular chain operad of Voronov's Swiss Cheese operad  $\mathrm{SC}_2$  (and the non-SC part of  $|\mathcal{O}|$  is quasi-isomorphic to the singular chain operad of the little disc operad).

It remains to show that the induced action of  $H_{-\bullet}(\mathrm{SC}_2)$  on the pair  $(HH^\bullet(A, A), A)$  coincides with the one given in Proposition 1.1. For this purpose we present operations on the pair

$$(9.9) \quad (C^\bullet(A, A), A)$$

which come from the action of  $|\mathcal{O}|$  and which induce on  $(HH^\bullet(A, A), A)$  the  $H_{-\bullet}(\mathrm{SC}_2)$ -algebra structure from Proposition 1.1.

These operations are the cup-product and the Gerstenhaber bracket [14] on  $C^\bullet(A, A)$ , the associative product on  $A$ , and the following contraction of a cochain  $P$  with elements of the algebra  $A$ :

$$(9.10) \quad i(P, a) = aP(1, 1, \dots, 1) : C^\bullet(A, A) \otimes A \rightarrow A.$$

We would like to remark that since  $C^\bullet(A, A)$  is the normalized Hochschild complex only degree zero cochains contribute to the contraction.

These operations induce the desired  $H_{-\bullet}(\mathrm{SC}_2)$ -algebra structure on  $(HH^\bullet(A, A), A)$  and they obviously come from the action of the SC operad  $|\mathcal{O}|$  on the pair (9.9).

Since the cohomology operad  $H^\bullet(|\mathcal{O}|)$  of  $|\mathcal{O}|$  is isomorphic to  $H_{-\bullet}(\mathrm{SC}_2)$  we conclude that the action of  $|\mathcal{O}|$  on (9.9) induce the desired  $H_{-\bullet}(\mathrm{SC}_2)$ -algebra structure on  $(HH^\bullet(A, A), A)$ .

Theorem 2.1 is proved.  $\square$

#### APPENDIX

Let  $[n]$  be the standard ordinal  $\{0, 1, 2, \dots, n\}$ .

Given a collection of  $k$  ordinals  $[n_1], [n_2], \dots, [n_k]$  we consider the following ordinal

$$(9.11) \quad \mathcal{I}_{n_1, \dots, n_k} = [n_1] \sqcup [n_2] \sqcup \dots \sqcup [n_k],$$

where the order is defined by the following rule: for  $i_1 \in [n_{l_1}]$  and  $i_2 \in [n_{l_2}]$   $i_1 < i_2$  if

- $l_1 < l_2$  or
- $l_1 = l_2$  and  $i_1 < i_2$  in  $[n_{l_1}]$ .

Given ordinals  $J, [n_1], [n_2], \dots, [n_k]$  the collection

$$(9.12) \quad (\Xi_k)_{n_1, \dots, n_k}^J = \mathrm{hom}_\Delta(\mathcal{I}_{n_1, \dots, n_k}, J)$$

form a polysimplicial/cosimplicial set. Indeed  $(\Xi_k)_{n_1, \dots, n_k}^J$  is simplicial in  $[n_1], [n_2], \dots, [n_k]$  and cosimplicial in  $J$ .

In this appendix we show that

**Proposition 9.2.** *The cochain complex  $|\Xi_k|$  is concentrated in nonnegative degrees. Furthermore,*

$$(9.13) \quad H^\bullet(|\Xi_k|) = \begin{cases} \mathbf{k}, & \text{if } \bullet = 0, \\ 0, & \text{otherwise.} \end{cases}$$

**Proof.** The first statement is very easy. Indeed, an element  $v \in \mathrm{hom}_\Delta(\mathcal{I}_{n_1, \dots, n_k}, J)$  will not contribute to the realization if it is degenerate. It is clear that if

$$|J| < \sum_{i=1}^k (n_i + 1) - k + 1$$

then  $v$  is degenerate. Therefore, elements  $v \in \text{hom}_\Delta(\mathcal{I}_{n_1, \dots, n_k}, J)$  with

$$|J| - 1 - \sum_{i=1}^k n_i < 0$$

will not contribute to the realization. Hence the cochain complex  $|\Xi_k|$  is indeed concentrated in nonnegative degrees.

The cochain complex  $|\Xi_k|$  can be considered as bicomplex

$$(9.14) \quad |\Xi_k| = |\Xi_k|^{\bullet, \bullet}.$$

The first degree is the total degree of the simplicial indices. According to our conventions this degree is nonpositive. The second degree is the degree in the cosimplicial index and this degree is nonnegative. Let us denote by  $\partial^s$  the part of the differential in  $|\Xi_k|$  which comes from the simplicial indices and by  $\partial^c$  the part of the differential in  $|\Xi_k|$  coming from the cosimplicial structure.

Fixing the second degree we get the cochain complex

$$(9.15) \quad |\Xi_k|^{\bullet, m}$$

which is the realization of the polysimplicial set

$$(9.16) \quad ([n_1], [n_2], \dots, [n_k]) \rightarrow \text{hom}_\Delta(\mathcal{I}_{n_1, n_2, \dots, n_k}, [m]).$$

It is not hard to see that the realization of (9.16) in the category of topological spaces is the following stretched  $m$ -simplex:

$$\{(x_0, x_1, \dots, x_m) \mid x_i \geq 0, \quad x_0 + x_1 + x_2 + \dots + x_m = k\}.$$

Therefore for each  $m$  the complex  $|\Xi_k|^{\bullet, m}$  has non-trivial cohomology only in degree 0 and

$$(9.17) \quad H^0(|\Xi_k|^{\bullet, m}) = \mathbf{k}.$$

The class which generates  $H^0(|\Xi_k|^{\bullet, m})$  is represented by the map

$$(9.18) \quad c \in \text{hom}_\Delta(\mathcal{I}_{0, \dots, 0}, [m]),$$

which sends all elements of  $\mathcal{I}_{0, \dots, 0}$  to the same element  $0 \in [m]$ . All other maps in  $\text{hom}_\Delta(\mathcal{I}_{0, \dots, 0}, [m])$  are cohomologous to the cocycle (9.18).

It is not hard to see that

$$(9.19) \quad \Theta = \bigoplus_{q < 0} |\Xi_k|^{q, \bullet} \oplus \partial^s(|\Xi_k|^{-1, \bullet})$$

is a subcomplex of the bicomplex  $|\Xi_k|$ .

Equation (9.17) implies that each term of the quotient complex  $|\Xi_k|/\Theta$  is  $\mathbf{k}$ . Using the explicit cocycle (9.18) it is not hard to see that the quotient complex  $|\Xi_k|/\Theta$  is

$$\mathbf{k} \xrightarrow{0} \mathbf{k} \xrightarrow{\text{id}} \mathbf{k} \xrightarrow{0} \mathbf{k} \xrightarrow{\text{id}} \mathbf{k} \xrightarrow{0} \dots$$

and hence

$$(9.20) \quad H^\bullet(|\Xi_k|/\Theta) = \begin{cases} \mathbf{k}, & \text{if } \bullet = 0, \\ 0, & \text{otherwise.} \end{cases}$$

We see from the construction that the bicomplex  $\Theta$  (9.19) is acyclic in the first degree. Therefore  $\Theta$  is acyclic as the total complex.

Thus  $H^\bullet(|\Xi_k|) = H^\bullet(|\Xi_k|/\Theta)$  and the proposition follows.  $\square$

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