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# Cyclic Formality and Index Theorems

*In memory of Moshé Flato*

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**Abstract.** The Letter announces the following results (the proofs will appear elsewhere). An operad acting on Hochschild chains and cochains of an associative algebra is constructed. This operad is formal. In the case when this algebra is the algebra of smooth function on a smooth manifold, the action of this operad on the corresponding Hochschild chains and cochains is formal. The induced map on the (periodic) cyclic homology is given by the formula involving the $\hat{A}$-genus. The index theorem for degenerate Poisson structures follows from the latter fact.

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## 1. Introduction

This letter is a written version of a talk given by the first author at the Euroconference 2000 Moshé Flato at Dijon. The first part is devoted to the problem of generalization of the notions of a differential form and a polyvector field for an arbitrary noncommutative algebra. It is well known that the noncommutative analogue of differential forms is the Hochschild chain complex, and the analogue of the polyvector fields is the Hochschild cochain complex. The question is how to define the standard differential geometric operations on forms and polyvector fields in terms of Hochschild (co)chains. This topic originates from [1, 7, 13] and is called the noncommutative differential calculus. Here we give a (hopefully) ultimate answer to this question. It provides us with a description of a certain algebraic structure on the Hochschild chains and cochains of an associative algebra. The proofs will be given in a subsequent paper. The formality of the Lie bracket part of this structure has been proven in [18].

In the case $A = C^\infty(M)$, where $M$ is a smooth manifold, this structure is formal in homological sense (we discuss the meaning of this formality below). For an

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interesting geometric variant of this problem, see [19]. In the second part of the Letter, we discuss an application of these results to index theory: the algebraic index theorem [2, 12] is generalized to all deformations of the associative algebra $C^\infty(M)$, not necessarily symplectic ones.

## 2. Noncommutative Differential Calculus

### 2.1. COMMUTATIVE DIFFERENTIAL CALCULUS

Let $M$ be a $C^\infty$-manifold. It possesses the following features:

- Smooth differential forms $\Omega^\bullet(M)$. They will be denoted by lower case Greek letters $\omega, \eta, \ldots$ and their degrees by $|\omega|, |\eta|, \ldots$, respectively.
- Smooth polyvector fields $V^\bullet(M) =_{\mathrm{def}} \Gamma(M, \Lambda^\bullet(TM))$. They will be denoted by capital Roman letters $X, Y, \ldots$ and their degrees will be denoted by $|X|, |Y|, \ldots$, respectively.

There are certain operations on these objects, of which the most important for us are:

- The de Rham differential d: $\Omega^\bullet(M) \to \Omega^{\bullet+1}(M)$;
- The cup (exterior) product of vector fields $V^i(M) \otimes V^j(M) \to V^{i+j}(M)$;
- The Schouten–Nijenhuis bracket $\{,\}$: $V^i(M) \otimes V^j(M) \to V^{i+j-1}(M)$;
- The contraction of a form $\omega$ and a polyvector field $X$ denoted by $i_X\omega \in \Omega^{|\omega|-|X|}(M)$. Note that in this definition the contraction is zero if $|\omega|<|X|$.
- The Lie derivative $L_X\omega = \mathrm{d}i_X\omega + (-1)^{|X|-1} i_X \mathrm{d}\omega$.

These operations satisfy the following properties:

(1) The graded vector space $V^\bullet(M)$ together with the cup product and the Schouten–Nijenhuis bracket is a *Gerstenhaber algebra*, meaning that the cup-product is graded commutative and associative, that the Schouten–Nijenhuis bracket defines a graded Lie algebra structure on the shifted space $V^\bullet(M)$ [1], and that the following graded Leibniz identity holds:

$$\{X, Y \wedge Z\} = \{X, Y\} \wedge Z + (-1)^\varepsilon Y \wedge \{X, Z\},$$

where $\varepsilon = (|X|-1)|Y|$.

(2) $i_X i_Y \omega = i_{X\wedge Y}\omega$;
(3) $L_X i_Y \omega = L_{\{X,Y\}}\omega + (-1)^\varepsilon i_Y L_X \omega$, where $\varepsilon=(|X|-1)|Y|$;
(4) $L_X L_Y \omega - (-1)^{(|X|-1)(|Y|-1)} L_Y L_X \omega = L_{\{X,Y\}}\omega$;
(5) $L_{X\wedge Y}\omega = L_X i_Y \omega + (-1)^{|X|} i_X L_Y \omega$.

A pair of graded vector spaces $V, \Omega$ with the structure specified above will be called a $T$-algebra.

*Remark*. Certainly there are other natural operations on vector fields and differential forms, but they are irrelevant for the sequel.

## 2.2. NONCOMMUTATIVE DIFFERENTIAL CALCULUS

Let $A$ be an arbitrary associative algebra. It is well known that one can define an analogue of differential forms and polyvector fields in this situation, the differential forms being replaced with the Hochschild chain complex $C_\bullet(A, A)$ of $A$, and polyvector fields with the Hochschild cochain complex $C^\bullet(A, A)$. If $A = C^\infty(M)$ and only support preserving Hochschild cochains are taken, the Hochschild (co)homology is isomorphic to the space of differential forms (polyvector fields). This suggests that there may be an analogue of $T$-algebra structure on Hochschild (co)chains.

In [13] it was shown that one can introduce the operations implied by the $T$-algebra structure on Hochschild (co)chains but the identities of $T$-algebra structure are satisfied only up to homotopy. It is natural in such a situation to try to write the so called 'higher homotopies'. Let us briefly recall the meaning of this term. The gist can already be seen from the consideration of the cup product on Hochschild cochains. The cup product $f \cup g \in C^{n+m}(A, A)$ of $f \in C^m(A, A)$ and $g \in C^n(A, A)$ reads as

$$(f \cup g)(a_1, \dots, a_{m+n}) = (-1)^{mn} f(a_1, \dots, a_m) \cdot g(a_{m+1}, \dots, a_{m+n}).$$

This defines an associative but non-commutative product and one can prove that there is no 'natural' formula for an associative *and* commutative product which is homotopy equivalent to this one. The good news is that the commutator $[f, g] = f \cup g - (-1)^{|f||g|} g \cup f$ is homotopy equivalent to zero. This means that there exists a map $\gamma$: $C^\bullet(A, A) \otimes C^\bullet(A, A) \to C^\bullet(A, A)$ of degree $-1$ such that

$$[f, g] = \partial\gamma(f, g) + \gamma(\partial f, g) + (-1)^{|f|}\gamma(f, \partial g),$$

$\partial$ being the differential on the Hochschild cochains. It follows that the symmetrized product $a \circ b = (a \cup b + (-1)^{mn} b \cup a)/2$ is commutative and

$$\begin{aligned} &a \circ (b \circ c) - (a \circ b) \circ c \\ &\quad = \partial m_3(a, b, c) + m_3(\partial a, b, c) + \\ &\qquad + (-1)^{|a|} m_3(a, \partial b, c) + (-1)^{|a|+|b|} m_3(a, b, \partial c). \end{aligned}$$

In other words $\circ$ is associative up to the homotopy $m_3$. Now we can consider the expression

$$\begin{aligned} &\mu_4(a, b, c, d) \\ &\quad = a \circ m_3(b, c, d) - (-1)^{|a|} m_3(a \circ b, c, d) + (-1)^{|a|+|b|} m_3(a, b \circ c, d) - \\ &\qquad - (-1)^{|a|+|b|+|c|} m_3(a, b, c \circ d) + (-1)^{|a|+|b|+|c|+|d|} m_3(a, b, c) \circ d. \end{aligned} \tag{1}$$

It turns out that the differential of $\mu_4$ is equal to 0, therefore it is natural to require that $\mu_4$ is homotopy equivalent to 0 by means of a homotopy $m_4$. Then we can take some other combinations of $\circ$, $m_3$ and $m_4$ differential of which is 0, and to require that these combinations be homotopy equivalent to 0, etc. To perform this procedure we need to know all such combinations, which is a nontrivial problem.

The most convenient way to express this informal definition rigorously is to use operads.

## 2.3. BRIEF REMINDER ON OPERADS

The detailed exposition of the theory of operads can be found in a variety of sources, e.g. [4, 5, 17]. We will recall the definition of the operad in the example of $T$-algebras. A $T$-structure on a pair of graded vector spaces $V$, $\Omega$ is defined as a collection of one unary (the de Rham differential $\Omega \to \Omega$) and many binary operations some of which are maps $V \otimes V \to V$, and the others are maps $V \otimes \Omega \to \Omega$. Taking all possible combinations of these operations we obtain graded vector spaces $T(n)$ describing all possible operations

$$V^{\otimes n} \to V \tag{2}$$

which are compositions of operations involved in the definition of $T$-structure, and $T(n,1)$ describing the operations

$$V^{\otimes n} \otimes \Omega \to \Omega. \tag{3}$$

Two such operations are equivalent if they are equal by virtue of the identities involved in the definition of the $T$-structure. The symmetric group $S_n$ acts naturally in the spaces $T(n)$, $T(n,1)$ by permutations of the arguments of the same type. We have natural composition maps
$\circ_i$: $T(n) \otimes T(m) \to T(n+m-1)$ such that

$$\circ_i(f, g)(a_1, \ldots, a_{n+m-1}) = f(a_1, \ldots, a_{i-1}, g(a_i, \ldots, a_{i+n-1}), \ldots, a_{m+n-1});$$

$\circ_i^1$ : $T(m, 1) \otimes T(n) \to T(n+m-1, 1)$ such that

$$\circ_i^1(f, g)(a_1, \ldots, a_{n+m-1}, \omega) = f(a_1, \ldots, a_{i-1}, g(a_i, \ldots, a_{i+n-1}), \ldots; \omega);$$

and $\circ^2$ : $T(n, 1) \otimes T(m, 1) \to T(n+m, 1)$ such that

$$\circ^2(f, g)(a_1, \ldots, a_{n+m}; \omega) = f(a_1, \ldots, a_n, g(a_{n+1}, \ldots; \omega)).$$

These compositions satisfy obvious 'associativity' laws and are compatible with the action of the symmetric groups.

*A partial colored operad* $\mathscr{T}$ $(PCO)$ is by definition an arbitrary collection of objects $\mathscr{T}(n)$ $\mathscr{T}(n, 1)$ endowed with and $S_n$-action for each $n = 0, 1,$ together with the composition laws $\circ$ subject to the associativity laws and compatible with the action of the symmetric groups. The word partial means that we allow the argument $\omega$ to enter only once and all our operations are as in (2), (3). $\mathscr{T}(n)$, $\mathscr{T}(n,1)$ may be graded vector spaces, complexes, and, more generally objects of any fixed symmetric monoidal category. It is clear that the collection $T(n)$, $T(n,1)$ forms an operad denoted by $T$. The definition of a morphism between two PCO's is straightforward.

Given two complexes $V$, $\Omega$, denote $F_{V,\Omega}(n) = \underline{\mathrm{Hom}}(V^{\otimes n}, V)$ and $F_{V,\Omega}(n,1) = \underline{\mathrm{Hom}}(V^{\otimes n} \otimes \Omega, \Omega)$, where $\underline{\mathrm{Hom}}$ refers to the internal hom. The collection $F_{V,\Omega}$ forms a PCO with the obvious composition maps and $S_n$-action. We can define a $\mathscr{T}$-structure on the pair $V$, $\Omega$ as a morphism of PCO $\mathscr{T} \to F_{V,\Omega}$. In this case we also say that the pair $V$, $\Omega$ is a $\mathscr{T}$-algebra.

The notion of PCO is a modification of the one of usual operad, in particular, the objects $\mathscr{T}(n)$ of a PCO $\mathscr{T}$ form a usual operad.

## 2.4. $T_\infty$-STRUCTURE

The meaning of the procedure of adding higher homotopies is that we want to replace the operad $T$ of graded vector spaces with an operad of complexes $T_\infty$. Since we want all the relations to be satisfied only up to a homotopy, we require that $T_\infty$ be *free*. The condition that all higher homotopies must be added expresses itself as follows: there exists a map of PCO $p: T_\infty \to T$ (graded spaces are viewed as complexes with differential), such that the corresponding maps $T_\infty(n) \to T(n)$; $T_\infty(n,1) \to T(n,1)$ are quasi-isomorphisms of complexes, in other words, $p$ is a quasi-isomorphism of operads. Recall that a map of complexes is called a quasi-isomorphism if it induces an isomorphism of their cohomology.

An operad $T_\infty$ with such properties is known to exist and for the sequel we fix one of them.*

## 2.5. A MODIFICATION

We will use a slightly more general definition than the one stated in the previous section.

DEFINITION 2.1. We say that a pair (V,Ω) has a structure of T-algebra up to higher homotopies (or simply that (V,Ω) is a homotopy T-algebra) if both a chain of quasi-isomorphisms of operads

$$p: T' \to T_1 \leftarrow T_2 \to T_3 \ldots \to T \tag{4}$$

and an action of T′ on (V,Ω) are specified.

*Remark*. Any two objects (such as operads, algebras, etc.) are called quasi-isomorphic if there exists a chain of quasi-isomorphisms connecting them.

The general homotopy theory ([4, 8]) implies that for any PCO $T_\infty$ as above there exists a collection of maps $f': T_\infty \to T'$ and $f_i: T_\infty \to T_i$ such that these maps with the map $p: T_\infty \to T$ added to them commute with all the maps from (4) and are

*For the results in the following section to be true one has to require that $T_\infty$ be cofibrant [8].

quasi-isomorphic. For any other such a collection $f'', f_i'$, there exists a quasi-isomorphism $q: T_\infty \to T_\infty$ such that $f'' = f' \circ q$.

### 2.6. HOMOTOPY EQUIVALENT HOMOTOPY $T$-ALGEBRAS

A morphism of two homotopy $T$-algebras $A$ and $B$ such that $A$ is a $T'$-algebra and $B$ is a $T''$-algebra, $T'$, $T''$ being quasi-isomorphic with $T$, is a collection of a quasi-isomorphism $T' \to T''$ and a map $j$: $A \to B$ compatible with the actions of $T'$, $T''$. If $j$ is a quasi-isomorphism, the corresponding map of algebras is called quasi-isomorphic. Two homotopy $T$-algebras are called quasi-isomorphic if they are connected by a chain of quasi-isomorphisms.

### 2.7. THE MAIN PROBLEM OF NONCOMMUTATIVE DIFFERENTIAL CALCULUS

The problem is, for any associative algebra $A$, to define a $T$-structure up to higher homotopies on the pair $(C^\bullet(A, A), C_\bullet(A, A))$. The existence of such a structure was conjectured in [14].

## 3. Idea of the Proof

For simplicity, consider first Hochschild cochains only and try to understand what is the richest possible algebraic structure on them. It is easier to explain the idea if we replace the associative algebra $A$ with a topological unital monoid $X$. Define $C^k(X, X) = C(X^k, X)$, the space of all continuous maps topologized by the open-compact topology. The collection $C^\bullet(X, X)$ has a natural structure of a co-simplicial set.

For each $k_1, \ldots, k_n, l$ we are going to define the set of all 'natural' operations

$$C^{k_1}(X, X) \times \cdots \times C^{k_n}(X, X) \to C^l(X, X).$$

Any such operation is by definition a composition of elementary operations, which are

- The insertion

$$o_i: C^{k_1}(X, X) \times C^{k_2}(X, X) \to C^{k_1+k_2-1}(X, X) \tag{5}$$

  such that

$$o_i(f, g)(x_1, \ldots, x_{k_1+k_2-1}) = f(x_1, \ldots, x_{i-1}, g(x_i, \ldots, x_{i+k_2-1}), \ldots, x_{k_1+k_2-1});$$

- Let $m \in C^2(X, X)$ be the associative product and $1 \in C^0(X)$ be the unit. Then the other elementary operations are $o_i(m, f), o_i(f, m), o_i(f, 1)$ for all $i$'s for which the insertions are defined.

Any two compositions of elementary operations are called equivalent if they produce the same map (5). The set of all such equivalence classes is denoted by

$F^l_{k_1\dots k_n}$. We have composition maps

$$F^l_{k_1\dots k_n} \times F^{k_l}_{m_1\dots m_p} \to F^l_{k_1\dots m_1\dots m_p\dots k_n}. \tag{6}$$

The collection of sets $F$ is simplicial with respect to all lower indices and co-simplicial with respect to the upper indices. In other words, $F$ has a structure of functor from $(\Delta^{\mathrm{op}})^n \times \Delta \to$ Sets, where an object of $\Delta$ is a set $[n]=\{0, 1, \dots, n\}$ and a morphism is a nondecreasing map between these sets. For $a_i \in \mathrm{Mor}(\Delta^{\mathrm{op}})$; $b \in \mathrm{Mor}(\Delta)$, $h_i \in C_i^{k'}(X, X)$; $g \in C^{l'}(X, X)$; $f \in F^l_{k_l\dots k_n}$, we have

$$[(a_1 \dots a_n, b)f](h_1, \dots, h_n) = b^{\mathrm{op}}[f(a_1^{\mathrm{op}}h_1, \dots, a_n^{\mathrm{op}}h_n)],$$

where $b^{\mathrm{op}} \in \mathrm{Mor}(\Delta^{\mathrm{op}})$ and $a_i \in \mathrm{Mor}(\Delta)$ are the corresponding morphisms and we assume that all the compositions are defined.

The topological realizations $E_n = \mathrm{Tot}(F^{\bullet}_{\bullet\dots\bullet})$ form a topological operad, the composition maps are defined by the composition maps (6). The operad $E$ acts on the realization $Z(X) = \mathrm{Tot}(C^{\bullet}(X, X))$.

For a functor $F$: $\Delta \times (\Delta^{\mathrm{op}})^n \to$ Sets, Tot($F$) means by definition the following. $F$ determines in the obvious way a functor $F'$ from $\Delta$ to the category $nS$ of functors from $(\Delta^{\mathrm{op}})^n$ to the category of sets ($nS$ is also called the category of $n$-simplicial sets). We have the functor of the topological realization $R$: $nS \to$Top: the composition $F'' = R \circ F'$ is a cosimplicial topological space. Tot($F$) is by definition the realization of $F''$.

### 3.1. DESCRIPTION OF $E$

The analysis of the above construction gives the following description of $E$. Let $\mathrm{U}(n) = C(I, I^n)$. $U$ is naturally a usual operad. $E$ can be realized as a sub-operad of U, such that

$$E(n) = \{\phi: I \to I^n; \phi = (\phi_1, \phi_2, \dots, \phi_n)\} \tag{7}$$

and

(1) $\phi_i(0) = 0$; $\phi_i(1) = 1$ for all $i$;
(2) for each pair of indices $p$, $q$ there exist numbers $0 < b < c < 1$ such that

*either*: $\phi_p$ nondecreases and $\phi_q$ is constant on $[o, b] \cup [c, 1]$; $\phi_p$ is constant and $\phi_q$ nondecreases on $[b, c]$

*or*: the same thing with $p$ and $q$ interchanged.

Let $S^1$ be a unit circle $|z| = 1$, $z \in \mathbb{C}$, $e : I \to S^1$ be the map $e(x) = e^{2\pi i x}$. Then any map $\phi : I \to I^n$ satisfying (1) determines by means of $e$ the corresponding map $\phi_S : S^1 \to (S^1)^n$. This way we can realize $E$ as a suboperad of another full operad $V$ such that $V(n) = C(S^1, (S^1)^n)$.

Note that if $\psi \in E(n) \subset V(n)$ and

$$\psi(z) = (\psi_1(z), \dots, \psi_n(z)),$$

then for any $a$ with absolute value 1,

$$\psi_a(z) = (\psi_1(az)/\psi_1(a), \dots, \psi_n(az)/\psi_n(a)) \tag{7}$$

remains in $E(n)$.

*Remark*. There is a sub-operad of this operad each space of which is homeomorphic to a finitely-dimensional piecewise-linear space: take only piecewise-linear maps $\phi$ that are linear on the intervals onto which $I$ is cut by all points $b, c$ for all $p, q$ (see property 2).

### 3.2. RECOGNITION OF $E$

First of all we check that there is a homotopy equivalence $E(k) = K(\mathrm{PB}(n),1)$, where $\mathrm{PB}(n)$ is the pure braid group in $n$ strands. Then using the recognition technique by Fiedorewicz [6], we prove that as an operad $E$ is homotopy equivalent to the little disks operad $E_2$. Thus, we prove the following topological version of the Deligne conjecture: *An operad homotopy equivalent to $E_2$ acts on $Z(X)$.*

For other proofs of this (or a similar) result, see [10, 11, 16].

### 3.3. ADDING $C_\bullet(X, X)$

Similarly, let us define $C_n(X, X) = X^{n+1}$. It is a simplicial topological space. The face maps are

$$d_0(x_0, \dots, x_n) = (x_0x_1, x_2, \dots, x_n), \dots, d_n(x_0, \dots, x_n) = (x_nx_0, \dots, x_{n-1}).$$

The degeneracies are the insertions of 1. Let $\mathrm{O}(X) = \mathrm{Tot}(C_\bullet(X, X))$.

The above results generalize straightforwardly onto this case, and we obtain a PCO $G$ on the pair $Z(X)$, $O(X)$.

### 3.4. DESCRIPTION OF $G$

The PCO $G$ can be realized as a suboperad of the following full PCO $W$. Set $W(n) = C(S^1, (S^1)^n)$; $W(n, 1) = C(S^1, (S^1)^n \times S^1)$ with the natural composition operations. Set $G(n) = E(n) \subset V(n) = W(n)$ (see Section 3.1); set $G(n, 1) \subset W(n, 1)$ to consist of all maps $\theta$ such that:

- $1 \times 1 \cdots \times 1 \times (S^1) \subset \mathrm{Im}\, \theta$;
- for any point $x \in S^1$, the map $\theta_x$ (see (7)) is in $E(n+1) \subset W(n, 1)$.

Note that it suffices to check the second condition just for one point $x$, the validity of this condition for all the other $x$'s will then follow.

### 3.5. HOMOTOPY EQUIVALENT DESCRIPTION

We know that the operad $E$ is homotopy equivalent to the operad of little disks $E_2$. We will extend this equivalence to $F$. We will define a PCO $F'$ which is homotopy equivalent to $F$. $F'(n) = E_2(n)$; $F'(n,1)$ is the configuration space of all cylinders (= surfaces in $\mathbb{R}^2 \times \mathbb{R}^1$ of the form $S^1 \times [a, b]$ with their base being a unit circle) with $n$ non-intersecting circles on the lateral surface, (we use the natural flat metric on it) and with 2 marked points $b$, $t$, $b$ on the bottom base, $t$ on the top base. Two such configurations are equivalent if one can obtain one of them from the other by a parallel shift or rotation.

The insertions of the type $F'(n)\times F'(m)\to F'(n+m-1)$ are performed in the same way as in $E_2$. The insertions $F'(n,1)\times F'(m)$ are just insertions of a compressed big disc from $F'(m) \cong E_2(m)$ into the corresponding 'little' disk on the lateral surface of an element from $F'(n,1)$. Finally, the insertions $F'(n,1)\times F'(m,1)\to F'(n+m,1)$ look like putting the first cylinder under the second one, rotation of the second cylinder so that the marked point on the bottom of the second cylinder coincide with the marked point on the top of the first one, and gluing the two cylinders into one.

M. Kontsevich has informed us that he had independently found the same description.

### 3.6. BACK TO HOCHSCHILD (CO)CHAINS OF AN ALGEBRA

A certain technique allows us to convert this topological result into an algebraic one. Namely, one can prove that the PCO $C_\bullet(F)$ of the singular chains of $F$ acts homotopically on the pair $(C^\bullet(A, A); C_\bullet(A, A))$.

### 3.7. FORMALITY OF $C_\bullet(F)$

Using the same technique as in [15], one proves that the PCO $C_\bullet(F)$ is formal, i.e. is quasi-isomorphic to the operad $H_\bullet(F)$. The latter PCO can be easily identified with $T$. This determines a canonical up to a homotopy $T_\infty$-structure on $(C^\bullet(A, A); C_\bullet(A, A))$.

## 4. Formality of the $T_\infty$-Structure on $(C^\bullet(A, A); C_\bullet(A, A))$, $A = C^\infty(M)$

Using the Gel'fand–Fuks technique we can show that our homotopy $T$-algebra structure is homotopy equivalent to the corresponding structure on the cohomology that is to the natural $T$-algebra structure on the pair $(V^\bullet(M), \Omega^\bullet(M))$.

### 4.1. CYCLIC HOMOLOGY

It is well known that on the Hochschild complex $C_\bullet(A, A)$ with the differential $b$ of degree $+1$ (we assume that $C_i(A, A)$ sits in degree $-i$), there is the Reinhart differential $B$ of degree $-1$ anti-commuting with $b$. In other words, $C_\bullet(A, A)$ is a differential graded module over the skew-commutative algebra $k[B] \cong k \oplus kB$.

This structure allows us to define all kinds of cyclic homology. For example, the usual cyclic homology $HC_\bullet(A, A)$ is isomorphic to the homology of $(k \otimes^L_{k[B]} C_\bullet(A, A))$, where $k$ is the trivial $k[B]$-module. The Bott periodicity map $S : HC_k(A, A) \to HC_{k-2}(A, A)$ is induced by the generator $s$ of the one dimensional space $H^2(R\mathrm{Hom}_{k[B]}(k, k))$. This generator determines a canonical morphisms $S'$: $K[-2] \to K$ in the derived category.

The periodic cyclic homology $HC^{\mathrm{per}}_\bullet(A, A)$ is defined as the homology of the derived projective limit of the sequence

$$K' \stackrel{S'}{\leftarrow} K'[-2] \stackrel{S'}{\leftarrow} \cdots.$$

To compute the limit one can take a complex $\tilde{K}$ and a morphism $\tilde{S}: \tilde{K} \to \tilde{K}$ which represent $K$ and $S'$ and to form a bicomplex

$$0 \to \prod_n \tilde{K}[-2n] \stackrel{\mu}{\to} \prod_n \tilde{K}[-2n] \to 0,$$

where

$$\mu(a_0, a_1, \ldots, a_n, \ldots) = (a_0 - \tilde{S}(a_1), a_1 - \tilde{S}(a_2), \ldots a_n - \tilde{S}(a_{n+1}) \ldots).$$

Note that we can take $\tilde{K} = HC_\bullet(A, A)$ with zero differential and $\tilde{S} = s$.

We have the $T_\infty$-algebra $(C^\bullet(A, A), (C_\bullet(A, A))$. Our task is to define the cyclic homology completely in terms of this $T_\infty$-structure. Let $D = T(0,1)$. The composition law makes it an associative algebra, $C_\bullet(A, A)$ being a module over it. Since $T_\bullet$ is quasi-isomorphic with $T$, $D$ is quasi-isomorphic with $T(0,1) \cong k[B]$. It follows that there exists a $D$-module $L$ quasi-isomorphic with the pair: $k[B]$ and the module $k$ over it. Define $\overline{CC_\bullet}(C_\bullet(A, A)) = L \otimes^L_D C_\bullet(A, A)$. One checks that this complex is canonically quasi-isomorphic to $CC_\bullet(A, A)$. Similarly, we have the elements $s \in R^2\mathrm{Hom}_D(L, L) \cong R^2\mathrm{Hom}_{k[B]}(k, k)$, which allows us to define $HC^{\mathrm{per}}_\bullet(C_\bullet(A, A))$. This homology is canonically isomorphic to $HC^{\mathrm{per}}_\bullet(A, A)$.

The quasi-isomorphism between the $T_\infty$-algebra $(C^\bullet(A, A), C_\bullet(A, A))$ and the $T$-algebra $(V^\bullet(M), \Omega^\bullet(M))$ induces a quasi-isomorphism between $\overline{CC_\bullet}(C_\bullet(A, A))$ and $(k \otimes^L_{k[B]} \Omega^\bullet(M))$, where $B$ acts on $\Omega^\bullet(M)$ as the de Rham differential. Therefore, we have an induced map on the cohomology. Denote this identification by $\nu$. The cohomology of the first complex is canonically identified with the cyclic homology $HC_\bullet(A)$, the cohomology of the second term is identified with

$$H_i = \oplus_{k>0} H^{i-2k}(M) \oplus \Omega^i(M)/d\Omega^{i-1}(M).$$

Similarly, the formality induces an identification $\nu^{\mathrm{per}}: HC^{\mathrm{per}}_i(A) \cong H^{\mathrm{per}}_i(M)$, where $H^{\mathrm{per}}_k(M) = \oplus_k H^{i+2k}_{DR}(M)$.

Compare this with the well-known Kostant–Hochschild–Rosenberg identification (cf. [1]) $\mu\colon HC_\bullet(A) \to H_\bullet$ and $\mu^{\mathrm{per}}\colon HC_\bullet^{\mathrm{per}}(A, A) \to H_\bullet^{\mathrm{per}}(M)$. It turns out that for any $x \in HC_\bullet(A)$ we have

THEOREM 4.1.

$$\nu(x) = \mu(x)\hat{A}(M); \tag{8}$$

$$\nu^{\mathrm{per}}(x) = \mu^{\mathrm{per}}(x)\hat{A}(M), \tag{9}$$

*where $\hat{A}(M)$ is the A-class.*

### 4.2. HOCHSCHILD, CYCLIC AND PERIODIC HOMOLOGY WITH COMPACT SUPPORT

All the results above remain true if we replace Hochschild cochains with Hochschild cochains with compact support, cyclic homology with cyclic homology with compact support, and periodic cyclic homology with periodic cyclic homology with compact support.

### 4.3. CYCLIC COHOMOLOGY

Consider a dual version of Hochschild chains. For a smooth variety $M$, let $A$ be the algebra of smooth functions. Let $C_d^i(M) \subset \mathrm{Hom}(A^{\otimes i}, \Omega^{\mathrm{top}}(M))$ be the subspace of support-preserving morphisms. Let $C_{i,c}(M) \subset C_i(M)$ be the subset of elements with compact support. We have a map

$$k\colon C_d^i(M) \otimes C_{i,c}(A, A) \to \mathbb{R},$$
$$k(e, a_0 \otimes \cdots \otimes a_n) = \int_M a_0 e(a_1, a_2, \ldots, a_n).$$

We can define the differential $b\colon C_d^i(M) \to C_d^{i+1}(M)$ such that

$$k(bx, y) + (-1)^{|x|}k(x, by) = 0,$$

it is uniquely determined by this condition.

## 5. Applications to Index Theory

### 5.1. ALGEBRAIC STRUCTURE ON THE PAIR $C^\bullet(A, A)$, $C_d^\bullet(M)$

For a PCO $X$ define the PCO $X'$ with $X'(n) = X(n)$; $X'(n,1) = X(n,1)$, the insertion operations are the same as in $X$ except the following ones. For $u \in X'(n, 1)$, $v \in X'(m, 1)$, $u \circ_{X'} v = v \circ_X u$.

By the duality, the $T_\infty$-structure on $C^\bullet(A, A), C_{\bullet,c}(A, A)$ defines a canonical $T_\infty'$-structure on $C^\bullet(A, A)$, $C_d^\bullet(M)$. We can also prove that this structure is formal.

### 5.2. DEFORMED ALGEBRA

A part of the formality of the $T_\infty$-algebra $C^\bullet(A, A)$, $C_{\bullet,c}(A, A)$ is the formality of $C^\bullet(A, A)$ as a Lie algebra. We can use this formality, following Kontsevich [9], to construct the deformation star-product $\star$ on $A[[h]]$ corresponding to any Poisson bi-vector field $\pi \in hV^2(M)[[h]]$. The resulting associative algebra will be denoted by $A_\pi$. Let $m_\pi \in hC^\bullet(A, A)[[h]]$ be the corresponding Maurer–Cartan element: $a \star b = ab + m_\pi(a, b)$.

### 5.3. TRACES ON $A_\pi$

One sees that by its duality with $C_{\bullet,c}(A, A)$, the $C_c^b(M)$ is a module over the Lie algebra $C^\bullet(A, A)$. Consider the complex $C_d^\bullet(A_\pi) = (C_d^b(M)[h^{-1}, h]], b + L_{m_\pi})$. One sees that the zeroth cohomology of this complex is equal to the space of such formal top forms $\omega$ on $M$ that $\int_M (f \star g - g \star f)\omega = 0$ whenever $f$ or $g$ has compact support. The formality of the $T'_\infty$-algebra $(C^\bullet(A, A), C_d^\bullet(M))$ implies that the cohomology of $C_d^\bullet(A_\pi)$ is isomorphic to the cohomology of the complex $(\Omega^\bullet(M)[h^{-1}, h]], L_\pi)$. The latter cohomology is isomorphic to the space of all top forms $\omega \in \Omega^{\mathrm{top}}[h^{-1}, h]]$ on $M$ such that $L_\pi\omega = 0$. Denote by $\mathrm{Tr}_\omega$ the trace corresponding to $\omega$. By definition, any such a trace produces a linear functional on $HH_0(A, A)$ also denoted by $\mathrm{Tr}_\omega$.

### 5.4. ALGEBRAIC INDEX THEOREM

Now we have all the necessary ingredients to formulate an algebraic index theorem in the spirit of the index theory from [2, 3, 12]. We have the following diagram of maps

$$\begin{array}{ccc}
HC_0^{\mathrm{per}} \text{ over } \mathbb{R}(A_\pi) & \to & HC_0(A_\pi) \xrightarrow{\mathrm{Tr}_\omega} \mathbb{R}[h^{-1}, h]] \\
\downarrow & & \\
CC_0^{\mathrm{per}}(A) & & \\
\downarrow & & \\
H_{DR}^{\mathrm{even}}(M) & &
\end{array}$$

The composition of the vertical arrows is called the Chern character map and is denoted by ch. Denote the composition of the horizontal maps by $I$. The map ch is an isomorphism. Therefore $I$ can be expressed in terms of ch, such an expression is called the algebraic index theorem. The formality theorems stated above imply the following algebraic index theorem:

$$I(D) = \int_M \mathrm{ch}(D)(\mathrm{e}^{i_\pi}\omega)\hat{A}(M).$$

Note that the condition $L_\pi\omega = 0$ implies that $\mathrm{e}^{i_\pi}\omega$ is a closed form.

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